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A HIGHER ORDER SPLIT LEAST-SQUARES CHARACTERISTIC MIXED ELEMENT METHOD FOR SOBOLEV EQUATIONS

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Abstract. In this paper, we introduce a higher order split least-squares characteristic mixed element scheme for Sobolev equations. First, we use a characteristic mixed element method to manipulate both convection term and time derivative term efficiently and obtain the system of equations in the primal unknown and the flux unknown. Second, we define a least-squares minimization problem and a least-squares characteristic mixed element scheme. Finally, we obtain a split least-squares characteristic mixed element scheme for the given problem whose system is uncoupled in the unknowns. We establish the convergence results for the primal unknown and the flux unknown with the second order in a time increment.

1. Introduction

In this paper, we consider the following Sobolev equation

$$\begin{cases} c(\boldsymbol{x})u_t + \boldsymbol{d}(\boldsymbol{x})\cdot\nabla u - \nabla\cdot(a(u)\nabla u_t + b(u)\nabla u) & \\ \qquad\qquad\qquad = f(u), & (\boldsymbol{x},t)\in\Omega\times(0,T], \\ u(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t)\in\Gamma_D\times(0,T], \\ (a(u)\nabla u_t + b(u)\nabla u)\cdot\mathbf{n} = 0, & (\boldsymbol{x},t)\in\Gamma_N\times(0,T], \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded convex domain in $\mathbb{R}^m$, $1 \le m \le 3$, with its boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \varnothing$ and $c(\boldsymbol{x}), \boldsymbol{d}(\boldsymbol{x}), a(u), b(u), f(u)$, and $u_0(\boldsymbol{x})$ are given functions. The applications of Sobolev equations were studied in [2, 22, 23] and the existence and uniqueness results of the solutions of (1.1) were given in [8].

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When $\boldsymbol{d}(\boldsymbol{x}) = \mathbf{0}$, many numerical methods, such as mixed finite element methods [12, 19, 21, 25], least-squares methods [13, 21, 24, 25], and discontinuous Galerkin methods [15, 16], were used to achieve the numerical results. And if we use a conventional (least-squares) mixed finite element method, then we will have the coupled system of equations in two unknowns and some difficulties of solving the coupled system. So, in [21], a split least-squares mixed finite element method for reaction-diffusion problems was first introduced to avoid the problem of solving the coupled systems of equations in the unknowns and in [11], a split least-squares mixed element method for pseudo-parabolic equations was introduced to avoid the problem of solving the coupled systems of equations in the unknowns.

When $\boldsymbol{d}(\boldsymbol{x}) \neq \mathbf{0}$, we generally use a characteristic (mixed) finite element method as one of the useful methods [1, 3, 4, 5, 6, 7, 10, 14] to reflect well the physical character of a convection term and to treat efficiently both convection term and time derivative term. Gao and Rui [9] introduced a split least-squares characteristic mixed finite element method to approximate the primal unknown u and the flux unknown $-a\nabla u$ of the equation (1.1). And Zhang and Guo [26] introduced a split least-squares characteristic mixed element method for nonlinear nonstationary convection-diffusion problem to approximate the primal unknown and the flux unknown. In [17], Ohm and Shin introduced a split least-squares characteristic mixed element method to approximate the primal unknown u and the flux unknown $\boldsymbol{\sigma} = -(a(\boldsymbol{x})\nabla u_t + b(\boldsymbol{x})\nabla u)$ and proved the optimal order of convergence in L^2 and H^1 norms for the primal unknown and the suboptimal order in L^2 norm for the flux unknown, with the first order in a time increment. And in [18], Ohm and Shin extended their previous result to the problem with the flux unknown $\boldsymbol{\sigma} = -(a(u)\nabla u_t + b(u)\nabla u)$.

In this paper, we introduce a split least-squares characteristic mixed element scheme based on the three point formula for the directional derivative and obtain the optimal order of convergence in L^2 and H^1 norms for the primal u and the suboptimal order in L^2 norm for the flux $\boldsymbol{\sigma}$, with the second order in a time increment which improves the results in in [18]. The outline of this paper is as follows. Section 2 is devoted to introduce some assumptions and notations and Section 3 to introduce finite element spaces with approximation properties. In Section 4, we construct a split least-squares characteristic mixed element scheme to approximate the primal unknown and the flux unknown. In Section 5 and Section 6, the convergence results for the primal unknown and the flux unknown are established.

