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BLOW-UP PHENOMENA OF ARBITRARY POSITIVE INITIAL ENERGY SOLUTIONS FOR A VISCOELASTIC WAVE EQUATION WITH NONLINEAR DAMPING AND SOURCE TERMS

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Abstract. In this paper, we considered the Dirichlet initial boundary value problem of a nonlinear viscoelastic wave equation with nonlinear damping and source terms, and investigated finite time blow-up phenomena of the solutions to the equation with arbitrary positive initial data, under suitable conditions.

1. Introduction

Consider the viscoelastic wave equation with nonlinear damping and source terms

$$\begin{aligned} & u_{tt} - \Delta u_t - \operatorname{div}(|\nabla u|^{\alpha-2}\nabla u) - \operatorname{div}(|\nabla u_t|^{\beta-2}\nabla u_t) + |u_t|^{m-2}u_t \\ (1.1) \quad & = |u|^{p-2}u, \quad (x,t) \in \Omega \times (0,\infty), \end{aligned}$$

under the homogeneous Dirichlet boundary and initial conditions

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and $\alpha, \beta, p, m > 2$ are constants and $u_0, u_1 : \Omega \to \mathbb{R}$ are given initial data.

The equation (1.1) appears in the models of nonlinear viscoelasticity such as the system governing the longitudinal motion of a viscoelastic

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configuration obeying a nonlinear Voight model, see [1] and [3]. The interaction between the damping and source terms was firstly investigated by Levine, considering the semilinear wave equation

$$u_{tt} - \Delta u + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.4}$$

where $m = 2$, and the author showed that the solutions with negative initial energy blow up in finite time (cf. [4], [5]). Later, Georgiev and Todorova [2] extended Levine's results to the nonlinear case, i.e., $m > 2$, and they introduced a different method and showed that global solutions of (1.4) exist if $2 < p \leq m$ and the initial energy is negative, and that the solution blows up in finite time if $2 < m < p$ and the initial energy is sufficiently negative. Vitillaro [9] considered an abstract equation, including linear and quasilinear cases, with nonlinear damping term, and presented a blow-up result of the solutions with small positive energy. Messaoudi [6] showed that any solution of (1.4) with negative initial energy blows up in finite time if $m < p$, whereas Georgiev and Todorova [2] showed that the solutions of (1.4) blow up in finite time if the initial energy is sufficiently negative.

Recently, Yang [10] studied problem (1.1)-(1.3) with $p > \max\{\alpha, m\}$ and $\alpha > \beta$, and established a blow-up result, when the initial energy is sufficiently negative. Afterwards, Messaoudi and Houari [7] extended the result to the case that the initial energy is negative. Besides, on the blow-up of solutions with arbitrary positive initial energy for other type wave equations, one can refer to [8] and [11].

In this paper, inspired by [7] and [8], we establish a blow-up result for the solutions to problem (1.1)-(1.3) with arbitrary positive initial energy. We first introduce the following function space, energy functional, and lemma:

$$\begin{aligned} Z = {} & L^\infty([0,T]; W_0^{1,\alpha}(\Omega)) \cap W^{1,\infty}([0,T]; L^2(\Omega)) \\ & \cap W^{1,\beta}([0,T]; W_0^{1,\beta}(\Omega)) \ \cap \ W^{1,m}([0,T]; L^m(\Omega)), \end{aligned}$$

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha - \frac{1}{p}\|u\|_p^p, \quad t > 0, \tag{1.5}$$

LEMMA 1.1. *Suppose that $\alpha, \beta, p, m > 2$, $\alpha > \beta$, and $max\{m, \alpha\} < p < r_\alpha$, where r_α is the Sobolev critical exponent of $W_0^{1,\alpha}(\Omega)$. If there exists a number $t_0 \geq 0$ such that $E(t_0) < 0$, then the solution $u \in Z$ of problem (1.1)-(1.3) blows up in finite time.*

REMARK 1.2. We remind that $r_\alpha = \dfrac{N\alpha}{N-\alpha}$, if $N > \alpha$, and $r_\alpha > \alpha$, if $N = \alpha$. In addition, $r_\alpha = \infty$, if $N < \alpha$.

Lemma 1.1 can be easily proved by a similar argument as in [7] with slight modification, and hence, we omit the proof here.

