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A NOTE ON FUNCTIONAL INEQUALITY AND ADDITIVE MAPPING

Ick-Soon Chang, Hyun-Wook Lee and Hark-Mahn Kim*

Abstract. In this note, we prove some theorems concerning the stability of functional inequality associated with additive mappings on quasi-β-normed spaces.

1. Introduction

The concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Ulam [12] brought up the question concerning the stability of group homomorphisms. Hyers [7] proved first this problem for the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces. Since then, many mathematicians have generalized the result of Hyers ; see, e.g., [3, 4, 9].

Gilányi [5] and Rätz [10], meanwhile, proved that if f fills the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\|, \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

Gilányi [6] and Fechner [2] studied the stability of the functional inequality (1.1).

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This work was supported by research fund of Chungnam National University.
*Correspondence should be addressed to hmkim@cnu.ac.kr.

In this paper, we consider the following functional inequality with several variables

$$\Big\|\sum_{i=1}^{l} a_i f(x_i)\Big\| \le \Big\|f\Big(\sum_{i=1}^{l} a_i x_i\Big)\Big\| + \phi(x_1, \cdots, x_l), \tag{1.2}$$

where the function ϕ is a perturbing term of the functional inequality

$$\Big\|\sum_{i=1}^{l} a_i f(x_i)\Big\| \le \Big\|f\Big(\sum_{i=1}^{l} a_i x_i\Big)\Big\|, \quad \prod_{i=1}^{l} a_i \neq 0, \; l \ge 3, \tag{1.3}$$

which is associated with Jordan-von Neumann type additive functional equations with several variables. Then, we obtain some theorems concerning the stability of the functional inequality (1.2).

2. Stability of functional inequality (1.2)

We first recall some basic concepts concerning quasi-β-normed spaces and some preliminary results. We fix a real number β with $0 < \beta \le 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A *quasi-β-norm* $\|\cdot\|$ is a real-valued function on X satisfying the following:

(1) $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = |\lambda|^{\beta} \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.
(3) There is a constant $K \ge 1$ such that $\|x + y\| \le K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-β-normed space* if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-β-Banach space* is a complete quasi-β-normed space. A quasi-β-norm $\|\cdot\|$ is called a *(β, p)-norm* $(0 < p \le 1)$ if $\|x+y\|^p \le \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case, a quasi-β-Banach space is called a *(β, p)-Banach space.* We can make reference to [1, 11] for the concept of quasi-normed spaces and p-Banach spaces. Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem in [11], each quasi-norm is equivalent to some p-norm. From now on, we assume that X is a quasi-β-normed linear space with quasi-β-norm and Y is a (β, p)-Banach space with p-norm. Let $K \ge 1$ be the modulus of concavity of $\|\cdot\|_Y$. Before taking up the main subject, we now obtain the following lemma through some elementary computations.

Lemma 2.1. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality* (1.3). *Then, the mapping f is additive.*

Now we prove the stability of a functional inequality (1.2) with a Jordan-von Neumann type l-variable additive functional equation. For a given function $\phi : X^l \to [0,\infty)$, we set

$$\phi_{ijk}(x,y,z) := \phi\Big(0,\cdots,0,\overbrace{x}^{i-th},0,\cdots,0,\overbrace{y}^{j-th},0,\cdots,0,\overbrace{z}^{k-th},0,\cdots,0\Big)$$

for notational convenience throughout the paper. In addition, we remark that for any nonnegative real numbers $u_1, u_2, \cdots, u_n$

$$(\sum_{i=1}^{n} u_i)^p \le \sum_{i=1}^{n} {u_i}^p$$

holds for p with $0 < p \le 1$ [8].

THEOREM 2.2. *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality* (1.2) *for which $\phi : X^l \to [0,\infty)$ is a function subject to*

$$\phi\Big(\frac{x_1}{2},\cdots,\frac{x_l}{2}\Big) \le \Big(\frac{L}{2}\Big)^{\beta}\phi(x_1,\cdots,x_l) \tag{2.1}$$

for some constant L with $0 < L < 1$. Then, there exists a unique additive mapping $A : X \to Y$ satisfying the estimation

$$\begin{aligned}\|f(x) - A(x)\| \quad &\le \quad \frac{K^2L^{\beta}}{2^{\beta}|a_k|^{\beta}\sqrt[p]{1-L^{p\beta}}}\Big[\phi_{ijk}(\frac{-a_kx}{a_i},\frac{-a_kx}{a_j},2x)\\ &\qquad + \phi_{ijk}(\frac{-a_kx}{a_i},0,x) + \phi_{ijk}(0,\frac{-a_kx}{a_j},x)\Big]\end{aligned} \tag{2.2}$$

for all $x \in X$ and any distinct $i,j,k \in \{1,\cdots,l\}$.