2. Assumptions and notations

For a nonnegative integer s and $1 \le p \le \infty$, we denote by $W^{s,p}(\Omega)$ the Sobolev space with the norm

$$\|\phi\|_{s,p} = \begin{cases} \Big(\sum\limits_{|\boldsymbol{k}|\le s} \int_\Omega |D^{\boldsymbol{k}}\phi|^p dx \Big)^{1/p}, & 1 \le p < \infty, \\ max_{|\boldsymbol{k}|\le s} \text{ ess sup}|D^{\boldsymbol{k}}\phi|, & p = \infty, \end{cases}$$

where $\boldsymbol{k} = (k_1, k_2, \cdots, k_m), k_i \ge 0$, is a multiindex of order $|\boldsymbol{k}| = k_1 + k_2 + \cdots + k_m$ and $D^{\boldsymbol{k}}\phi = \frac{\partial^{|\boldsymbol{k}|}\phi}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_m^{k_m}}$. When $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $\|\phi\|_s = \|\phi\|_{s,2}$. And we simply denote $\|\phi\| = \|\phi\|_0$ for $s = 0$. Let $\boldsymbol{H}^s(\Omega) = \{\boldsymbol{u} = (u_1, u_2, \cdots, u_m) \mid u_i \in H^s(\Omega), 1 \le i \le m\}$ with the norm $\|\boldsymbol{u}\|_s = \Big(\sum\limits_{i=1}^m \|u_i\|_s^2\Big)^{1/2}$. And let $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ and $\boldsymbol{W} = \{\boldsymbol{w} \in H(div, \Omega) : \boldsymbol{w} \cdot \mathbf{n} = 0 \text{ on } \Gamma_N\}$.

When a function $\phi(x,t)$ belongs to a Sobolev space equipped with a norm $\|\cdot\|_X$ for each t, we let

$$\|\phi(x,t)\|^p_{L^p(0,t_0:X)} = \int_0^{t_0} \|\phi(x,t)\|^p_X dt, \text{ for } 1 \le p < \infty,$$
$$\|\phi(x,t)\|_{L^\infty(0,t_0:X)} = \text{ess sup}_{0\le t\le t_0} \|\phi(x,t)\|_X.$$

When $t_0 = T$, we write $L^p(X) = L^p(0, T : X)$ and $L^\infty(X) = L^\infty(0, T : X)$, respectively.

Assumptions on the coefficients and the function f in (1.1) are as follows:
(A1) There exist c_*, c^*, and d^* such that $0 < c_* < c(\boldsymbol{x}) \le c^*$ and $0 < |d(\boldsymbol{x})| \le d^*$ for all $\boldsymbol{x} \in \Omega$, where $|d(\boldsymbol{x})| = \sum_{i=1}^m d_i^2(\boldsymbol{x})$.
(A2) There exist a_*, a^*, b_*, and b_* such that $0 < a_* < a(p) \le a^*$ and $0 < b_* < b(p) \le b^*$ for all $p \in \mathbb{R}$.
(A3) $a_p(p), a_{pp}(p), b_p(p)$, and $b_{pp}(p)$ are bounded .
(A4) $f(p)$ is Lipschitz continuous.

3. Finite element spaces

To begin with, we let $\mathcal{E}_h = \{E_1, E_2, \cdots, E_{N_h}\}$ be a family of regular finite element subdivision of Ω. We let h denote the maximum of the diameters of the elements of $\mathcal{E}_h$. If $m = 2$, then E_i is a triangle or a quadrilateral, and if $m = 3$, then E_i is a 3-simplex or 3-rectangle. Boundary elements are allowed to have one curvilinear edge (or one curved surface).

We denote by $V_h \times \boldsymbol{W}_h$ the Raviart-Thomas-Nedlec space of index $k \ge 0$

associated with $\mathcal{E}_h$. And let $P_h \times \mathbf{\Pi}_h : V \times \boldsymbol{W} \to V_h \times \boldsymbol{W}_h$ denote the Raviart-Thomas projection [20] which satisfies

$$\begin{cases} (\nabla \cdot \boldsymbol{w} - \nabla \cdot \mathbf{\Pi}_h \boldsymbol{w}, \chi) = 0, & \forall \chi \in V_h, \\ (v - P_h v, \chi) = 0, & \forall \chi \in V_h. \end{cases} \tag{3.1}$$