2. Main result and an example

THEOREM 2.1. *Suppose that* $\alpha, \beta, p, m > 2$, $\alpha > \beta$, *and* $max\{m, \alpha\} < p < r_\alpha$. *If* $u(t) \in Z$ *is a solution of problem (1.1)-(1.3) satisfying*

$$E(0) > 0 \text{ and } \int_\Omega u_0 u_1 \mathrm{d}x > ME(0), \tag{2.1}$$

then $u(t)$ *blows up in finite time, where*

$$M = \eta(\varepsilon_0),$$

$$\eta(\varepsilon) = C_2 \left[\frac{p(1-\varepsilon)}{\alpha} - 1\right]^{-1} + \frac{\beta - 1}{\beta}\left\{\frac{\beta}{4C_1}\left[\frac{p(1-\varepsilon)}{\alpha} - 1\right]\right\}^{-\frac{1}{\beta-1}} + \frac{m-1}{m}\left(\frac{p\varepsilon}{1-\theta}\right)^{-\frac{1}{m-1}},$$

$$\theta = \frac{p-m}{p-2},$$

and $\varepsilon_0 \in (0, 1)$ *is the root of the equation* $\dfrac{p(1-\varepsilon_0)}{\gamma(\varepsilon_0)} = \eta(\varepsilon_0)$,

$$\gamma(\varepsilon) = 2\sqrt{\left[1 + \frac{p(1-\varepsilon)}{2}\right]\left\{\frac{\lambda_1}{2C_2}\left[\frac{p(1-\varepsilon)}{\alpha} - 1\right] - \frac{p\theta\varepsilon}{2(1-\theta)}\right\}},$$

$\lambda_1 > 0$ *is the first eigenvalue of* $-\Delta$, *and* C_1 *and* C_2 *are positive constants.*

Proof. By contradiction, we assume that $u(t) \in Z$ is a global solution of problem (1.1)-(1.3). Multiplying the both sides of equation (1.1) by u_t and integrating the result over Ω, one can see that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\|\nabla u_t\|_2^2 - \|\nabla u_t\|_\beta^\beta - \|u_t\|_m^m \leq 0. \tag{2.2}$$

It follows from Green's formula that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u u_t \mathrm{d}x &= \|u_t\|_2^2 + \int_\Omega u u_{tt} \mathrm{d}x \\ &= \|u_t\|_2^2 - \|\nabla u\|_\alpha^\alpha - \int_\Omega |\nabla u_t|^{\beta-2}\nabla u_t \cdot \nabla u \mathrm{d}x \\ &\quad - \int_\Omega \nabla u_t \cdot \nabla u \mathrm{d}x - \int_\Omega |u_t|^{m-2} u_t u \mathrm{d}x + \|u\|_p^p, \end{aligned} \tag{2.3}$$

by multiplying the both sides of (1.1) by u and integrating the result over Ω.

In order to estimate the right side of (2.3), we use Hölder's inequality and Young's inequality and obtain the inequalities

