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FLEXIBILITY OF AFFINE CONES OVER SINGULAR DEL PEZZO SURFACES WITH DEGREE 4

Joonyeong Won

Abstract. For an ample divisor A of birational type on a singular del Pezzo surface S of degree 4 with A_1-singularity, we show that the affine cone of S defined by A is flexible

All considered varieties are assumed to be algebraic and defined over an algebraically closed field of characteristic 0 throughout this article.

1. Introduction

An affine algebraic variety X is called *flexible* if the tangent space of X at any smooth point is spanned by the tangent vectors to the orbits of one-parameter unipotent group actions [1].

For a positive integer m, a group G is called to act m-transitively on a set X if the action is transitive on m-tuples of distinct points of X. In addition, an action is called infinitely transitive on X if the action is m-transitive for each integer $m > 0$. For an algebraic variety X, the subgroup of $\mathrm{Aut}(X)$ generated by all algebraic one parameter unipotent subgroups of $\mathrm{Aut}(X)$ is denoted by $\mathrm{SAut}(X)$. The group $\mathrm{SAut}(X)$ is called the special automorphism group of X. The following theorem from [1] shows the relations between flexibility and infinite transitivity.

It is well known that any effective action of one-dimensional unipotent group $\mathbb{G}_a = \mathbb{G}_a(\mathbb{K})$ on X defines a locally nilpotent derivation $\delta \in \mathrm{LND}(\mathbb{K}[X])$ of the algebra of regular functions on X. All such actions generate a subgroup of *special automorphisms* $\mathrm{SAut}\, X \subset \mathrm{Aut}\, X$.

Theorem 1 ([1, Theorem 0.1]). *Let X be an affine algebraic variety of dimension ≥ 2. Then the following conditions are equivalent:*

(1) *The variety X is flexible;*
(2) *the group $\mathrm{SAut}\, X$ acts transitively on X;*
(3) *the group $\mathrm{SAut}\, X$ acts infinitely transitively on X.*

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Three classes of flexible affine varieties are showed in [2],that is, affine cones over flag varieties, non-degenerate toric varieties of dimension ≥ 2, and suspensions over flexible varieties. For example, affine cones over del Pezzo varieties of degree ≥ 6 are toric, thus they are flexible. In addition, [1] proves that any flexible variety is unirational. On the other hand, in [3], it is conjectured that any unirational variety is stably birational to an infinitely transitive variety and is proved in some cases.

In this paper we consider affine cones over singular del Pezzo surfaces with A_1 singularity polarized by ample divisors. For smooth del Pezzo surfaces of degrees at most 3, the non-existence of $\mathbb{G}_a$-actions on affine cones by their anticanonical divisors was proved in [4], [10]. In [5], the existence and non-existence of $\mathbb{G}_a$-actions on affine cones over anticanonically polarized del Pezzo surfaces with du Val singularities were fully classified according to their singularities and degrees.

In [13], [14] it is also shown that affine cones over the smooth del Pezzo surface of degree 4 and 5 polarized by arbitrary ample divisors are flexible.

Theorem 2. *Let S be a smooth del Pezzo surface of degree at least 4. For any ample divisor H, the affine cone*

$$\mathrm{Affcone}_H(S) = \mathrm{Spec} \bigoplus_{m=0}^{\infty} H^0(S, \mathcal{O}(mH))$$

is flexible.

Therefore, it is natural to extend the flexibility problem to del Pezzo surfaces with mild singularity. As a first step to the direction of the problem, we prove the following

Main Theorem 1. *Let S be a singular del Pezzo surface of degree* 4 *with A_1 singularity. For an ample divisor H of birational type on S, the affine cone* $\mathrm{Affcone}_H(S)$ *is flexible.*

2. Cylinders, flexibility of affine cones

Let Y be a projective variety and H be a very ample divisor on Y. A polarization of (Y, H) provides an embedding $Y \hookrightarrow \mathbb{P}^n$. Consider an affine cone $X = \mathrm{AffCone}_H Y \subset \mathbb{A}^{n+1}$ with vertex at the origin $0 \in \mathbb{A}^{n+1}$ corresponding to this embedding.