Then, $(\nabla \cdot \boldsymbol{w}, v - P_h v) = 0$ holds for each $v \in V$ and each $\boldsymbol{w} \in \boldsymbol{W}_h$ and $div\ \mathbf{\Pi}_h = P_h\ div$ is a function from $\boldsymbol{W}$ onto V_h. The following approximation properties are proved in [20]:

$$\begin{aligned} \|v - P_h v\| + h\|v - P_h v\|_1 &\le K h^r \|v\|_r,\ \forall v \in V \cap H^r(\Omega), 1 \le r \le k+1, \\ \|\boldsymbol{w} - \mathbf{\Pi}_h \boldsymbol{w}\| &\le K h^r \|\boldsymbol{w}\|_r,\ \forall \boldsymbol{w} \in \boldsymbol{W} \cap \boldsymbol{H}^r(\Omega), 1 \le r \le k+1, \\ \|\nabla \cdot (\boldsymbol{w} - \mathbf{\Pi}_h \boldsymbol{w})\| &\le K h^r \|\nabla \cdot \boldsymbol{w}\|_r,\ \forall \boldsymbol{w} \in \boldsymbol{W} \cap \boldsymbol{H}^r(\Omega), 0 \le r \le k+1. \end{aligned} \tag{3.2}$$

4. A split least-squares characteristic mixed element scheme

Let $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x}, t)$ be the unit vector in the direction of $(\boldsymbol{d}(\boldsymbol{x}), c(\boldsymbol{x}))$. Then, the directional derivative of u in the direction of $\boldsymbol{\nu}$ is given as follows:

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{c(\boldsymbol{x})}{\psi(\boldsymbol{x})} \frac{\partial u}{\partial t} + \frac{\boldsymbol{d}(\boldsymbol{x})}{\psi(\boldsymbol{x})} \cdot \nabla u$$

where $\psi(\boldsymbol{x}) = \left(c^2(\boldsymbol{x}) + |\boldsymbol{d}(\boldsymbol{x})|^2\right)^{\frac{1}{2}}$ and $|\boldsymbol{d}(\boldsymbol{x})|^2 = \sum_{i=1}^{m} d_i^2(\boldsymbol{x})$. So the problem (1.1) becomes

$$\begin{cases} \psi(\boldsymbol{x}) \frac{\partial u}{\partial \boldsymbol{\nu}} - \nabla \cdot (a(u) \nabla u_t + b(u) \nabla u) = f(u), & \text{in } \Omega \times (0, T], \\ u(\boldsymbol{x}, t) = 0, & \text{on } \Gamma_D \times (0, T], \\ (a(u) \nabla u_t + b(u) \nabla u) \cdot \mathbf{n} = 0, & \text{on } \Gamma_N \times (0, T], \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \text{in } \Omega. \end{cases} \tag{4.1}$$

By denoting $\boldsymbol{\sigma} = -(a(u)\nabla u_t + b(u)\nabla u)$, we can rewrite the problem (4.1) as follows:

$$\begin{cases} \psi(\boldsymbol{x}) \frac{\partial u}{\partial \boldsymbol{\nu}} + \nabla \cdot \boldsymbol{\sigma} = f(u), & \text{in } \Omega \times (0, T], \\ \boldsymbol{\sigma} + a(u) \nabla u_t + b(u) \nabla u = 0, & \text{in } \Omega \times (0, T], \\ u(\boldsymbol{x}, t) = 0, & \text{on } \Gamma_D \times (0, T], \\ \boldsymbol{\sigma} \cdot \mathbf{n} = 0, & \text{on } \Gamma_N \times (0, T], \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \text{in } \Omega. \end{cases} \tag{4.2}$$

To discretize the problem (4.2), let $\Delta t = T/N$ be a time increment for a given positive integer N and $t^n = n\Delta t$ for $n = 0, 1, \cdots, N$. Discretizing $\psi(\boldsymbol{x}) \frac{\partial u}{\partial \boldsymbol{\nu}}$ at $(\boldsymbol{x}, t^n)$ by applying the backward three point formula along the direction of ν, we get

$$\psi(\boldsymbol{x}) \frac{\partial u}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, t^n) \cong c(\boldsymbol{x}) \frac{\frac{3}{2} u(\boldsymbol{x}, t^n) - 2u(\check{\boldsymbol{x}}, t^{n-1}) + \frac{1}{2} u(\hat{\boldsymbol{x}}, t^{n-2})}{\Delta t},$$

