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ON EIGENSHARPNESS AND ALMOST EIGENSHARPNESS OF LEXICOGRAPHIC PRODUCTS OF SOME GRAPHS

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Abstract. The minimum number of complete bipartite subgraphs needed to partition the edges of a graph G is denoted by $b(G)$. A known lower bound on $b(G)$ states that $b(G) \geq \max\{p(G), q(G)\}$, where $p(G)$ and $q(G)$ are the numbers of positive and negative eigenvalues of the adjacency matrix of G, respectively. When equality is attained, G is said to be eigensharp and when $b(G) = \max\{p(G), q(G)\} + 1$, G is called an almost eigensharp graph. In this paper, we investigate the eigensharpness and almost eigensharpness of lexicographic products of some graphs.

1. Introduction

All graphs in this paper are finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *distance* between two distinct vertices x and y of G, denoted by $d(x, y)$, is the length of a shortest path connecting them, if such a path exists; otherwise, we set $d(x, y) := \infty$. If $x = y$, we assume that $d(x, y) = 0$. The *diameter* of a graph G is the supremum of the set $\{d(x, y) \,|\, x$ and y are distinct vertices of $G\}$, which is denoted by $diam(G)$. G is *connected* if there is a path between any two distinct vertices and is *complete* if it is connected with diameter one. A *clique* in a graph is a set of pairwise adjacent vertices while an *independent* set is a set of pairwise non-adjacent vertices. A *perfect matching* of G is a set of independent edges which covers all vertices of G. G is *bipartite* if $V(G)$ is the union of two disjoint independent sets called *partite* sets of G. A complete bipartite graph or a *biclique* is a special kind of bipartite graph where every vertex of the first set is adjacent to every vertex of the second set. When the sets have size r and s, the biclique is denoted by $K_{r,s}$. The complete bipartite graphs $K_{1,s}$ are called *stars*. The *adjacency matrix* of G denoted by A, is a square matrix of order n, with ij-th entry equals to 1 if v_iv_j is an edge of G and 0, otherwise. The eigenvalues and the spectrum of A are called the *eigenvalues* and the *spectrum* of G, respectively. Since A is real and symmetric, its eigenvalues are real. If

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all of the eigenvalues of G are integers, then G is called an *integral graph*. We denote the spectrum of G by

$$\sigma_A(G) = \{\gamma_1(G)^{[g_1]}, \gamma_2(G)^{[g_2]}, \ldots, \gamma_s(G)^{[g_s]}\},$$

where $\gamma_1(G) > \gamma_2(G) > \cdots > \gamma_s(G)$ are distinct eigenvalues of G and $\gamma_i(G)^{[g_i]}$ means that γ_i has multiplicity g_i. The *biclique partition number*, the minimum number of bicliques needed to partition the edges of a graph G is denoted by $b(G)$. It is an important invariant in graph theory and has been the topic of many investigations of researchers in the last few years, see [3–5] and [9]. This parameter have numerous applications to automata and language theories, partial orders, artificial intelligence and biology. In 1971, Graham and Pollak [4], proved that $b(K_n) = n-1$. Witsenhausen (c.f. [4]) showed that $b(G) \geq h(G)$ where $h(G) = \max\{p(G), q(G)\}$ and $p(G)$ and $q(G)$ are the numbers of positive and negative eigenvalues of G, respectively. Let $s(G)$ denote the number of zero eigenvalues of G. We say that G is *eigensharp* if the eigenvalue bound is sharp, i.e., $b(G) = h(G)$, and it is *almost eigensharp* if $b(G) = h(G)+1$. Several classes of graphs including trees, cycles C_n with $n = 4$ or $n \neq 4k$, prisms $C_n \square K_2$ with $n \neq 3k$ and some Cartesian products of cycles are shown to be eigensharp (c.f. [9]). Ghorbani and Maimani in [3], have studied the eigensharpness of some graphs with at most one cycle and products of some families of graphs. They also showed that $P_m \vee P_n$, $C_m \vee P_n$ for $m \equiv 2, 3 \pmod 4$ and Q_n, when n is an odd number, are eigensharp.