Proof. We note without loss of generality that any three distinct entries $i,j,k \in \{1,\cdots,l\}$ of the terms in the approximation (2.2) can be replaced by the first, second and third entries because the proof is verified in the same manner. Thus, we prove our assertion for distinct indices $\{i,j,k\} = \{1,2,3\} \subset \{1,\cdots,l\}$. Letting $x_1 := x, x_3 := \frac{-a_1x}{a_3}$ and $x_i := 0$ for all $i \ne 1,3$ in (1.2), we get

$$\Big\|a_1f(x) + a_3f\Big(\frac{-a_1x}{a_3}\Big)\Big\| \le \phi_{123}(x,0,\frac{-a_1x}{a_3}) \tag{2.3}$$

for all $x \in X$. By putting $x_2 := y, x_3 := \frac{-a_2y}{a_3}$ and $x_i := 0$ for all $i \ne 2,3$ in (1.2), one obtains

$$\Big\|a_2f(y) + a_3f\Big(\frac{-a_2y}{a_3}\Big)\Big\| \le \phi_{123}(0,y,\frac{-a_2y}{a_3}) \tag{2.4}$$

for all $y \in X$. Setting $x_1 := x, x_2 := y$, $x_i := 0$ for all $i \neq 1, 2, 3$ and then replacing x_3 by $\frac{-a_1x-a_2y}{a_3}$ in (1.2), we obtain

$$\Big\|a_1 f(x) + a_2 f(y) + a_3 f\Big(\frac{-a_1x - a_2y}{a_3}\Big)\Big\| \leq \phi_{123}(x, y, \frac{-a_1x - a_2y}{a_3}) \tag{2.5}$$

for all $x, y \in X$. It follows from (2.3), (2.4) and (2.5) that

$$\begin{aligned}|a_3|^\beta \Big\| f\Big(\frac{-a_1x - a_2y}{a_3}\Big) - f\Big(\frac{-a_1x}{a_3}\Big) - f\Big(\frac{-a_2y}{a_3}\Big)\Big\| \\ \leq K^2\Big[\phi_{123}(x, y, \frac{-a_1x - a_2y}{a_3}) + \phi_{123}(x, 0, \frac{-a_1x}{a_3}) + \phi_{123}(0, y, \frac{-a_2y}{a_3})\Big],\end{aligned}$$

which yields the classical Cauchy difference operator controlled by the function ϕ as follows:

$$\begin{aligned}\|f(x + y) - f(x) - f(y)\| \leq \frac{K^2}{|a_3|^\beta}\Big[\phi_{123}\Big(\frac{-a_3x}{a_1}, \frac{-a_3y}{a_2}, x + y\Big) \\ + \phi_{123}\Big(\frac{-a_3x}{a_1}, 0, x\Big) + \phi_{123}\Big(0, \frac{-a_3y}{a_2}, y\Big)\Big]\end{aligned}$$

for all $x, y \in X$. Therefore, we get

$$\begin{aligned}\|f(2x) - 2f(x)\| \leq \frac{K^2}{|a_3|^\beta}\Big[\phi_{123}\Big(\frac{-a_3x}{a_1}, \frac{-a_3x}{a_2}, 2x\Big) \\ + \phi_{123}\Big(\frac{-a_3x}{a_1}, 0, x\Big) + \phi_{123}\Big(0, \frac{-a_3x}{a_2}, x\Big)\Big]\end{aligned} \tag{2.6}$$

for all $x \in X$. Thus, we have by (2.6) that

$$\begin{aligned}\Big\|2^m f\Big(\frac{x}{2^m}\Big) - 2^n f\Big(\frac{x}{2^n}\Big)\Big\|^p &\leq \sum_{s=m}^{n-1}\Big\|2^s f\Big(\frac{x}{2^s}\Big) - 2^{s+1} f\Big(\frac{x}{2^{s+1}}\Big)\Big\|^p \\ &\leq \frac{K^{2p}}{2^{p\beta}|a_3|^{p\beta}}\sum_{s=m}^{n-1} L^{(s+1)p\beta}\Big[\phi_{123}\Big(\frac{-a_3x}{a_1}, \frac{-a_3x}{a_2}, 2x\Big) \\ &\quad + \phi_{123}\Big(\frac{-a_3x}{a_1}, 0, x\Big) + \phi_{123}\Big(0, \frac{-a_3x}{a_2}, x\Big)\Big]^p\end{aligned}$$