where $\check{\boldsymbol{x}} = \boldsymbol{x} - \tilde{\boldsymbol{d}}(\boldsymbol{x})\Delta t$ and $\hat{\boldsymbol{x}} = \boldsymbol{x} - 2\tilde{\boldsymbol{d}}(\boldsymbol{x})\Delta t$ with $\tilde{\boldsymbol{d}}(\boldsymbol{x}) = \frac{\boldsymbol{d}(\boldsymbol{x})}{c(\boldsymbol{x})}$. Therefore, from (4.2), we know that for $n \geq 2$, $(u^n, \boldsymbol{\sigma}^n)$ satisfies

$$\begin{cases} c(\boldsymbol{x})\frac{\frac{3}{2}u^n - 2\check{u}^{n-1} + \frac{1}{2}\hat{u}^{n-2}}{\Delta t} + \nabla\cdot\boldsymbol{\sigma}^n \\ \qquad\qquad = f(Eu^n) + Q_1^n + Q_2^n, \qquad \text{in } \Omega, \\ \boldsymbol{\sigma}^n + a(Eu^n)\frac{\frac{3}{2}\nabla u^n - 2\nabla u^{n-1} + \frac{1}{2}\nabla u^{n-2}}{\Delta t} + b(Eu^n)\nabla u^n \\ \qquad\qquad = Q_3^n + Q_4^n, \qquad \text{in } \Omega, \\ u^n = 0, \qquad \text{on } \Gamma_D, \\ \boldsymbol{\sigma}^n\cdot\mathbf{n} = 0, \qquad \text{on } \Gamma_N, \end{cases} \tag{4.3}$$

where $u^n = u(\boldsymbol{x}, t^n)$, $\boldsymbol{\sigma}^n = \boldsymbol{\sigma}(\boldsymbol{x}, t^n)$, $\check{u}^{n-1} = u(\check{\boldsymbol{x}}, t^{n-1})$, $\hat{u}^{n-2} = u(\hat{\boldsymbol{x}}, t^{n-2})$, $Eu^n = 2u^{n-1} - u^{n-2}$, $Q_1^n = c(\boldsymbol{x})\frac{\frac{3}{2}u^n - 2\check{u}^{n-1} + \frac{1}{2}\hat{u}^{n-2}}{\Delta t} - \psi(\boldsymbol{x})\frac{\partial u}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, t^n)$, $Q_2^n = f(u^n) - f(Eu^n)$, $Q_3^n = a(Eu^n)\frac{\frac{3}{2}\nabla u^n - 2\nabla u^{n-1} + \frac{1}{2}\nabla u^{n-2}}{\Delta t} - a(u^n)\nabla u_t^n$, and $Q_4^n = b(Eu^n)\nabla u^n - b(u^n)\nabla u^n$. Notice that we need u^0 and u^1 in (4.3), which will be given later. For first and second equations of (4.3), we obtain the equivalent system of equations

$$\begin{cases} \frac{3}{2}c(\boldsymbol{x})u^n + \Delta t\ \nabla\cdot\boldsymbol{\sigma}^n = 2c(\boldsymbol{x})\check{u}^{n-1} - \frac{1}{2}c(\boldsymbol{x})\hat{u}^{n-2} + \Delta t(f(Eu^n) + Q_1^n + Q_2^n), \\ \Delta t\ \boldsymbol{\sigma}^n + \big(\frac{3}{2}a(Eu^n) + \Delta t\ b(Eu^n)\big)\nabla u^n \\ \qquad\qquad = a(Eu^n)(2\nabla u^{n-1} - \frac{1}{2}\nabla u^{n-2}) + \Delta t(Q_3^n + Q_4^n) \end{cases}$$

and hence

$$\begin{cases} \frac{3}{2}c(\boldsymbol{x})u^n + \Delta t\ \nabla\cdot\boldsymbol{\sigma}^n \\ \qquad = 2c(\boldsymbol{x})\check{u}^{n-1} - \frac{1}{2}c(\boldsymbol{x})\hat{u}^{n-2} + \Delta t(f(Eu^n) + Q_1^n + Q_2^n), \\ \Delta t\ \boldsymbol{\sigma}^n + \frac{3}{2}A(Eu^n)\nabla u^n \\ \qquad = a(Eu^n)(2\nabla u^{n-1} - \frac{1}{2}\nabla u^{n-2}) + \Delta t(Q_3^n + Q_4^n), \end{cases} \tag{4.4}$$