It is shown in [9] that $b(G * H) \leq 2b(G)b(H)$, where $*$ is the weak product. In this paper, we are interested in finding bounds on biclique partition number of lexicographic product of some graphs from their spectrum. For two graphs G and H, their *lexicographic product* (also known as *composition*) $G[H]$ is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$, with two vertices (x_1, y_1) and (x_2, y_2) being adjacent whenever $x_1 \sim x_2$ in G, or $x_1 = x_2$ and $y_1 \sim y_2$ in H. For more details about lexicographic product see [6,8]. In Section 2, first we obtain a bound for the biclique partition number of lexicographic product of two graphs and then characterize the eigensharp property for lexicographic products of some special graphs. We also show that the *Cocktail Party graph* CP_{2m}, a circulant graph obtained by removing $2m$ disjoint edges from K_{4m} is eigensharp. In Section 3, we discuss the eigensharpness of power of some graph compositions.

2. Eigensharpness of lexicographic products of some graphs

The *lexicographic product* of graphs was introduced in 1959 by Harary in [7], and independently, in the same year, by Sabidussi [10]. It is clear from the definition that if G and H are two nontrivial graphs with at least two vertices, then $G[H]$ is connected if and only if G is connected. The lexicographic product of graphs is a binary operation which may not be commutative, even when both factors are connected; but it satisfies the associative law. Also it is clear that if $|V(G)| = m$ and $|V(H)| = n$, then $|V(G[H])| = mn$ and $|E(G[H])| = m|E(H)| + n^2|E(G)|$. Recently, Abreu et al. [1] determined the spectrum of $G[H]$ when H is regular. We start with the following result. See [1, Corollary 2.2].

Lemma 2.1. *If G is a graph of order m with the spectrum*

$$\sigma_A(G) = \{\gamma_1(G)^{[g_1]}, \gamma_2(G)^{[g_2]}, \ldots, \gamma_s(G)^{[g_s]}\}$$

and if H is an r-regular graph of order n with the spectrum

$$\sigma_A(H) = \{r, \gamma_2(H)^{[h_2]}, \ldots, \gamma_t(H)^{[h_t]}\},$$

then

$$\sigma_A(G[H]) = \{(n\gamma_1(G) + r)^{[g_1]}, \ldots, (n\gamma_s(G) + r)^{[g_s]}\} \cup \{\gamma_2(H)^{[mh_2]}, \ldots, \gamma_t(H)^{[mh_t]}\}.$$

Definition. A *decomposition* of a graph G is a set of subgraphs $G_1, G_2, \ldots, G_r$ that partitions the edges of G such that $\bigcup_{1\leq i\leq r} E(G_i) = E(G)$ and $E(G_i) \cap E(G_j) = \emptyset$ for all $i \neq j$. If there is a decomposition $G_1, G_2, \ldots, G_r$ for G, we say that G is decomposed by $G_1, G_2, \ldots, G_r$ and denote it by $G = G_1 + G_2 + \cdots + G_r$.

Lemma 2.2. *Let G and H be two graphs on m and n vertices, respectively. Then*

$$b(G[H]) \leq b(G) + mb(H).$$

Proof. Let $G_i \subseteq V(G)$ and $H_i \subseteq V(H)$ such that $g_i \in G_i$, $h_i \in H_i$ for $i = 1, 2$ and $b(G), b(H)$ be the minimum number of bicliques of G and H, respectively. Here, $K_{|G_1|,|G_2|}$ denotes a biclique with G_1 and G_2 as partite sets. In order to get the upper bound, we only need to show that $G[H]$ can be decomposed by $b(G) + mb(H)$ bicliques. It follows from the definition of $G[H]$ that every edge g_1g_2 in G determines four edges $(g_1, h_1)(g_2, h_1)$, $(g_1, h_2)(g_2, h_2)$, $(g_1, h_1)(g_2, h_2)$ and $(g_1, h_2)(g_2, h_1)$ in $G[H]$. If $K_{|G_1|,|G_2|}$ is a biclique of G which contains g_1g_2, then the complete bipartite subgraph $K_{|G_1\times V(H)|,|G_2\times V(H)|}$ of $G[H]$ contains all these four edges except those edges $(g, h_1)(g, h_2)$ where $h_1h_2 \in E(H)$. Furthermore, if $g_1 = g_2$ and h_1h_2 is an edge of H which belongs to the subgraph $K_{|H_1|,|H_2|}$ of H, then the biclique $K_{|g_1\times H_1|,|g_1\times H_2|}$ of $G[H]$ contains the associated edge $(g_1, h_1)(g_1, h_2)$. Therefore, $b(G) + mb(H)$ bicliques of $G[H]$ include all edges of $G[H]$. □