Definition 3. An open subset U of a variety Y is called a *cylinder* if $U \cong Z \times \mathbb{A}^1$, where Z is a smooth variety with $\mathrm{Pic}\, Z = 0$. Given a divisor $H \subset Y$, we say that a cylinder U is *H-polar* if $U = Y \setminus \mathrm{supp}\, D$ for some effective divisor D which is $\mathbb{Q}$-eqivalent to H.

Definition 4. We call a subset $W \subset Y$ *invariant* with respect to a cylinder $U = Z \times \mathbb{A}^1$ if $W \cap U = \pi_1^{-1}(\pi_1(W))$, where $\pi_1 \colon U \to Z$ is the first projection of the direct product. In other words, every $\mathbb{A}^1$-fiber of the cylinder is either contained in W or does not meet W.

Definition 5. We say that a variety Y is *transversally covered* by cylinders U_i, $i = 1, \dots, s$, if $Y = \bigcup U_i$ and there is no proper subset $W \subset Y$ invariant with respect to all U_i.

Clearly, any cylinder U_i is smooth. Thus, a singular variety Y does not admit a transversal covering by cylinders. But if a singular variety Y can be covered by transversal cylinders except for singular points.

Theorem 6. *The affine cone* $\mathrm{Affcone}_H(Y)$ *admits an effective* $\mathbb{G}_a$*-action if and only if* Y *contains an* H *-polar cylinder*

Theorem 7. *If* Y_{smooth} *has a transversal covering by* H*-polar cylinders , then the affine cone* $\mathrm{Affcone}_H(Y)$*is flexible.*

Corollary 8. *Let* Y *be a singular del Pezzo surface with* A_1 *singularity, Then we have minimal resolution* $Z \to Y$*, where* Z *is weak del Pezzo surface. Then if* Z *has a transversal covering by* H*-polar cylinders, then the affine cone* $\mathrm{Affcone}_H(Y)$*is flexible.*

3. Ample divisors on weak del Pezzo surface of degree 4

From now on, the divisor A is always assumed to be ample, unless otherwise stated. The following method to express the divisor A in terms of $-K_S$ and (-1)-curves is adopted from [6], [14]. For the log pair (S, A), we define an invariant of (S, A) by

$$\mu := \inf\left\{\lambda \in \mathbb{Q}_{>0} \;\middle|\; \text{the } \mathbb{Q}\text{-divisor } K_S + \lambda A \text{ is pseudo-effective}\right\}.$$

The invariant μ is always obtained by a positive rational number. Let $\Delta_{(S,A)}$ be the smallest extremal face of the boundary of the Mori cone $\overline{\mathbb{NE}}(S)$ that contains $K_S + \mu A$.

Let $\phi\colon S \to Z$ be the contraction given by the face $\Delta_{(S,A)}$. Then either ϕ is a birational morphism or a conic bundle with $Z \cong \mathbb{P}^1$. In the former case $\Delta_{(S,A)}$ is generated by r disjoint (-1)-curves contracted by ϕ, where $r \leq 8$. In the later case, $\Delta_{(S,A)}$ is generated by the (-1)-curves in the eight reducible fibers of ϕ. Each reducible fiber consists of two (-1)-curves that intersect transversally at one point.

Suppose that ϕ is birational. Let $E_1, \dots, E_r$ be all (-1)-curves contained in $\Delta_{(S,A)}$. These are disjoint and generate the face $\Delta_{(S,A)}$. Therefore,

$$K_S + \mu A \sim_{\mathbb{Q}} \sum_{i=1}^{r} a_i E_i$$

for some positive rational numbers $a_1, \dots, a_r$. We have $a_i < 1$ for every i because $A \cdot E_i > 0$. Vice versa, for every positive rational numbers $a_1, \dots, a_r < 1$, the divisor

$$-K_S + \sum_{i=1}^{r} a_i E_i$$

is ample.