where $A(\cdot) = a(\cdot) + \frac{2}{3}b(\cdot)\Delta t$. Therefore, from (4.4), we get

$$\begin{cases} c(\boldsymbol{x})^{-1/2}[\frac{3}{2}c(\boldsymbol{x})u^n + \Delta t\ \nabla\cdot\boldsymbol{\sigma}^n - 2c(\boldsymbol{x})\check{u}^{n-1} \\ \qquad + \frac{1}{2}c(\boldsymbol{x})\hat{u}^{n-2} - \Delta t(f(Eu^n) + Q_1^n + Q_2^n)] = 0, \\ A(Eu^n)^{-1/2}[\Delta t\ \boldsymbol{\sigma}^n + \frac{3}{2}A(Eu^n)\nabla u^n \\ \qquad - a(Eu^n)(2\nabla u^{n-1} - \frac{1}{2}\nabla u^{n-2}) - \Delta t(Q_3^n + Q_4^n)] = 0. \end{cases} \tag{4.5}$$

For $(v, \boldsymbol{\tau}) \in V \times \boldsymbol{W}$, we define a least-squares functional $J^n(v, \boldsymbol{\tau})$ as follows:

$$\begin{aligned} J^n(v, \boldsymbol{\tau}) = \|c(\boldsymbol{x})^{-1/2}[\frac{3}{2}c(\boldsymbol{x})v + \Delta t\ \nabla\cdot\boldsymbol{\tau}^n - 2c(\boldsymbol{x})\check{u}^{n-1} + \frac{1}{2}c(\boldsymbol{x})\hat{u}^{n-2} \\ - \Delta t(f(Eu^n) + Q_1^n + Q_2^n)]\|^2 \\ + \|A(Eu^n)^{-1/2}[\Delta t\ \boldsymbol{\tau} + \frac{3}{2}A(Eu^n)\nabla v \\ - a(Eu^n)(2\nabla u^{n-1} - \frac{1}{2}\nabla u^{n-2}) - \Delta t(Q_3^n + Q_4^n)]\|^2. \end{aligned}$$

Then the least-squares minimization problem corresponding to (4.5) is given as follows: find $(u^n, \boldsymbol{\sigma}^n) \in V \times \boldsymbol{W}$, $n \geq 2$, such that

$$J^n(u^n, \boldsymbol{\sigma}^n) = \inf_{(v,\boldsymbol{\tau}) \in V \times \boldsymbol{W}} J^n(v, \boldsymbol{\tau}). \tag{4.6}$$

Define the bilinear form B on $(V \times \boldsymbol{W})^2$ by

$$\begin{aligned} B(w:\ u, \boldsymbol{\sigma}; v, \boldsymbol{\tau}) = &\Big(c(\boldsymbol{x})^{-1}(\frac{3}{2}c(\boldsymbol{x})u + \Delta t\ \nabla \cdot \boldsymbol{\sigma}),\ \frac{3}{2}c(\boldsymbol{x})v + \Delta t\ \nabla \cdot \boldsymbol{\tau}\Big) \\ &+ \Big(A(w)^{-1}(\frac{3}{2}A(w)\nabla u + \Delta t \boldsymbol{\sigma}),\ \frac{3}{2}A(w)\nabla v + \Delta t \boldsymbol{\tau}\Big). \end{aligned} \tag{4.7}$$

Then the weak formulation of the minimization problem (4.6) is given as follows: find $(u^n, \boldsymbol{\sigma}^n) \in V \times \boldsymbol{W}$, $n \geq 2$, such that

$$\begin{aligned} B(Eu^n\ :\ &u^n, \boldsymbol{\sigma}^n; v, \boldsymbol{\tau}) = \\ &\Big(c(\boldsymbol{x})^{-1}\big(2c(\boldsymbol{x})\check{u}^{n-1} - \frac{1}{2}c(\boldsymbol{x})\hat{u}^{n-2} + \Delta t(f(Eu^n) + Q_1^n + Q_2^n)\big), \\ &\qquad \frac{3}{2}c(\boldsymbol{x})v + \Delta t\ \nabla \cdot \boldsymbol{\tau}\Big) \\ &+ \Big(A(Eu^n)^{-1}\big(a(Eu^n)(2\nabla u^{n-1} - \frac{1}{2}\nabla u^{n-2}) + \Delta t(Q_3^n + Q_4^n)\big), \\ &\qquad \frac{3}{2}A(Eu^n)\nabla v + \Delta t \boldsymbol{\tau}\Big) \end{aligned} \tag{4.8}$$