Example 2.3. Let $G = P_3$ and $H = K_3$. Then $b(P_3) = 1$ and K_3 has two bicliques $K_{1,2}$ and K_2. As it is shown in Figure 1, the following seven bicliques partition the edges of $P_3[K_3]$.

$$\begin{gathered}\{(2,a),(2,b),(2,c)\} \cup \{(1,a),(1,b),(1,c),(3,a),(3,b),(3,c)\},\\ \{(1,b)\} \cup \{(1,a),(1,c)\}, \quad \{(1,a)\} \cup \{(1,c)\},\\ \{(2,b)\} \cup \{(2,a),(2,c)\}, \quad \{(2,a)\} \cup \{(2,c)\},\\ \{(3,b)\} \cup \{(3,a),(3,c)\}, \quad \{(3,a)\} \cup \{(3,c)\}.\end{gathered}$$

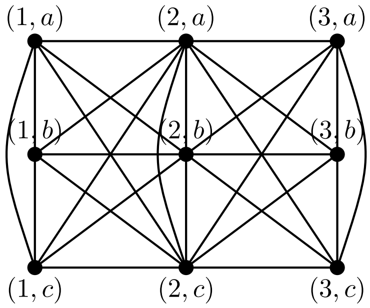

FIGURE 1. A biclique decomposition of $P_3[K_3]$

The following theorems provide a detailed exposition of eigenshapness of some graph compositions.

Theorem 2.4. *$K_m[K_n]$ is eigensharp for all $m, n \in \mathbb{N}$.*

Proof. From Lemma 2.1 we have

$$\sigma_A(K_m[K_n]) = \{mn-1, (-1)^{[mn-1]}\}.$$

So, $h(K_m[K_n]) = mn-1$. On the other hand, by definition of lexicographic product, $mK_n + K_{n,(m-1)n} + K_{n,(m-2)n} + \cdots + K_{n,n}$ is a decomposition of $K_m[K_n]$. Also, $b(K_n) = n-1$ by Graham-Pollak [4]. Therefore, $mn-1$ complete bipartite subgraphs partition the edges of $K_m[K_n]$ and $b(K_m[K_n]) = h(K_m[K_n])$. □

Remark 2.5. Note that according to Graham-Pollak [4], Theorem 2.4 may be summarized by saying that $K_m[K_n] \simeq K_{mn}$.

Theorem 2.6. *$K_{n,m}[K_t]$ is eigensharp for all $m, n \in \mathbb{N}$.*

Proof. It is shown in [2] that $\sigma_A(K_{n,m}) = \{\sqrt{nm}, 0^{[n+m-2]}, -\sqrt{nm}\}$ and $\sigma_A(K_t) = \{t-1, (-1)^{[t-1]}\}$. So, by Lemma 2.1,

$$\begin{aligned}\sigma_A(K_{n,m}[K_t]) = \{&(t\sqrt{nm}+t-1), (t-1)^{[n+m-2]}, (-t\sqrt{nm}+t-1)\}\\ &\cup \{(-1)^{[(n+m)(t-1)]}\}.\end{aligned}$$

Hence, we have $h(K_{n,m}[K_t]) = q(K_{n,m}[K_t]) = (n+m)(t-1)+1$. On the other hand, the edge set of $K_{n,m}[K_t]$ can be decomposed by $n+m$ complete subgraphs K_t and one subgraph $K_{tn,tm}$. Therefore, $(n+m)(t-1)+1$ bicliques partition the edges of $K_{n,m}[K_t]$. Thus, $b(K_{n,m}[K_t]) = h(K_{n,m}[K_t]) = (n+m)(t-1)+1$ and $K_{n,m}[K_t]$ is eigensharp. □

Remark 2.7. We know that $\sigma_A(C_n) = \{2\cos(\frac{2\pi j}{n}) : 0 \leq j < n\}$ by Brouwer [2]. Kratzke et al. in [9] proved that cycles C_n with $n = 4$ or $n \neq 4k$ are eigensharp.

In [3], Ghorbani and Maimani rephrased the theorem by determining $p(C_n)$ and $q(C_n)$. We summarize the results in Table 1.