$$
\begin{aligned}
\left|\int_\Omega \nabla u_t \cdot \nabla u \mathrm{d}x\right| &\leq \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u_t|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&= \left[(2\mu)^{\frac{1}{2}}\|\nabla u\|_2\right]\left[\frac{1}{(2\mu)^{\frac{1}{2}}}\|\nabla u_t\|_2\right] \\
&\leq \frac{1}{2}\left[(2\mu)^{\frac{1}{2}}\|\nabla u\|_2\right]^2 + \frac{1}{2}\left[\frac{1}{(2\mu)^{\frac{1}{2}}}\|\nabla u_t\|_2\right]^2 \\
&= \mu\|\nabla u\|_2^2 + \frac{1}{4\mu}\|\nabla u_t\|_2^2,
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
\left|\int_\Omega |\nabla u_t|^{\beta-2}\nabla u_t \cdot \nabla u \mathrm{d}x\right| &\leq \left(\int_\Omega |\nabla u|^\beta \mathrm{d}x\right)^{\frac{1}{\beta}} \left(\int_\Omega |\nabla u_t|^\beta \mathrm{d}x\right)^{\frac{\beta-1}{\beta}} \\
&= \left(\lambda\|\nabla u\|_\beta\right)\left(\lambda^{-1}\|\nabla u_t\|_\beta^{\beta-1}\right) \\
&\leq \frac{1}{\beta}\left(\lambda\|\nabla u\|_\beta\right)^\beta + \frac{\beta-1}{\beta}\left(\lambda^{-1}\|\nabla u_t\|_\beta^{\beta-1}\right)^{\frac{\beta}{\beta-1}} \\
&= \frac{\lambda^\beta}{\beta}\|\nabla u\|_\beta^\beta + \frac{\beta-1}{\beta}\lambda^{-\frac{\beta}{\beta-1}}\|\nabla u_t\|_\beta^\beta,
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
\left|\int_\Omega |u_t|^{m-2}u_t u \, \mathrm{d}x\right| &\leq \left(\int_\Omega |u|^m \mathrm{d}x\right)^{\frac{1}{m}} \left(\int_\Omega |u_t|^m \mathrm{d}x\right)^{\frac{m-1}{m}} \\
&= \left(\delta\|u\|_m\right)\left(\delta^{-1}\|u_t\|_m^{m-1}\right) \\
&\leq \frac{1}{m}\left(\delta\|u\|_m\right)^m + \frac{m-1}{m}\left(\delta^{-1}\|u_t\|_m^{m-1}\right)^{\frac{m}{m-1}} \\
&= \frac{\delta^m}{m}\|u\|_m^m + \frac{m-1}{m}\delta^{-\frac{m}{m-1}}\|u_t\|_m^m,
\end{aligned}
\tag{2.6}
$$

where $\mu, \lambda, \delta > 0$ are constants which will be specified later. Substituting inequalities (2.4)-(2.6) into (2.3), one can have the inequality

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega uu_t\,\mathrm{d}x \geq & \|u_t\|_2^2 - \|\nabla u\|_\alpha^\alpha - \mu\|\nabla u\|_2^2 - \frac{1}{4\mu}\|\nabla u_t\|_2^2 \\ & - \frac{\lambda^\beta}{\beta}\|\nabla u\|_\beta^\beta - \frac{\beta-1}{\beta}\lambda^{-\frac{\beta}{\beta-1}}\|\nabla u_t\|_\beta^\beta - \frac{\delta^m}{m}\|u\|_m^m \\ & - \frac{m-1}{m}\delta^{-\frac{m}{m-1}}\|u_t\|_m^m + \|u\|_p^p. \end{aligned} \tag{2.7}$$

On the other hand, it follows from (1.5) that

$$\|u\|_p^p = -pE(t) + \frac{p}{2}\|u_t\|_2^2 + \frac{p}{\alpha}\|\nabla u\|_\alpha^\alpha,$$

and from (2.2) one can see that

$$-\|\nabla u_t\|_2^2 \geq \frac{\mathrm{d}}{\mathrm{d}t}E(t),$$

$$-\|\nabla u_t\|_\beta^\beta \geq \frac{\mathrm{d}}{\mathrm{d}t}E(t),$$

and

$$-\|u_t\|_m^m \geq \frac{\mathrm{d}}{\mathrm{d}t}E(t).$$

Then inequality (2.7) can be written as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega uu_t\mathrm{d}x \geq & \|u_t\|_2^2 - \|\nabla u\|_\alpha^\alpha - \mu\|\nabla u\|_2^2 - \frac{1}{4\mu}\|\nabla u_t\|_2^2 \\ & - \frac{\lambda^\beta}{\beta}\|\nabla u\|_\beta^\beta - \frac{\beta-1}{\beta}\lambda^{-\frac{\beta}{\beta-1}}\|\nabla u_t\|_\beta^\beta - \frac{\delta^m}{m}\|u\|_m^m \\ & - \frac{m-1}{m}\delta^{-\frac{m}{m-1}}\|u_t\|_m^m + \varepsilon\|u\|_p^p + (1-\varepsilon)\|u\|_p^p \\ \geq & \left(\frac{1}{4\mu} + \frac{\beta-1}{\beta}\lambda^{-\frac{\beta}{\beta-1}} + \frac{m-1}{m}\delta^{-\frac{m}{m-1}}\right)\frac{\mathrm{d}}{\mathrm{d}t}E(t) \\ & + \left[1 + \frac{p(1-\varepsilon)}{2}\right]\|u_t\|_2^2 + \left[\frac{p(1-\varepsilon)}{\alpha} - 1\right]\|\nabla u\|_\alpha^\alpha \\ & - \mu\|\nabla u\|_2^2 - \frac{\lambda^\beta}{\beta}\|\nabla u\|_\beta^\beta - \frac{\delta^m}{m}\|u\|_m^m + \varepsilon\|u\|_p^p - p(1-\varepsilon)E(t), \end{aligned}$$