for all $x \in X$ and all integers $n, m \geq 0$ with $n > m$. Since the right hand side tends to zero as $m \to \infty$, a sequence $\{2^n f(\frac{x}{2^n})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So, one can define a mapping $A : X \to Y$ by $A(x) := \lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big)$ for all $x \in X$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in the last inequality, we get the estimation (2.2) of the approximate mapping f for the functional inequality (1.3) by the mapping A.

In addition, we intend to claim that $A : X \to Y$ is exactly an additive mapping satisfying (1.3). It follows by (1.2) and the condition of ϕ that

$$\Big\|\sum_{i=1}^{l} a_i A(x_i)\Big\| = \lim_{n\to\infty} 2^{n\beta}\Big\|\sum_{i=1}^{l} a_i f\Big(\frac{x_i}{2^n}\Big)\Big\| \le \Big\|A\Big(\sum_{i=1}^{l} a_i x_i\Big)\Big\|.$$

Thus, the mapping A satisfies the functional inequality (1.3), and thus A is an additive mapping by Lemma 2.1.

Now, let $T : X \longrightarrow Y$ be another additive mapping satisfying the approximation (2.2). Then, we obtain that

$$\begin{aligned}
&\Big\|2^n f\Big(\frac{x}{2^n}\Big) - T(x)\Big\| = 2^{n\beta}\Big\|f\Big(\frac{x}{2^n}\Big) - T\Big(\frac{x}{2^n}\Big)\Big\| \\
&\le \frac{K^2 L^\beta}{2^\beta |a_3|^\beta \sqrt[p]{1-L^{p\beta}}} L^{n\beta}\Big[\phi_{123}\Big(\frac{-a_3 x}{a_1}, \frac{-a_3 x}{a_2}, 2x\Big) \\
&\qquad\qquad + \phi_{123}\Big(\frac{-a_3 x}{a_1}, 0, x\Big) + \phi_{123}\Big(0, \frac{-a_3 x}{a_2}, x\Big)\Big],
\end{aligned}$$

which tends to zero as $n \to \infty$. So, we conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. □

The following theorem is an alternative stability result of the previous theorem

THEOREM 2.3. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality (1.2). Suppose that $\phi : X^l \to [0, \infty)$ is a function subject to*

$$\phi(2x_1, \cdots, 2x_l) \le (2L)^\beta \phi(x_1, \cdots, x_l)$$

for some constant L with $0 < L < 1$. Then, there exists a unique additive mapping $A : X \to Y$ satisfying the estimation

$$\begin{aligned}
(2.7)\quad \|f(x) - A(x)\| \quad \le \quad &\frac{K^2}{2^\beta |a_k|^\beta \sqrt[p]{1-L^{p\beta}}}\Big[\phi_{ijk}(\frac{-a_k x}{a_i}, \frac{-a_k x}{a_j}, 2x) \\
&+ \phi_{ijk}(\frac{-a_k x}{a_i}, 0, x) + \phi_{ijk}(0, \frac{-a_k x}{a_j}, x)\Big]
\end{aligned}$$

for all $x \in X$ and any distinct $i, j, k \in \{1, \cdots, l\}$.

Proof. It follows from (2.6) that

$$
\begin{aligned}
(2.8)\qquad \Big\|\frac{f(2^m x)}{2^m}-\frac{f(2^n x)}{2^n}\Big\|^p &\le \sum_{s=m}^{n-1}\Big\|\frac{f(2^s x)}{2^s}-\frac{f(2^{s+1}x)}{2^{s+1}}\Big\|^p \\
&\le \frac{K^{2p}}{2^{p\beta}|a_3|^{p\beta}}\sum_{s=m}^{n-1}L^{sp\beta}\Big[\phi_{123}\Big(\frac{-a_3x}{a_1},\frac{-a_3x}{a_2},2x\Big) \\
&\qquad + \phi_{123}\Big(\frac{-a_3x}{a_1},0,x\Big)+\phi_{123}\Big(0,\frac{-a_3x}{a_2},x\Big)\Big]^p
\end{aligned}
$$

for all integers $n,m \ge 0$ with $n > m$ and all $x \in X$. As in the proof of Theorem 2.2, we see that $\big\{\frac{f(2^n x)}{2^n}\big\}$ is a convergent sequence. Thereby, we can define a mapping $A : X \to Y$ by $A(x) := \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ for all $x \in X$. Also, letting $m = 0$ and passing the limit $n \to \infty$ in (2.8), we get (2.7).