for any $(v, \boldsymbol{\tau}) \in V \times \boldsymbol{W}$. Based on (4.8), we derive the following least-squares characteristic mixed element scheme: find $(u_h^n, \boldsymbol{\sigma}_h^n) \in V_h \times \boldsymbol{W}_h, n \geq 2$ such that

$$\begin{aligned} B(Eu_h^n\ :\ &u_h^n, \boldsymbol{\sigma}_h^n; v_h, \boldsymbol{\tau}_h) = \\ &\Big(c(\boldsymbol{x})^{-1}(2c(\boldsymbol{x})\check{u}_h^{n-1} - \frac{1}{2}c(\boldsymbol{x})\hat{u}_h^{n-2} + \Delta t f(Eu_h^n)), \\ &\qquad \frac{3}{2}c(\boldsymbol{x})v_h + \Delta t\ \nabla \cdot \boldsymbol{\tau}_h\Big) \\ &+ \Big(A(Eu_h^n)^{-1}(2a(Eu_h^n)\nabla u_h^{n-1} - \frac{1}{2}a(Eu_h^n)\nabla u_h^{n-2}), \\ &\qquad \frac{3}{2}A(Eu_h^n)\nabla v_h + \Delta t\ \boldsymbol{\tau}_h\Big) \end{aligned} \tag{4.9}$$

for any $(v_h, \boldsymbol{\tau}_h) \in V_h \times \boldsymbol{W}_h$.

Lemma 4.1. *For any* $(u, \boldsymbol{\sigma})$, $(v, \boldsymbol{\tau}) \in V \times \boldsymbol{W}$, *we have*

$$\begin{aligned} B(w\ :\ u, \boldsymbol{\sigma}; v, \boldsymbol{\tau}) = &\frac{9}{4}(c(\boldsymbol{x})u, v) + (\Delta t)^2\ (c(\boldsymbol{x})^{-1}\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\tau}) \\ &+ \frac{9}{4}(A(w)\nabla u, \nabla v) + (\Delta t)^2\ (A(w)^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}). \end{aligned}$$

Proof. The proof is straightforward from the definition of of the bilinear form B in (4.7). □

Letting $v_h = 0$ in (4.9) and applying the definition of the bilinear form B, we have

$$\begin{aligned}
&(\Delta t)^2 \left((c(\boldsymbol{x})^{-1}\nabla\cdot\boldsymbol{\sigma}_h^n, \nabla\cdot\boldsymbol{\tau}_h) + (A(Eu_h^n)^{-1}\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h^n)\right)\\
=&\Big(c(\boldsymbol{x})^{-1}(2c(\boldsymbol{x})\check{u}_h^{n-1} - \frac{1}{2}c(\boldsymbol{x})\hat{u}_h^{n-2} + \Delta t f(Eu_h^n)),\ \Delta t\ \nabla\cdot\boldsymbol{\tau}_h\Big)\\
&\quad + \Big(A(Eu_h^n)^{-1}(2a(Eu_h^n)\nabla u_h^{n-1} - \frac{1}{2}a(Eu_h^n)\nabla u_h^{n-2}),\ \Delta t\ \boldsymbol{\tau}_h\Big)
\end{aligned}$$