where ε is a constant small enough, from which we can have the inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega uu_t\mathrm{d}x - \left(\frac{1}{4\mu} + \frac{\beta-1}{\beta}\lambda^{-\frac{\beta}{\beta-1}} + \frac{m-1}{m}\delta^{-\frac{m}{m-1}}\right)E(t)\right] \\
(2.8)\quad &\geq \left[1 + \frac{p(1-\varepsilon)}{2}\right]\|u_t\|_2^2 + \left[\frac{p(1-\varepsilon)}{\alpha} - 1\right]\|\nabla u\|_\alpha^\alpha - \mu\|\nabla u\|_2^2 \\
&- \frac{\lambda^\beta}{\beta}\|\nabla u\|_\beta^\beta - \frac{\delta^m}{m}\|u\|_m^m + \varepsilon\|u\|_p^p - p(1-\varepsilon)E(t).
\end{aligned}
$$

Using Hölder's inequality, one can see that there exists a constant $C_1 > 0$ such that

$$
\|\nabla u\|_\beta^\beta \leq |\Omega|^{1-\frac{\beta}{\alpha}}\|\nabla u\|_\alpha^\beta \leq C_1\|\nabla u\|_\alpha^\alpha. \tag{2.9}
$$

In fact, we have $\|\nabla u\|_\alpha^\beta \leq \|\nabla u\|_\alpha^\alpha$, if $\|\nabla u\|_\alpha \geq 1$, and there exists a constant $C > 0$ such that $\|\nabla u\|_\alpha^\beta \leq C\|\nabla u\|_\alpha^\alpha$, if $0 < \|\nabla u\|_\alpha < 1$. The inequality (2.9) clearly holds, if $\|\nabla u\|_\alpha = 0$. Similarly, it can be seen that there exists a constant $C_2 > 0$ such that

$$
\|\nabla u\|_2^2 \leq C_2\|\nabla u\|_\alpha^\alpha.
$$

Then, from (2.8), we have the inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega uu_t\mathrm{d}x - \left(\frac{1}{4\mu} + \frac{\beta-1}{\beta}\lambda^{-\frac{\beta}{\beta-1}} + \frac{m-1}{m}\delta^{-\frac{m}{m-1}}\right)E(t)\right] \\
(2.10)\quad &\geq \left[1 + \frac{p(1-\varepsilon)}{2}\right]\|u_t\|_2^2 + \left[\frac{p(1-\varepsilon)}{\alpha} - 1 - \frac{\lambda^\beta C_1}{\beta} - \mu C_2\right]\|\nabla u\|_\alpha^\alpha \\
&- \frac{\delta^m}{m}\|u\|_m^m + \varepsilon\|u\|_p^p - p(1-\varepsilon)E(t).
\end{aligned}
$$

Meanwhile, from the convexity of the function $\frac{u^y}{y}$ in y for $u \geq 0$ and $y > 0$, we have the inequality

$$
\frac{1}{m}\|u\|_m^m \leq \frac{\theta}{2}\|u\|_2^2 + \frac{1-\theta}{p}\|u\|_p^p, \tag{2.11}
$$

where $\theta = \dfrac{p-m}{p-2}$. It then follows from inequalities (2.10) and (2.11) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega uu_t \mathrm{d}x - \left(\frac{1}{4\mu} + \frac{\beta-1}{\beta}\lambda^{-\frac{\beta}{\beta-1}} + \frac{m-1}{m}\delta^{-\frac{m}{m-1}}\right)E(t)\right] \\ &\geq \left[1+\frac{p(1-\varepsilon)}{2}\right]\|u_t\|_2^2 + \left[\frac{p(1-\varepsilon)}{\alpha} - 1 - \frac{\lambda^\beta C_1}{\beta} - \mu C_2\right]\|\nabla u\|_\alpha^\alpha \\ &\quad - \frac{\theta\delta^m}{2}\|u\|_2^2 + \left[\varepsilon - \frac{(1-\theta)\delta^m}{p}\right]\|u\|_p^p - p(1-\varepsilon)E(t). \end{aligned} \tag{2.12}$$