The remaining proof goes through by the similar way to Theorem 2.2. $\square$

THEOREM 2.4. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality* (1.2). *Suppose that $\phi : X^l \to [0,\infty)$ is a function subject to*

$$
\Phi_1(x_1,\cdots,x_l) := \sum_{s=1}^{\infty}2^{sp\beta}\phi\Big(\frac{x_1}{2^s},\cdots,\frac{x_l}{2^s}\Big)^p < \infty
$$

for all $x_1,\cdots,x_l \in X$. Then, there exists a unique additive mapping $A : X \to Y$ satisfying the estimation

$$
\begin{aligned}
(2.9)\quad \|f(x)-A(x)\| \le\ & \frac{K^2}{2^\beta|a_k|^\beta}\Big[\sum_{s=1}^{\infty}2^{sp\beta}\Big\{\phi_{ijk}\big(\frac{-a_kx}{2^s a_i},\frac{-a_kx}{2^s a_j},\frac{2x}{2^s}\big) \\
&+\ \phi_{ijk}\big(\frac{-a_kx}{2^s a_i},0,\frac{x}{2^s}\big)+\phi_{ijk}\big(0,\frac{-a_kx}{2^s a_j},\frac{x}{2^s}\big)\Big\}^p\Big]^{1/p}
\end{aligned}
$$

for all $x \in X$ and any distinct $i,j,k \in \{1,\cdots,l\}$.

Proof. It follows from (2.6) and $0 < p \leq 1$ that

$$
\begin{aligned}
(2.10)\quad \Big\|2^m f(\frac{x}{2^m}) - 2^n f(\frac{x}{2^n})\Big\|^p &\leq \sum_{s=m+1}^{n} \Big\|2^s f\Big(\frac{x}{2^s}\Big) - 2^{s-1} f\Big(\frac{x}{2^{s-1}}\Big)\Big\|^p \\
&\leq \frac{K^{2p}}{2^{p\beta}|a_3|^{p\beta}} \sum_{s=m+1}^{n} 2^{sp\beta}\Big\{\phi_{123}\Big(\frac{-a_3x}{2^s a_1}, \frac{-a_3x}{2^s a_2}, \frac{2x}{2^s}\Big) \\
&\qquad + \phi_{123}\Big(\frac{-a_3x}{2^s a_1}, 0, \frac{x}{2^s}\Big) + \phi_{123}\Big(0, \frac{-a_3x}{2^s a_2}, \frac{x}{2^s}\Big)\Big\}^p
\end{aligned}
$$

for all $x \in X$ and for all nonnegative integers n and m with $n > m$. Since the right hand side tends to zero by the convergence of Φ_1 as $m \to \infty$, a sequence $\{2^n f(\frac{x}{2^n})\}$ is Cauchy in Y. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. Therefore, one can define a mapping $A : X \to Y$ by $A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ for all $x \in X$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.10), we get the approximation (2.9) of f by A.

It follows easily from (1.2) and the condition of ϕ that

$$
\Big\|\sum_{i=1}^{l} a_i A(x_i)\Big\|^p = \lim_{n\to\infty} 2^{np\beta}\Big\|\sum_{i=1}^{l} a_i f\Big(\frac{x_i}{2^n}\Big)\Big\|^p \leq \Big\|A\Big(\sum_{i=1}^{l} a_i x_i\Big)\Big\|^p
$$

Thus, the mapping A satisfies the functional inequality (1.3). According to Lemma 2.1, we see that the mapping A is additive .

The rest of proof goes through by the similar argument to the corresponding part of Theorem 2.2. □

The next theorem is an alternative stability result of Theorem 2.4.