TABLE 1. Eigensharpness of the cycle C_n.

n	$p(C_n)$	$q(C_n)$	$h(C_n)$	$b(C_n)$	Type
4	1	1	1	1	Eigensharp
$4k$	$\frac{n-2}{2}$	$\frac{n-2}{2}$	$\frac{n-2}{2}$	$\frac{n}{2}$	Almost eigensharp
$4k+1$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	Eigensharp
$4k+2$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	Eigensharp
$4k+3$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	Eigensharp

The next theorem is an analogue of the result stated in the remak.

Theorem 2.8. *$K_m[C_n]$ is eigensharp for $n = 4$ and $n \equiv 2, 3 \pmod 4$.*

Proof. Let $G = K_m[C_n]$. If $n = 4$, then by Lemma 2.1, the spectrum of $K_m[C_4]$ is $\{4m - 2, -2^{[2m-1]}, 0^{[2m]}\}$. Hence, $h(K_m[C_4]) = 2m - 1$. Also, it follows from Lemma 2.2 and Table 1 that $b(K_m[C_4]) \leq 2m - 1$. Therefore, $h(K_m[C_4]) = b(K_m[C_4])$ and $K_m[C_4]$ is an eigensharp graph.

For $n \neq 4$, again by Lemma 2.2 and Table 1, $b(G) \leq m - 1 + m\lceil \frac{n}{2} \rceil$. In other words, $b(G) \leq \frac{mn+2m-2}{2}$ for $n = 4k, 4k + 2$ and $b(G) \leq \frac{mn+3m-2}{2}$ for $n = 4k + 1, 4k + 3$. What is left is to evaluate the number of positive and negative eigenvalues of G. Set $n = 4k + 2$, Lemma 2.1 shows that

$$\sigma_A(G) = \{mn - n + 2, (-n + 2)^{[m-1]}\} \bigcup \{(\sigma_A(C_n) \setminus \{2\})^{[m]}\}.$$

In view of Table 1, $p(C_n) = q(C_n) = \frac{n}{2}$. Thus, it is evident that $q(G) = mq(C_n) + m - 1 = \frac{mn+(2m-2)}{2}$ and $p(G) = m(p(C_n) - 1) + 1 = \frac{mn-(2m-2)}{2}$. Thus, $q(G) = h(G) = b(G)$ and G is eigensharp. Similar arguments apply to the other cases of n. We arranged the amount of $p(G)$ and $q(G)$ in Table 2.

TABLE 2. The number of $p(G)$ and $q(G)$ for $G = K_m[C_n]$.

n	$p(G)$	$q(G)$	$b(G)$
4	1	$2m-1$	$\leq 2m-1$
$4k$	$\frac{mn-4m+2}{2}$	$\frac{mn-2}{2}$	$\leq \frac{mn+2m-2}{2}$
$4k+1$	$\frac{mn-(m-2)}{2}$	$\frac{mn+(m-2)}{2}$	$\leq \frac{mn+3m-2}{2}$
$4k+2$	$\frac{mn-(2m-2)}{2}$	$\frac{mn+(2m-2)}{2}$	$\leq \frac{mn+2m-2}{2}$
$4k+3$	$\frac{mn-(3m-2)}{2}$	$\frac{mn+(3m-2)}{2}$	$\leq \frac{mn+3m-2}{2}$

Now it is easy to check the eigensharpness of G from Table 2. For instance, if $n = 4k + 3$, then

$$\frac{mn + (3m - 2)}{2} = h(G) \leq b(G) \leq \frac{mn + (3m - 2)}{2}.$$

So, $h(G) = b(G)$ and G is eigensharp. □

Corollary 2.9. *The Cocktail Party graph CP_{2m} is eigensharp.*

Proof. First note that by definition of lexicographic product, $K_m[K_4]$ is the complete graph K_{4m}. Let $V(K_m) = \{1, 2, \ldots, m\}$ and $a{-}b{-}c{-}d{-}a$ be a 4-cycle of K_4. Consider two edge sets $E_1 = \{(i,a)(i,c)\}_{i=1}^m$ and $E_2 = \{(i,b)(i,d)\}_{i=1}^m$ of $E(K_m[K_4])$. It is clear that $E_1 \cup E_2$ is a perfect matching of $K_m[K_4]$. Furthermore, removing $2m$ edges of E_1 and E_2 from $K_m[K_4]$ induces the subgraph $K_m[C_4]$. Hence, $K_m[C_4]$ is the Cocktail Party graph CP_{2m} which is eigensharp by Theorem 2.8. □