Setting $\delta = \left(\frac{\varepsilon p}{1-\theta}\right)^{\frac{1}{m}}$, $\lambda = \left\{\frac{\beta}{4C_1}[\frac{p(1-\varepsilon)}{\alpha} - 1]\right\}^{\frac{1}{\beta}}$, and $\mu = \frac{1}{4C_2}\left[\frac{p(1-\varepsilon)}{\alpha} - 1\right]$ in (2.12), and using Poincaré's inequality, inequality (2.12) becomes

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega uu_t \mathrm{d}x - \eta(\varepsilon)E(t)\right] \\ &\geq \left[1+\frac{p(1-\varepsilon)}{2}\right]\|u_t\|_2^2 + \left\{\frac{\lambda_1}{2C_2}\left[\frac{p(1-\varepsilon)}{\alpha} - 1\right] - \frac{p\theta\varepsilon}{2(1-\theta)}\right\}\|u\|_2^2 \\ &\quad - p(1-\varepsilon)E(t), \end{aligned} \tag{2.13}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$.

Let ε be a constant small enough such that $0 < \varepsilon < 1$ and

$$\frac{\lambda_1}{2C_2}\left[\frac{p(1-\varepsilon)}{\alpha} - 1\right] - \frac{p\theta\varepsilon}{2(1-\theta)} > 0.$$

We then have the inequality

$$\begin{aligned} &\left[1+\frac{p(1-\varepsilon)}{2}\right]\|u_t\|_2^2 + \left\{\frac{\lambda_1}{2C_2}\left[\frac{p(1-\varepsilon)}{\alpha} - 1\right] - \frac{p\theta\varepsilon}{2(1-\theta)}\right\}\|u\|_2^2 \\ &\geq 2\sqrt{[1+\frac{p(1-\varepsilon)}{2}]\left\{\frac{\lambda_1}{2C_2}[\frac{p(1-\varepsilon)}{\alpha} - 1] - \frac{p\theta\varepsilon}{2(1-\theta)}\right\}}\int_\Omega uu_t\,\mathrm{d}x, \end{aligned} \tag{2.14}$$

by Cauchy's inequality, and it follows from (2.13) and (2.14) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega uu_t\,\mathrm{d}x - \eta(\varepsilon)E(t)\right] &\geq \gamma(\varepsilon)\int_\Omega uu_t\,\mathrm{d}x - p(1-\varepsilon)E(t) \\ &= \gamma(\varepsilon)\left[\int_\Omega uu_t\,\mathrm{d}x - \frac{p(1-\varepsilon)}{\gamma(\varepsilon)}E(t)\right]. \end{aligned} \tag{2.15}$$

It can be easily seen that

$$
\begin{aligned}
&\left[1+\frac{p(1-\varepsilon)}{2}\right]\left\{\frac{\lambda_1}{2C_2}\left[\frac{p(1-\varepsilon)}{\alpha}-1\right]-\frac{p\theta\varepsilon}{2(1-\theta)}\right\}\\
&\to \frac{\lambda_1}{2C_2}\left(1+\frac{p}{2}\right)\left(\frac{p}{\alpha}-1\right),\\
&\gamma(\varepsilon)\to\sqrt{\frac{2\lambda_1}{C_2}\left(1+\frac{p}{2}\right)\left(\frac{p}{\alpha}-1\right)},\\
&\frac{p(1-\varepsilon)}{\gamma(\varepsilon)}\to\frac{p}{\sqrt{\frac{2\lambda_1}{C_2}(1+\frac{p}{2})(\frac{p}{\alpha}-1)}},\quad \eta(\varepsilon)\to\infty,\ \text{as } \varepsilon\to 0^+,
\end{aligned}
\tag{2.16}
$$

and

$$
\left[1+\frac{p(1-\varepsilon)}{2}\right]\left\{\frac{\lambda_1}{2C_2}\left[\frac{p(1-\varepsilon)}{\alpha}-1\right]-\frac{p\theta\varepsilon}{2(1-\theta)}\right\}\to-\left[\frac{\lambda_1}{2C_2}+\frac{p\theta}{2(1-\theta)}\right],
$$