and hence

$$\begin{aligned}
&(c(\boldsymbol{x})^{-1}\nabla\cdot\boldsymbol{\sigma}_h^n, \nabla\cdot\boldsymbol{\tau}_h) + (A(Eu_h^n)^{-1}\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h^n)\\
=&\frac{2}{\Delta t}(\check{u}_h^{n-1},\ \nabla\cdot\boldsymbol{\tau}_h) - \frac{1}{2\Delta t}(\hat{u}_h^{n-2},\ \nabla\cdot\boldsymbol{\tau}_h) + (c(\boldsymbol{x})^{-1}f(Eu_h^n),\ \nabla\cdot\boldsymbol{\tau}_h)\\
&+ \frac{2}{\Delta t}(A(Eu_h^n)^{-1}a(Eu_h^n)\nabla u_h^{n-1},\ \boldsymbol{\tau}_h) - \frac{1}{2\Delta t}(A(Eu_h^n)^{-1}a(Eu_h^n)\nabla u_h^{n-2},\ \boldsymbol{\tau}_h)\\
=&\frac{2}{\Delta t}(\nabla u_h^{n-1} - \nabla\check{u}_h^{n-1},\ \boldsymbol{\tau}_h) - \frac{1}{2\Delta t}(\nabla u_h^{n-2} - \nabla\hat{u}_h^{n-2},\ \boldsymbol{\tau}_h)\\
&- \frac{2}{\Delta t}(\nabla u_h^{n-1},\ \boldsymbol{\tau}_h) + \frac{1}{2\Delta t}(\nabla u_h^{n-2},\ \boldsymbol{\tau}_h) + (c(\boldsymbol{x})^{-1}f(Eu_h^n),\ \nabla\cdot\boldsymbol{\tau}_h)\\
&+ \frac{2}{\Delta t}(A(Eu_h^n)^{-1}a(Eu_h^n)\nabla u_h^{n-1},\ \boldsymbol{\tau}_h) - \frac{1}{2\Delta t}(A(Eu_h^n)^{-1}a(Eu_h^n)\nabla u_h^{n-2},\ \boldsymbol{\tau}_h).
\end{aligned}$$

Since

$$\begin{aligned}
1 - A(Eu_h^n)^{-1}a(Eu_h^n) &= A(Eu_h^n)^{-1}(A(Eu_h^n) - a(Eu_h^n))\\
&= \frac{2}{3}\Delta t A(Eu_h^n)^{-1}b(Eu_h^n),
\end{aligned}$$

we have

$$\begin{aligned}
&(c(\boldsymbol{x})^{-1}\nabla\cdot\boldsymbol{\sigma}_h^n, \nabla\cdot\boldsymbol{\tau}_h) + (A(Eu_h^n)^{-1}\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h^n)\\
=&\frac{2}{\Delta t}(\nabla u_h^{n-1} - \nabla\check{u}_h^{n-1},\ \boldsymbol{\tau}_h) - \frac{1}{2\Delta t}(\nabla u_h^{n-2} - \nabla\hat{u}_h^{n-2},\ \boldsymbol{\tau}_h)\\
&- \frac{4}{3}(A(Eu_h^n)^{-1}b(Eu_h^n)\nabla u_h^{n-1},\ \boldsymbol{\tau}_h) + \frac{1}{3}(A(Eu_h^n)^{-1}b(Eu_h^n)\nabla u_h^{n-2},\ \boldsymbol{\tau}_h)\\
&+ (c(\boldsymbol{x})^{-1}f(Eu_h^n),\ \nabla\cdot\boldsymbol{\tau}_h).
\end{aligned}$$

Letting $\boldsymbol{\tau}_h = 0$ in (4.9) and applying the definition of the bilinear form B, we have

$$\begin{aligned}
&\frac{9}{4}(c(\boldsymbol{x})u_h^n, v_h) + \frac{9}{4}(A(Eu_h^n)\nabla u_h^n, \nabla v_h)\\
&= \frac{3}{2}\Big(2c(\boldsymbol{x})\check{u}_h^{n-1} - \frac{1}{2}c(\boldsymbol{x})\hat{u}_h^{n-2} + \Delta t f(Eu_h^n),\ v_h\Big)\\
&+ \frac{3}{2}\Big(2a(Eu_h^n)\nabla u_h^{n-1} - \frac{1}{2}a(Eu_h^n)\nabla u_h^{n-2},\ \nabla v_h\Big)
\end{aligned}$$

and hence

$$\begin{aligned}&(c(\boldsymbol{x})u_h^n, v_h) + (A(Eu_h^n)\nabla u_h^n, \nabla v_h)\\ &= \frac{2}{3}\Big(2c(\boldsymbol{x})\check{u}_h^{n-1} - \frac{1}{2}c(\boldsymbol{x})\hat{u}_h^{n-2} + \Delta t f(Eu_h^n),\ v_h\Big)\\ &+ \frac{2}{3}\Big(2a(Eu_h^n)\nabla u_h^{n-1} - \frac{1}{2}a(Eu_h^n)\nabla u_h^{n-2},\ \nabla v_h\Big).\end{aligned}$$