Suppose that ϕ is a conic bundle. Then there are a 0-curve B and seven disjoint (-1)-curves $E_1, E_2, E_3, E_4, E_5, E_6, E_7$, each of which is contained in a distinct fiber of ϕ, such that

$$K_S + \mu A \sim_{\mathbb{Q}} aB + \sum_{i=1}^{7} a_i E_i$$

for some positive rational number a and non-negative rational numbers a_1, a_2, a_3, a_4, a_5, a_6, $a_7 < 1$. In particular, these curves generate the face $\Delta_{(S,A)}$. Vice versa, for every positive rational number a and non-negative rational numbers a_1, a_2, a_3, a_4, a_5, a_6, $a_7 < 1$ the divisor

$$-K_S + aB + \sum_{i=1}^{7} a_i E_i$$

is ample.

The followings describe the notations that we will use in the rest of the present paper. Unless otherwise mentioned, these notations are fixed from now until the end of the paper.

Let p_1, p_2, p_3, p_4 and p_5 be points of $\mathbb{P}^2$ in almost general position. We obtain a general del Pezzo surface S with degree 4 by the blow-up π of p_i, $i = 1, \ldots, 5$. It is a sequence of surfaces

$$\mathbb{P}^2 = S_0 \leftarrow S_1 \leftarrow \cdots \leftarrow S_5 = S$$

where each S_i is the blow-up π_i of S_{i-1} at p_i, $i = 1, \ldots 5$. Then there are the exceptional divisors E_i with $\pi_i(E_i) = \mathsf{p}_i$.

In this situation we consider the case that $\pi_5(E_5) \in E_4$.

Let l_{ij} be lines in $\mathbb{P}^2$ passing through p_i and p_j where $i \neq j$. Then for the birational morphism π the strict transform L_{ij} of l_{ij} is a (-1)-curve.

Let T_1 be the surface given by the contraction ϕ_1 of the curves L_{23}, L_{24}, L_{34} and E_1. Next, the contraction ψ_1 of E_5 in T_1 gives a surface isomorphic to $\mathbb{P}^2$. Similarly, we can find T_2 and T_3.

4. Proof of Main Theorem

4.1. Construction of cylinder

We set $\mathsf{q}_1 = \phi_1(E_1)$, $e_2 = \phi_1(E_2)$, $e_3 = \phi_1(E_3)$ and $e_4 = \phi_1(E_4)$. And we set $\mathsf{q}_{ij} = e_i \cap e_j$ with $i \neq j$. We consider the line m_{13} passing through q_{24} and q_1. Then $m_{13} = \phi_1(L_{13})$. In this situation

$$\mathbb{P}^2 \setminus (\psi_1(m_{13} \cup e_3))$$

is the cylinder that is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$. Thus

$$U_{13} =: S \setminus (E_1 \cup E_3 \cup L_{13} \cup L_{23} \cup L_{24} \cup L_{34} \cup C_{31})$$

is the cylinder of S that is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$ where $\psi_1 \circ \phi_1(C_{31})$ is the line passing through $\psi_1(m_{13} \cap e_3)$ and $\psi_1(E_5)$. Similarly we can construct the following cylinders.

$$\begin{aligned}
U_{12} &=: S \setminus (E_1 \cup E_2 \cup L_{12} \cup L_{23} \cup L_{24} \cup L_{34} \cup C_{21}),\\
U_{23} &=: S \setminus (E_2 \cup E_3 \cup L_{23} \cup L_{13} \cup L_{14} \cup L_{34} \cup C_{32}),\\
U_{21} &=: S \setminus (E_2 \cup E_1 \cup L_{12} \cup L_{13} \cup L_{14} \cup L_{34} \cup C_{12}),\\
U_{32} &=: S \setminus (E_3 \cup E_2 \cup L_{23} \cup L_{12} \cup L_{14} \cup L_{24} \cup C_{23}),\\
U_{31} &=: S \setminus (E_3 \cup E_1 \cup L_{13} \cup L_{12} \cup L_{14} \cup L_{24} \cup C_{13}).
\end{aligned}$$

It is easy to see that

$$S \setminus (U_{13} \cup U_{12} \cup U_{23} \cup U_{21} \cup U_{32} \cup U_{31})$$

is the set R of $L_{34} \cap L_{12}$, $L_{13} \cap L_{24}$, $L_{24} \cap L_{13}$, $L_{14} \cap E_1$, $L_{24} \cap E_2$ and $L_{34} \cap E_3$.