THEOREM 2.5. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality* (1.2). *Suppose that $\phi : X^l \to [0,\infty)$ is a function subject to*

$$
\Phi_2(x_1, \cdots, x_l) := \sum_{s=0}^{\infty} \frac{1}{2^{sp\beta}} \phi(2^s x_1, \cdots, 2^s x_l)^p < \infty
$$

for all $x_1, \cdots, x_l \in X$. Then, there exists a unique additive mapping $A : X \to Y$ satisfying the estimation

$$
\begin{aligned}
\|f(x) - A(x)\| \leq \frac{K^2}{2^\beta |a_k|^\beta}\Big[\sum_{s=0}^{\infty} \frac{1}{2^{sp\beta}}\Big\{\phi_{ijk}(\frac{-2^s a_k x}{a_i}, \frac{-2^s a_k x}{a_j}, 2^{s+1}x) \\
+ \phi_{ijk}(\frac{-2^s a_k x}{a_i}, 0, 2^s x) + \phi_{ijk}(0, \frac{-2^s a_k x}{a_j}, 2^s x)\Big\}^p\Big]^{1/p}
\end{aligned}
$$

for all $x \in X$ *and any distinct* $i, j, k \in \{1, \cdots, l\}$.

Proof. It follows from (2.6) and $0 < p \le 1$ that

$$\left\|\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}\right\|^p \le \sum_{s=m}^{n-1}\left\|\frac{f(2^s x)}{2^s} - \frac{f(2^{s+1}x)}{2^{s+1}}\right\|^p \tag{2.11}$$
$$\le \frac{K^{2p}}{2^{p\beta}|a_3|^{p\beta}}\sum_{s=m}^{n-1}\frac{1}{2^{sp\beta}}\Big\{\phi_{123}\Big(\frac{-2^s a_3 x}{a_1}, \frac{-2^s a_3 x}{a_2}, 2^{s+1}x\Big)$$
$$+ \phi_{123}\Big(\frac{-2^s a_3 x}{a_1}, 0, 2^s x\Big) + \phi_{123}\Big(0, \frac{-2^s a_3 x}{a_2}, 2^s x\Big)\Big\}^p$$

for all nonnegative integers n and m with $n > m$ and all $x \in X$, which implies that a sequence $\{\frac{f(2^n x)}{2^n}\}$ is Cauchy in Y for all $x \in X$. Since Y is complete, the sequence $\{\frac{f(2^n x)}{2^n}\}$ converges. So, we can define a mapping $A : X \to Y$ by $A(x) := \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ for all $x \in X$. Moreover, letting $m = 0$ and then taking the limit $n \to \infty$ in (2.11), we get the desired approximation.

The remaining proof goes through by the similar argument to Theorem 2.4. □

As a result, we have the following corollary by applying Theorem 2.3 and Theorem 2.5.

Corollary 2.6. *Assume that there exists a nonnegative number* δ *such that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the functional inequality*

$$\Big\|\sum_{i=1}^{l} a_i f(x_i)\Big\| \le \Big\|f\Big(\sum_{i=1}^{l} a_i x_i\Big)\Big\| + \delta$$

for all $x_1, \cdots, x_l \in X$. *Then, there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \le \frac{3K^2\delta}{\max\{|a_k|^\beta \sqrt[p]{2^{p\beta} - 1} : k = 1, \cdots, l\}} \tag{2.12}$$

for all $x \in X$.

3. Alternative stability results of (1.2)

In this section, we investigate stability results different from those of the section 2 for the functional inequality (1.2) with $\lambda := \frac{-a_i - a_j}{a_k} \ne 0$.

Theorem 3.1. *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality* (1.2) *and that $\phi : X^l \to [0, \infty)$ is a function subject to*

$$\widetilde{\Psi}(x_1, \cdots, x_l) := \sum_{s=1}^{\infty} |\lambda|^{sp\beta} \phi\Big(\frac{x_1}{\lambda^s}, \cdots, \frac{x_l}{\lambda^s}\Big)^p < \infty$$

for all $x_1, \cdots, x_l \in X$, where $\lambda \neq 0$ for some $i, j, k \in \{1, \cdots, l\}$. Then, there exists a unique additive mapping $A : X \to Y$ satisfying the estimation

$$\|A(x) - f(x)\| \le \frac{1}{|a_i + a_j|^\beta} \widetilde{\Psi}_{ijk}(x, x, \lambda x)^{\frac{1}{p}} \tag{3.1}$$

for all $x \in X$ and any distinct $i, j, k \in \{1, \cdots, l\}$, where $A(x) := \lim_{n\to\infty} \lambda^n f(\frac{x}{\lambda^n})$ for all $x \in X$.