Theorem 2.10. *$K_{n,m}[C_t]$ is eigensharp for $t = 4$ and $t \equiv 2, 3 \pmod 4$.*

Proof. Let $G = K_{n,m}[C_t]$. According to Lemma 2.2 and Table 1, $\sigma_A(G) = \{t\sqrt{mn}+2, -t\sqrt{mn}+2, 2^{[n+m-2]}\} \bigcup \{(\sigma_A(C_t) \setminus \{2\})^{[n+m]}\}$ and $b(G) \le 1 + (n+m)\lceil \frac{t}{2} \rceil$. Hence, we obtain Table 3.

Table 3. $p(G)$ and $q(G)$ for $G = K_{n,m}[C_t]$.

t	$p(G)$	$q(G)$	$b(G) \le$
4	$n+m-1$	$n+m+1$	$n+m+1$
$4k$	$\frac{(t-2)(n+m)-2}{2}$	$\frac{(t-2)(n+m)+2}{2}$	$\frac{t(n+m)+2}{2}$
$4k+1$	$\frac{(t+1)(n+m)-2}{2}$	$\frac{(t-1)(n+m)+2}{2}$	$\frac{(t+1)(n+m)+2}{2}$
$4k+2$	$\frac{t(n+m)-2}{2}$	$\frac{t(n+m)+2}{2}$	$\frac{t(n+m)+2}{2}$
$4k+3$	$\frac{(t-1)(n+m)-2}{2}$	$\frac{(t+1)(n+m)+2}{2}$	$\frac{(t+1)(n+m)+2}{2}$

For example, if $t = 4k{+}2$, then by Table 1, $p(G) = n{+}m{-}1{+}(n{+}m)(\frac{t}{2}{-}1) = \frac{t(n+m)-2}{2}$ and $q(G) = 1 + (n+m)(\frac{t}{2}) = \frac{t(n+m)+2}{2}$. Hence, $\frac{t(n+m)+2}{2} = h(G) \le b(G) \le \frac{t(n+m)+2}{2}$ and G is eigensharp. The same argument applies to cases $t = 4$ and $t = 4k+3$. □

Lemma 2.11. *Let G and H be two graphs on m and n vertices, respectively. Assume that H is r-regular. If G is integral or has no eigenvalues in $[\frac{-r}{n}, 0)$, then*

$$s(G[H]) = ms(H), q(G[H]) = mq(H) + q(G) \text{ and } p(G[H]) = mp(H) - q(G).$$

Proof. Lemma 2.1 shows that the number of eigenvalues of H distributes between $s(G[H])$, $p(G[H])$ and $q(G[H])$. Similarly, zero and positive eigenvalues of G enumerate the positive eigenvalues of $G[H]$ while negative eigenvalues of G do not so. Therefore, we have the following bounds for $s(G[H])$, $p(G[H])$ and $q(G[H])$:

$$s(G[H]) \ge ms(H), \quad q(G[H]) \le mq(H) + q(G),$$

and

$$p(G[H]) \geq p(G) + s(G) + m(p(H) - 1) = m - q(G) + mp(H) - m$$
$$= mp(H) - q(G).$$

Now, since G has no eigenvalues in $[\frac{-r}{n}, 0)$, if λ is a negative eigenvalue of G, then $n\lambda + r$ is a negative eigenvalue of $G[H]$. Therefore, each negative eigenvalue of G creates a negative eigenvalue for $G[H]$, which completes the proof. □

Theorem 2.12. *Let G and H be two graphs of order m and n, respectively. Let H be r-regular and G have no eigenvalue in $[\frac{-r}{n}, 0)$. Then $G[H]$ is*

(a) *eigensharp if G and H are eigensharp with $b(G) = q(G)$ and $b(H) = q(H)$,*
(b) *eigensharp or almost eigensarp if G is almost eigensharp with $b(G) = q(G) + 1$ and H is eigensharp with $b(H) = q(H)$.*