as $\varepsilon\to 1^-$. Hence, there exists a constant $\varepsilon_*\in(0,1)$ such that

$$
\gamma(\varepsilon_*)=0 \text{ and } \gamma(\varepsilon)>0 \text{ for all } \varepsilon\in(0,\varepsilon_*),
$$

which implies that

$$
\begin{aligned}
&\gamma(\varepsilon)\to 0,\quad \frac{p(1-\varepsilon)}{\gamma(\varepsilon)}\to\infty,\\
&\eta(\varepsilon)\to C_2\left[\frac{p(1-\varepsilon_*)}{\alpha}-1\right]^{-1}+\frac{\beta-1}{\beta}\left\{\frac{\beta}{4C_1}[\frac{p(1-\varepsilon_*)}{\alpha}-1]\right\}^{-\frac{1}{\beta-1}}\\
&\qquad +\frac{m-1}{m}\left(\frac{p\varepsilon_*}{1-\theta}\right)^{-\frac{1}{m-1}},
\end{aligned}
\tag{2.17}
$$

as $\varepsilon\to\varepsilon_*^-$.

By virtue of (2.16), (2.17), and the continuity of $\dfrac{p(1-\varepsilon)}{\gamma(\varepsilon)}$ in $\varepsilon\in(0,\varepsilon_*)$, there exists a constant $\varepsilon_0\in(0,\varepsilon_*)\subset(0,1)$ such that

$$
\frac{p(1-\varepsilon_0)}{\gamma(\varepsilon_0)}=\eta(\varepsilon_0).
$$

From (2.15), one can obtain the inequality

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega uu_t\,\mathrm{d}x-\eta(\varepsilon_0)E(t)\right]\ge\gamma(\varepsilon_0)\left[\int_\Omega uu_t\,\mathrm{d}x-\eta(\varepsilon_0)E(t)\right].
\tag{2.18}
$$

We now define a function $H : [0, \infty) \to R$ as

$$H(t) = \int_\Omega u u_t \, \mathrm{d}x - \eta(\varepsilon_0) E(t).$$

Then, from condition (2.1) and (2.18), we have

$$H(0) = \int_\Omega u_0 u_1 \, \mathrm{d}x - \eta(\varepsilon_0) E(0) > 0,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) \geq \gamma(\varepsilon_0) H(t), \tag{2.19}$$

and inequality (2.19) implies that

$$H(t) \geq e^{\gamma(\varepsilon_0)t} H(0) \text{ for all } t \geq 0.$$

Since $u(t)$ is a global solution, one can see that $0 \leq E(t) \leq E(0)$ for all $t \geq 0$ by Lemma 1.1 and (2.2), and hence, we have the inequalities

$$\int_\Omega u u_t \, \mathrm{d}x \geq H(t) \geq e^{\gamma(\varepsilon_0)t} H(0).$$

Therefore, we obtain the estimates

$$\begin{aligned} \|u(t)\|_2^2 &= \|u(0)\|_2^2 + 2 \int_0^t \int_\Omega u u_\tau \, \mathrm{d}x \mathrm{d}\tau \\ &\geq \|u(0)\|_2^2 + 2 \int_0^t e^{\gamma(\varepsilon_0)t} H(0) \, \mathrm{d}\tau \\ &= \|u(0)\|_2^2 + \frac{2}{\gamma(\varepsilon_0)} (e^{\gamma(\varepsilon_0)t} - 1) H(0). \end{aligned} \tag{2.20}$$

On the other hand, by Hölder's inequality and (2.2), we have

$$\begin{aligned} \|u(t)\|_2 &\leq \|u(0)\|_2 + \int_0^t \|u_\tau\|_2 \, \mathrm{d}\tau \\ &\leq \|u(0)\|_2 + C_0 \int_0^t \|u_\tau\|_m \, \mathrm{d}\tau \\ &\leq \|u(0)\|_2 + C_0 t^{\frac{m-1}{m}} \left(\int_0^t \|u_\tau\|_m^m \, \mathrm{d}\tau \right)^{\frac{1}{m}} \\ &\leq \|u(0)\|_2 + C_0 t^{\frac{m-1}{m}} [E(0) - E(t)]^{\frac{1}{m}} \\ &\leq \|u(0)\|_2 + C_0 t^{\frac{m-1}{m}} E(0)^{\frac{1}{m}}, \end{aligned} \tag{2.21}$$

where C_0 is a positive constant, which contradicts with (2.20). The proof is completed. □