Next we consider the cylinder given by

$$\mathbb{P}^2 \setminus (l_{14} \cup l_{23})$$

that is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$. Then

$$V_1 = S \setminus (L_{14} \cup L_{23} \cup E_1 \cup \cdots \cup E_5)$$

is the cylinder of S that is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$. Similarly we can find the following cylinder

$$V_2 = S \setminus (L_{13} \cup L_{24} \cup E_1 \cup \cdots \cup E_5).$$

It is easy to see that $V_1 \cup V_2$ contains R. Therefore S is covers by the above cylinders.

4.2. On the -1-curve E_4

In this section we consider E_4. We set $\mathsf{p} = L_{23} \cap L_{45} = l_{23} \cap l_{45}$. Let $l_{1\mathsf{p}}$ be the line in $\mathbb{P}^2$ passing through the two points p_1 and p. Let A be an ample divisor of S. For the Fujita invariant μ of A we have the following.

$$\mu A \sim_{\mathbb{Q}} 3e_0 + (a_1 - 1)E_1 + \cdots + (a_4 - 1)E_4 + (a_5 - 2)E_5$$

where e_0 is the pullback of the class of lines in $\mathbb{P}^2$ and a_i are positive rational numbers. For

$$\begin{aligned}
(1 - a_4)\pi^*(l_{45}) &= (1 - a_4)(L_{45} + E_4 + 2E_5),\\
\frac{2 + a_4}{2}\pi^*(l_{23}) &= \frac{2 + a_4}{2}(L_{23} + E_2 + E_3),\\
\frac{2 + a_4}{2}\pi^*(l_{1\mathsf{p}}) &= \frac{2 + a_4}{2}(L_{1\mathsf{p}} + E_1)
\end{aligned}$$

we have

$$\begin{aligned}
\mu A \sim_{\mathbb{Q}} (1 - a_4)\pi^*(l_{45}) + \frac{2 + a_4}{2}(\pi^*(l_{23}) + \pi^*(l_{1\mathsf{p}})) + (a_1 - 1)E_1 + \cdots + (a_5 - 2)E_5\\
= (1 - a_4)L_{45} + \frac{2 + a_4}{2}(L_{23} + L_{1\mathsf{p}}) +
\end{aligned}$$

$$\mu A \sim_{\mathbb{Q}} (1-a_4)L_{45} + \frac{2+a_4}{2}(L_{23}+L_{1\mathsf{p}})$$
$$+(a_1+\frac{a_4}{2})E_1 + (a_2+\frac{a_4}{2})E_2 + (a_3+\frac{a_4}{2})E_3 + (a_5-2a_4)E_5$$

Thus there is a divisor D such that $D \sim_{\mathbb{Q}} A$ and $\mathrm{Supp}(D) = L_{45} \cup L_{23} \cup L_{1\mathsf{p}} \cup E_1 \cup E_2 \cup E_3 \cup E_5$.

We can see that $S \setminus \mathrm{Supp}(D) \cong \mathbb{A}^1 \times \mathbb{A}^1_{**}$.

Therefore, the curve E_4 must contract.

4.3. Almost general positions

In this section we consider the case that a weak del Pezzo surface X of degree 4 is given by a blow-up of $\mathbb{P}^2$.

Let p_1, p_2 and p_3 be colinear points in $\mathbb{P}^2$. Let l_{123} be the line passing through the points p_1, p_2 and p_3. And let p_4 and p_5 be distinct points in $\mathbb{P}^2$ which do not contained in the line l_{123}. Then $\mathsf{p}_1, \dots \mathsf{p}_5$ are almost general position. Let X be the weak del Pezzo surface given by the blow-up $\pi\colon X \to \mathbb{P}^2$ along $\mathsf{p}_1, \dots \mathsf{p}_5$. Let A be an ample divisor on X related to π. Then we have

$$\mu A + K_X \equiv \sum_{i=1}^{5} a_i E_i$$

where μ is the Fujita invariant of A, a_i is positive real number and E_i are exceptional curve with $\pi(E_i) = \mathsf{p}_i$. It is easy to see that $a_i < 1$. Then we have

$$\mu A \equiv -K_X + \sum_{i=1}^{5} a_i E_i.$$

In this situation we can construct the following cylinders.