Proof. As an immediate consequence of Lemma 2.1 and Lemma 2.11 we have $mq(H) + q(G) = h(G[H]) \leq b(G[H]) \leq mb(H) + b(G) = mq(H) + q(G)$. So $b(G[H]) = h(G[H])$ and the graph is eigensharp. Similarly, for the other case $h(G[H]) \leq b(G[H]) \leq h(G[H]) + 1$ and the graph is eigensharp or almost eigensarp. □

Corollary 2.13. *$K_m[Q_n]$ is eigensharp if and only if n is odd or $n = 2$.*

Proof. It is shown in [3] that the n-cube Q_n is an eigensharp graph with $b(Q_n) = q(Q_n) = 2^{n-1}$ if and only if n is odd or $n = 2$. Thus, the result follows by Theorem 2.12. □

Definition. We say that G is *t-eigensharp* or *almost t-eigensharp* if $b(G) = q(G) + t$ or $b(G) = q(G) + 1 + t$, respectively.

Theorem 2.14. *There exists an even integer $0 \leq t \leq \lceil \frac{m}{2} \rceil$ such that $C_m[C_n]$ is*

(a) *at most t-eigensharp for $n, m \in \{4k+2, 4k+3; k = 0, 1, 2, \ldots\} \cup \{4\}$,*
(b) *at most almost t-eigensharp for $m \in \{4k+1; k = 0, 1, 2, \ldots\}$ and $n \in \{4k+2, 4k+3; k = 0, 1, 2, \ldots\} \cup \{4\}$,*
(c) *at most $t+m$-eigensharp for $m \in \{4k+2, 4k+3; k = 0, 1, 2, \ldots\} \cup \{4\}$ and $n \in \{4k; k = 0, 1, 2, \ldots\}$,*
(d) *at most almost $t+m$-eigensharp for $m \in \{4k+1; k = 0, 1, 2, \ldots\}$ and $n \in \{4k; k = 0, 1, 2, \ldots\}$.*

Proof. Let t be the number of roots of $\chi_{C_m}(x) = \prod_{j=0}^{m-1}(x - 2\cos\frac{2\pi j}{m})$, the characteristic polynomial of C_m, in $[\frac{-2}{n}, 0)$. As it is shown in [3], one has

$$\chi_{C_m}(0) = \begin{cases} -2 & m \text{ is odd}, \\ 0 & m = 4k, \\ -4 & m = 4k+2. \end{cases}$$

TABLE 4. The number of positive and negative eigenvalues of $C_m[C_n]$.

$C_m[C_n]$	$m=4$	$m=4k$	$m=4k+1$	$m=4k+2$	$m=4k+3$
$n=4$	$p=3$ $q=5$	$p=\frac{m+2}{2}+s$ $q=\frac{3m-2}{2}-s$	$p=\frac{m+1}{2}+t$ $q=\frac{3m-1}{2}-t$	$p=\frac{m}{2}+t$ $q=\frac{3m}{2}-t$	$p=\frac{m-2}{2}+t$ $q=\frac{3m+1}{2}-t$
$n=4k$	$p=2n-5$ $q=2n-3$	$p=\frac{mn-3m+2}{2}+s$ $q=\frac{mn-m-2}{2}-s$	$p=\frac{mn-3m+1}{2}+t$ $q=\frac{mn-m-1}{2}-t$	$p=\frac{mn-3m}{2}+t$ $q=\frac{mn-m}{2}-t$	$p=\frac{mn-3m-1}{2}+t$ $q=\frac{mn-m+1}{2}-t$
$n=4k+1$	$p=2n+1$ $q=2n-1$	$p=\frac{mn+2}{2}+s$ $q=\frac{mn-2}{2}-s$	$p=\frac{mn+1}{2}+t$ $q=\frac{mn-1}{2}-t$	$p=\frac{mn}{2}+t$ $q=\frac{mn}{2}-t$	$p=\frac{mn-1}{2}+t$ $q=\frac{mn+1}{2}-t$
$n=4k+2$	$p=2n-1$ $q=2n+1$	$p=\frac{mn-m+2}{2}+s$ $q=\frac{mn+m-2}{2}-s$	$p=\frac{mn-m+1}{2}+t$ $q=\frac{mn+m-1}{2}-t$	$p=\frac{mn-m}{2}+t$ $q=\frac{mn+m}{2}-t$	$p=\frac{mn-m-1}{2}+t$ $q=\frac{mn+m+1}{2}-t$
$n=4k+3$	$p=2n-3$ $q=2n+3$	$p=\frac{mn-2m+2}{2}+s$ $q=\frac{mn+2m-2}{2}-s$	$p=\frac{mn-2m+1}{2}+t$ $q=\frac{mn+2m-1}{2}-t$	$p=\frac{mn-2m}{2}+t$ $q=\frac{mn+2m}{2}-t$	$p=\frac{mn-2m-1}{2}+t$ $q=\frac{mn+2m+1}{2}-t$