Example 2.2. *As an application of Theorem 2.1, we consider the following example in one-dimensional space:*

Let $\Omega = [0, 2\pi] \subset R$ and assume that $\beta < \alpha = 4$ and $m < p = 5$. The initial data u_0 and u_1 are given by

$$u(x,0) = a\sin(bx) \text{ and } u_t(x,0) = a^2b^2\sin(bx),$$

where a and b are positive integers.

Then one can see that

$$\begin{aligned} E(0) =& \frac{1}{2}\|u_t(0)\|_2^2 + \frac{1}{4}\|\nabla u(0)\|_4^4 - \frac{1}{5}\|u(0)\|_5^5 \\ =& \frac{1}{2}\int_0^{2\pi} |a^2b^2\sin(bx)|^2 \, \mathrm{d}x + \frac{1}{4}\int_0^{2\pi} |ab\cos(bx)|^4 \, \mathrm{d}x \\ & - \frac{1}{5}\int_0^{2\pi} |a\sin(bx)|^5 \, \mathrm{d}x \\ =& \frac{5}{8}a^4b^4\pi - \frac{32}{75}a^5, \end{aligned} \tag{2.22}$$

and

$$\int_\Omega u(0)u_t(0) \, \mathrm{d}x = a^3b^2\pi.$$

It can be easily seen that for any given constant $c > 0$, there exist constants $a, b > 0$ such that $\frac{a^3b^2\pi}{M} > c$ and

$$\frac{5}{8}a^4b^4\pi - \frac{32}{75}a^5 - \frac{a^3b^2\pi}{2M} = 0,$$

where M is a constant defined in (2.1), which implies

$$\int_\Omega u_0u_1\mathrm{d}x > ME(0).$$

By Theorem 2.1, we conclude that the solution of problem (1.1)-(1.3) blows up in finite time.

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References

[1] D. D. Ang and A. P. N. Dinh, *Strong solutions of quasilinear wave equation with non-linear damping,* SIAM J. Math. Anal., **19** (1988), 337-347.

[2] V. Georgiev and G. Todorova, *Existence of solutions of the wave equation with non-linear damping and source terms,* J. Differntial Equations, **109** (1994), 295-308.

[3] S. Kavashima and Y. Shibata, *Global existence and exponential stability of small solutions to non-linear viscoelasticity,* Commun. Math. Phys., **148** (1992), 189-208.

[4] H. A. Levine, *Instability and nonexistene of global solutions of nonlinear wave equation of the form $Pu_{tt} = Au + F(u)$,* Trans. Amer. Math. Soc., **192** (1974), 1-21.

[5] H. A. Levine, *Some additional remarks on the nonexistence of global solutions to nonlinear wave equation,* SIAM J. Math. Anal., **5** (1974), 138-146.

[6] S. A. Messaoudi, *Blow up in a nonlinearly damped wave equation,* Math. Nachr., **231** (2001), no. 1, 1-7.

[7] S. A. Messaoudi and B. S. Houari, *Global non-existence of solutions of a class of wave equations with non-linear damping and source terms,* Math. Method. Appl. Sci., **27** (2004), no. 14, 1687-1696.

[8] H. Song, *Blow up of arbitrarily positive initial energy solutions for a viscoelastic wave equation,* Nonlinear Anal-RWA, **26** (2015), 306-314.

[9] E. Vitillaro, *Global non-existence theorems for a class of evolution equations with dissipation,* Arch. Ration. Mech. Anal., **149** (1999), 155-182.

[10] Z. Yang, *Blow up of solutions for a class of non-linear evolution equations with non-linear damping and source terms,* Math. Meth. Appl. Sci., **25** (2002), 825-833.

[11] R. Zeng, C. Mu and S. Zhou, *A blow-up result for Kirchhoff-type equations with high energy,* Math. Method. Appl. Sci., **34** (2011), no. 4, 479-486.

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