Let l_{45} be the line passing through p_4 and p_5. And let l be a general line passing through $l_{123} \cap l_{45}$. Then $\mathbb{P}^2 \setminus l_{123} \cup l_{45} \cup l$ is the cylinder isomorphic to $\mathbb{A}^1_{**} \times \mathbb{A}^1$. Meanwhile, we have

$$-K_{\mathbb{P}^2} \equiv l_{123} + l_{45} + l.$$

It implies that

$$\begin{array}{rcl} \mu A & \equiv & -K_X + \sum_{i=1}^{5} a_i E_i \\ & \equiv & \pi^*(-K_{\mathbb{P}^2}) + \sum_{i=1}^{5}(a_i - 1)E_i \\ & \equiv & L_{123} + L_{45} + L + \sum_{i=1}^{5} a_i E_i \end{array}$$

where L_* be the strict transform of l_*. Since $U_1 = X \setminus L_{123} \cup L_{45} \cup L \cup \bigcup_{i=1}^{5} E_i$ is isomorphic to $\mathbb{P}^2 \setminus l_{123} \cup l_{45} \cup l$, it is an A-polar cylinder.

Next we consider the following. Let l_{14} and l_{25} be the curves in $\mathbb{P}^2$ where l_{ij} is the line passing through p_i and p_j. And let l_3 be the curve passing through p_3 and $l_{14} \cap l_{25}$. Then we have

$$\begin{array}{rcl} \mu A & \equiv & \pi^*(-K_{\mathbb{P}^2}) + \sum_{i=1}^{5}(a_i - 1)E_i \\ & \equiv & L_{14} + L_{25} + L_3 + \sum_{i=1}^{5} a_i E_i. \end{array}$$

Thus $U_2 = X \setminus L_{14} \cup L_{25} \cup L_3 \cup \bigcup_{i=1}^{5} E_i$ is an A-polar cylinder. Similarly we can see that $U_3 = X \setminus L_{15} \cup L_{24} \cup \bar{L}_3 \cup \bigcup_{i=1}^{5} E_i$ is an A-polar cylinder where $\bar{L}_3$ is the strict transform of the line passing through p_3 and $l_{15} \cup l_{24}$. Thus we have

$$X \setminus U_1 \cup U_2 \cup U_3 = \bigcup_{i=1}^{5} E_i.$$

Meanwhile, we consider another birational morphism $\psi\colon X \to \mathbb{P}^2$ contracted by L_{14}, L_{15}, L_{45}, E_2 and E_3. We set $\psi(L_{14}) = \mathsf{q}_1$, $\psi(L_{15}) = \mathsf{q}_2$, $\psi(E_2) = \mathsf{q}_3$, $\psi(E_3) = \mathsf{q}_4$ and $\psi(L_{45}) = \mathsf{q}_5$. Using this morphsim we can see that $X \setminus (E_1 \cup E_2 \cup E_3 \cup L_{123} \cup L_{14} \cup L_{15} \cup L_{45} \cup L)$ is the cylinder where L is the strict transform of a general line passing through $\psi(E_1) \cap \psi(L_{123})$. Thus we must prove that it is an A-polar cylinder. We consider the following.

$$K_X \equiv \psi^*(K_{\mathbb{P}^2}) + E_2 + E_3 + L_{14} + L_{15} + L_{45}.$$

Thus we have

$$\mu A \equiv -\psi^*(K_{\mathbb{P}^2}) - L_{14} - L_{15} - L_{45} + a_1E_1 + (a_2-1)E_2 + (a_3-1)E_3 + a_4E_4 + a_5E_5.$$

Without loss of generality we can assume that $a_4 \leq a_5$. We have

$$\begin{array}{rcl} -K_{\mathbb{P}^2} & \equiv & (1-a_5-2\epsilon)L + (1+a_4+a_5+\epsilon)\psi(\bar{L}_{123}) + (1+a_5+\epsilon)\psi(E_1) \\ & - & a_4\psi(E_4) - a_5\psi(E_5) \end{array}$$

where $\epsilon \ll 1$ is a positive real number. From this we obtain the following.