Let $m \neq 4k$ be an even integer. Then,

$$\chi_{C_m}(\frac{-2}{n}) = (-2)^m(\frac{1}{n}+1)(\frac{1}{n}-1)\prod_{j=1}^{\frac{m-2}{2}}(\frac{1}{n}+\cos\frac{2\pi j}{m})^2 < 0.$$

Similarly, for an odd integer m, $\chi_{C_m}(\frac{-2}{n}) < 0$. Therefore, C_m has even negative eigenvalues in $(\frac{-r}{n}, 0)$ which create t positive eigenvalues of $C_m[C_n]$. Thus, $p(C_m[C_n]) = mp(C_n) - q(C_m) + t$ and $q(C_m[C_n]) = mq(C_n) + q(C_m) - t$ by Lemma 2.11 and Table 1. Consequently, part (a) follows by Lemma 2.2 and Table 1. That is,

$$\begin{aligned} q(C_m[C_n]) = mq(C_n) + q(C_m) - t &\leq b(C_m[C_n]) \\ &\leq mq(C_n) + q(C_m) = q(C_m[C_n]) + t. \end{aligned}$$

For part (b), it follows from Table 1 that

$$\begin{aligned} q(C_m[C_n]) &\leq b(C_m[C_n]) \\ &\leq mq(C_n) + p(C_m) = mq(C_n) + q(C_m) + 1 = q(C_m[C_n]) + 1 + t. \end{aligned}$$

Thus, $C_m[C_n]$ is at most almost t-eigensharp. An analogical argument may apply to the other cases.

For $m = 4k$, $C_m[C_n]$ is at most almost $s+m$-eigensharp for $n \equiv 0, 1 \pmod 4$ and almost s-eigensharp otherwise, for some $0 \leq s \leq \lceil \frac{m}{2} \rceil - 2$. $\square$

Corollary 2.15. *$C_4[C_n]$ is eigensharp for $n = 4$ or $n \equiv 2, 3 \pmod 4$ and at most 4-eigensharp for $n = 4k, k \neq 1$.*

Theorem 2.16. *Let $m \neq 4k$. Then there exists an even integer $0 \leq t \leq \lceil \frac{m}{2} \rceil$ such that $C_m[K_n]$ is at most t-eigensharp for $m = 4$ or $m \equiv 2, 3 \pmod 4$ and at most almost t-eigensharp for $m \equiv 1 \pmod 4$. In particular, it is eigensharp for $m = 4$.*

Proof. By a similar argument as in the proof of Theorem 2.14, C_m has no eigenvalues in $[\frac{1-n}{n}, 0)$. We arranged the values of $p(C_m[K_n])$ and $q(C_m[K_n])$ in Table 5. According to Table 1, for $m = 4$ or $m \equiv 2, 3 \pmod 4$, C_m is eigensharp with $b(C_m) = q(C_m)$. Thus, we have

$$q(C_m[K_n]) \leq b(C_m[K_n]) \leq mq(K_n) + q(C_m) = q(C_m[K_n]) + t.$$

If $m = 4k + 1$, then

$$\begin{aligned} q(C_m[K_n]) = \frac{2mn - m - 1}{2} - t \leq b(C_m[K_n]) \\ \leq \frac{m+1}{2} + m(n-1) = q(C_m[K_n]) + 1 + t. \end{aligned}$$

So, $C_m[K_n]$ is at most almost t-eigensharp. For $m = 4k$, $C_m[K_n]$ is at most almost s-eigensharp where s is the number of eigenvalues of C_m in $[\frac{1-n}{n}, 0)$.