Proof. Without loss of generality, we prove our assertion for distinct indices $\{i, j, k\} = \{1, 2, 3\}$. Replacing $(x, x, \frac{-a_1 - a_2}{a_3} x, \overbrace{0, \cdots, 0}^{(l-3)th})$ instead of $(x_1, x_2, x_3, \overbrace{0, \cdots, 0}^{(l-3)th})$ in the relation (1.2), we get

$$\Big\| f(x) - \frac{f(\lambda x)}{\lambda} \Big\| \le \frac{1}{|a_1 + a_2|^\beta} \phi_{123}\Big(x, x, \lambda x\Big). \tag{3.2}$$

Then, it follows from both (3.2) and p-norm that

$$\begin{aligned}
&\Big\| \lambda^m f\Big(\frac{x}{\lambda^m}\Big) - \lambda^n f\Big(\frac{x}{\lambda^n}\Big) \Big\|^p \\
&\le \sum_{s=m}^{n-1} \Big\| \lambda^s f\Big(\frac{x}{\lambda^s}\Big) - \lambda^{s+1} f\Big(\frac{x}{\lambda^{s+1}}\Big) \Big\|^p \\
&\le \frac{1}{|a_1 + a_2|^{p\beta}} \sum_{s=m}^{n-1} |\lambda|^{(s+1)p\beta} \phi_{123}\Big(\frac{x}{\lambda^{s+1}}, \frac{x}{\lambda^{s+1}}, \frac{\lambda x}{\lambda^{s+1}}\Big)^p
\end{aligned}$$

for all $x \in X$ and for all nonnegative integers n and m with $n > m$.

The rest of proof is similar to the corresponding part of Theorem 2.4. □

The approximation of f by A in (3.1) has much simpler upper bound than that of (2.9) according to the perturbation ϕ.

Theorem 3.2. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality* (1.2). *Suppose that $\phi : X^l \to [0, \infty)$ is* *a function subject to*

$$\tilde{\tilde{\Psi}}(x_1, \cdots, x_l) := \sum_{s=0}^{\infty} \frac{1}{|\lambda|^{sp\beta}} \phi(\lambda^s x_1, \cdots, \lambda^s x_l)^p < \infty$$

for all $x_1, \cdots, x_l \in X$, *where* $\lambda \neq 0$ *for some* $i, j, k \in \{1, \cdots, l\}$. *Then, there exists a unique additive mapping* $A : X \to Y$ *satisfying the estimation*

$$\|A(x) - f(x)\| \leq \frac{1}{|a_i + a_j|^\beta} \tilde{\tilde{\Psi}}_{123}(x, x, \lambda x)^{\frac{1}{p}}$$

for all $x \in X$ *and any distinct* $i, j, k \in \{1, \cdots, l\}$, *where* $A(x) := \lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x)$ *for all* $x \in X$.

Corollary 3.3. *Assume that there exists a nonnegative number* δ *such that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the functional inequality*

$$\Big\| \sum_{i=1}^{l} a_i f(x_i) \Big\| \leq \Big\| f\Big(\sum_{i=1}^{l} a_i x_i \Big) \Big\| + \delta$$

for all $x_1, \cdots, x_l \in X$. *Then, there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leq \frac{\delta}{\sqrt[p]{\Big| |a_i + a_j|^{p\beta} - |a_k|^{p\beta} \Big|}} \tag{3.3}$$

for all $x \in X$ *and any distinct* $i, j, k \in \{1, \cdots, l\}$ *with* $0 < |\frac{-a_i - a_j}{a_k}| \neq 1$.

We observe that the best approximation between (2.12) and (3.3) of f by A is determined by constants a_i, a_j and a_k.

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Ick-Soon Chang
Department of Mathematics,
Chungnam National University
99 Daehangno, Yuseong-gu, Daejeon 34134, Korea.
E-mail: ischang@cnu.ac.kr

Hyun-Wook Lee
Department of Mathematics,
Chungnam National University
99 Daehangno, Yuseong-gu, Daejeon 34134, Korea.
E-mail: hihyunww@naver.com

Hark-Mahn Kim
Department of Mathematics,
Chungnam National University
99 Daehangno, Yuseong-gu, Daejeon 34134, Korea.
E-mail: hmkim@cnu.ac.kr