$$\begin{array}{rcl} \psi^*(-K_{\mathbb{P}^2}) & \equiv & (1-a_5-2\epsilon)\bar{L} + (1+a_4+a_5+\epsilon)\bar{L}_{123} + (1+a_5+\epsilon)E_1 \\ & - & a_4E_4 - a_5E_5 + (1+a_4+a_5+\epsilon)(E_2+E_3) \\ & + & (1+\epsilon)\bar{L}_{45} + (1+a_5-a_4+\epsilon)\bar{L}_{14} + (1+\epsilon)\bar{L}_{15}. \end{array}$$

It implies that

$$\begin{array}{rcl} \mu A & \equiv & (1-a_5-2\epsilon)\bar{L} + (1+a_4+a_5+\epsilon)\bar{L}_{123} + (a_5-a_4+\epsilon)\bar{L}_{14} + \epsilon\bar{L}_{15} \\ & + & \epsilon\bar{L}_{45} + (1+a_1+a_5+\epsilon)E_1 + (a_2+a_4+a_5+\epsilon)E_2 \\ & + & (a_3+a_4+a_5+\epsilon)E_3 \end{array}$$

We consider the biraional morphsim $\phi\colon X \to \mathbb{P}^2$ given by the sequence of contraction

$$L_{15} \to L_{25} \to L_{35} \to L_{45} \to L_{123}.$$

Then $\phi(E_5)$ is the conic and $\phi(E_4)$ is the tangent line at the point $\phi(L_{45} \cup L_{123})$. Thus $X \setminus E_4 \cup E_5 \cup L_{123} \cup L_{15} \cup L_{25} \cup L_{35} \cup L_{45}$ is a cylinder which is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$. Thus we have to see that it is A-polar cylinder. Using the biraional morphism ϕ we have

$$K_X \equiv \phi^*(K_{\mathbb{P}^2}) + L_{15} + L_{25} + L_{35} + L_{123} + 2L_{45}.$$

Then we have

$$\mu A \equiv \phi^*(-K_{\mathbb{P}^2}) - L_{15} - L_{25} - L_{35} - L_{123} - 2L_{45} + \sum_{i=1}^{5} a_iE_i.$$

Meanwhile we have

$$
\begin{aligned}
-a_1\phi^*(e_1) &= -a_1(E_1 + L_{123} + 2L_{45} + L_{15}),\\
-a_2\phi^*(e_2) &= -a_2(E_2 + L_{123} + 2L_{45} + L_{25}),\\
-a_3\phi^*(e_3) &= -a_3(E_3 + L_{123} + 2L_{45} + L_{35}).
\end{aligned}
$$

They imply that

$$
\begin{array}{rcl}
\mu A & \equiv & (3 + a_1 + a_2 + a_3)e_0 - (1 + a_1)L_{15} - (1 + a_2)L_{25} - (1 + a_3)L_{35}\\
& - & (1 + a_1 + a_2 + a_3)L_{123} - 2(1 + a_1 + a_2 + a_3)L_{45}\\
& + & a_4E_4 + a_5E_5
\end{array}
$$

where e_0 is the pullback of the class of line in $\mathbb{P}^2$. Without loss of generality we can assume that $a_1 \leq a_2 \leq a_3$. We have

$$
(1 + a_3 + \epsilon)\phi^*(e_5) = (1 + a_3 + \epsilon)(E_5 + L_{15} + L_{25} + L_{35} + L_{123} + 2L_{45}),
$$

$$
(1 + a_1 + a_2 - a_3 - 2\epsilon)\phi^*(e_4) = (1 + a_1 + a_2 - a_3 - 2\epsilon)(E_4 + L_{123} + 2L_{45}).
$$

Then

$$
\begin{array}{rcl}
\mu A & \equiv & (1 + a_3 + a_4 + \epsilon)E_5 + (1 + a_1 + a_2 - a_3 + a_5 - 2\epsilon)E_4\\
& + & (a_3 - a_1 + \epsilon)L_{15} + (a_3 - a_2 + \epsilon)L_{25} + \epsilon L_{35}\\
& - & (1 - a_3 - \epsilon)L_{123} - 2(1 - a_3 - \epsilon)L_{45}.
\end{array}
$$

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Joonyeong Won
Department of Mathematics, Ewha Womans University, Seoul, Korea
E-mail address: leonwon@ewha.ac.kr