Table 5. $p(G)$ and $q(G)$ for $G = C_m[K_n]$.

m	$p(C_m[K_n])$	$q(C_m[K_n])$	Type
4	3	$4n - 3$	Eigensharp
$4k$	$\frac{m+2}{2} + s$	$\frac{2mn-m-2}{2} - s$	At most almost s-eigensharp
$4k + 1$	$\frac{m+1}{2} + t$	$\frac{2mn-m-1}{2} - t$	At most almost t-eigensharp
$4k + 2$	$\frac{m}{2} + t$	$\frac{2mn-m}{2} - t$	At most t-eigensharp
$4k + 3$	$\frac{m-1}{2} + t$	$\frac{2mn-m+1}{2} - t$	At most t-eigensharp

□

3. Eigensharpness of powers of lexicographic products of some graphs

Recently, Abreu et al. [1, Corollary 3.4] determined the spectrum of G^k, the kth power of a regular graph G with respect to the lexicographic product. That is, $G^0 = K_1$, $G^1 = G$ and $G^k = G^{k-1}[G]$ for $k \geq 2$.

Lemma 3.1. *Let G be a connected q-regular graph of order m with*

$$\sigma_A(G) = \{q, \gamma_2(G)^{[g_2]}, \ldots, \gamma_s(G)^{[g_s]}\}.$$

Then, for each integer $k \geq 1$, G^k is r_k-regular of order v_k, with

$$v_k = m^k, \qquad r_k = q\frac{m^k - 1}{m - 1},$$

and

$$\sigma_A(G^k) = \Big(\bigcup_{i=0}^{k-1}\{(m^i\gamma_2(G)+r_i)^{[g_2m^{k-1-i}]},\ldots,(m^i\gamma_s(G)+r_i)^{[g_sm^{k-1-i}]}\}\Big)\cup\{r_k\}.$$

Lemma 3.2. *Let G be a connected q-regular graph of order m. If G has no eigenvalue in the interval $[\frac{-r_i}{m^i},0]$ for $0\le i\le k-1$, where $r_i = q\frac{m^i-1}{m-1}$, then*

$$p(G^k) = m^k - q(G^k), \qquad q(G^k) = q(G)\frac{m^k-1}{m-1}.$$

Proof. Let $\{\gamma_1(G)^{[g_1]},\ldots,\gamma_t(G)^{[g_t]}\}$ be the negative eigenvalues of G. Since G has no eigenvalues in $[\frac{-r_i}{m^i},0]$ for $0\le i\le k-1$,

$$\bigcup_{i=0}^{k-1}\{(m^i\gamma_1(G)+r_i)^{[g_1m^{k-1-i}]},\ldots,(m^i\gamma_t(G)+r_i)^{[g_tm^{k-1-i}]}\},$$

is the set of negative eigenvalues of G^k by Lemma 3.1. Therefore,

$$\begin{aligned} q(G^k) &= g_1(1+\cdots+m^{k-2}+m^{k-1})+\cdots+g_t(1+\cdots+m^{k-2}+m^{k-1})\\ &= (g_1+\cdots+g_t)(1+\cdots+m^{k-2}+m^{k-1})\\ &= q(G)\frac{m^k-1}{m-1},\end{aligned}$$

and similarly,

$$\begin{aligned} p(G^k) &= (p(G)-1)\frac{m^k-1}{m-1}+s(G)\frac{m^k-1}{m-1}+1\\ &= \frac{(m-1-q(G))(m^k-1)+m-1}{m-1}\\ &= m^k - q(G)\frac{m^k-1}{m-1}\\ &= m^k - q(G^k). \end{aligned}$$

□

Theorem 3.3. *Let G be a connected q-regular graph of order m. Assume that G has no eigenvalues in $[\frac{-r_i}{m^i},0]$ for $0\le i\le k-1$, where $r_i = q\frac{m^i-1}{m-1}$. If G is an eigensharp graph with $b(G)=q(G)$, then so is G^k.*

Proof. By Lemma 2.2 and induction on k, one has $b(G^k)\le\frac{m^k-1}{m-1}b(G)$. On the other hand,

$$\frac{m^k-1}{m-1}q(G) = q(G^k)\le h(G^k)\le b(G^k)\le\frac{m^k-1}{m-1}q(G)$$

by Lemma 3.2. Hence, G^k is an eigensharp graph. □

Corollary 3.4. K_n^k *and* C_4^k *are eigensharp.*

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