Therefore, we finally derive a split least-squares characteristic mixed element scheme: find approximations $\{u_h^n, \boldsymbol{\sigma}_h^n\} \in V_h \times \boldsymbol{W}_h$ such that for $n \geq 2$

$$\begin{aligned}&(c(\boldsymbol{x})u_h^n, v_h) + (A(Eu_h^n)\nabla u_h^n, \nabla v_h)\\ =&\frac{2}{3}\Big(2c(\boldsymbol{x})\check{u}_h^{n-1} - \frac{1}{2}c(\boldsymbol{x})\hat{u}_h^{n-2} + \Delta t f(Eu_h^n),\ v_h\Big)\\ &+ \frac{2}{3}\Big(2a(Eu_h^n)\nabla u_h^{n-1} - \frac{1}{2}a(Eu_h^n)\nabla u_h^{n-2},\ \nabla v_h\Big), \qquad v_h \in V_h,\end{aligned} \tag{4.10}$$

$$\begin{aligned}&(c(\boldsymbol{x})^{-1}\nabla\cdot\boldsymbol{\sigma}_h^n, \nabla\cdot\boldsymbol{\tau}_h) + (A(Eu_h^n)^{-1}\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h^n)\\ =&\frac{2}{\Delta t}(\nabla u_h^{n-1} - \nabla\check{u}_h^{n-1},\ \boldsymbol{\tau}_h) - \frac{1}{2\Delta t}(\nabla u_h^{n-2} - \nabla\hat{u}_h^{n-2},\ \boldsymbol{\tau}_h)\\ &- \frac{4}{3}(A(Eu_h^n)^{-1}b(Eu_h^n)\nabla u_h^{n-1},\ \boldsymbol{\tau}_h)\\ &+ \frac{1}{3}(A(Eu_h^n)^{-1}b(Eu_h^n)\nabla u_h^{n-2},\ \boldsymbol{\tau}_h)\\ &+ (c(\boldsymbol{x})^{-1}f(Eu_h^n),\ \nabla\cdot\boldsymbol{\tau}_h), \qquad \boldsymbol{\tau}_h \in \boldsymbol{W}_h.\end{aligned} \tag{4.11}$$

As we know, we need u_h^1 in the definition of Eu_h^2 and so we discretize $\psi(\boldsymbol{x})\frac{\partial u}{\partial \boldsymbol{\nu}}$ at $(\boldsymbol{x}, t^{\frac{1}{2}})$ by applying the centered difference formula along the direction of ν to get

$$\psi(\boldsymbol{x})\frac{\partial u}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, t^{\frac{1}{2}}) \cong c(\boldsymbol{x})\frac{u(\bar{\boldsymbol{x}}, t^1) - u(\bar{\bar{\boldsymbol{x}}}, t^0)}{\Delta t},$$

where $\tilde{\boldsymbol{d}}(\boldsymbol{x}) = \frac{\boldsymbol{d}(\boldsymbol{x})}{c(\boldsymbol{x})}$, $\bar{\boldsymbol{x}} = \boldsymbol{x} + \tilde{\boldsymbol{d}}(\boldsymbol{x})\frac{\Delta t}{2}$, $\bar{\bar{\boldsymbol{x}}} = \boldsymbol{x} - \tilde{\boldsymbol{d}}(\boldsymbol{x})\frac{\Delta t}{2}$, and $t^{\frac{1}{2}} = \frac{1}{2}(t^0 + t^1)$. Then we have

$$\begin{cases} c(\boldsymbol{x})\frac{\bar{u}^1 - \bar{\bar{u}}^0}{\Delta t} + \nabla\cdot\boldsymbol{\sigma}(t^{\frac{1}{2}}) = f(u^{\frac{1}{2}}) + E_1 + E_2, & \text{in } \Omega,\\ \boldsymbol{\sigma}(t^{\frac{1}{2}}) + a(u^{\frac{1}{2}})\frac{\nabla u^1 - \nabla u^0}{\Delta t} + b(u^{\frac{1}{2}})\nabla u^{\frac{1}{2}} & \\ \qquad\qquad = E_3 + E_4 + E_5 + E_6, & \text{in } \Omega,\\ u(t^1) = 0, & \text{on } \Gamma_D,\\ \boldsymbol{\sigma}(t^{\frac{1}{2}})\cdot\mathbf{n} = 0, & \text{on } \Gamma_N, \end{cases} \tag{4.12}$$

where $\boldsymbol{\sigma}(t^{\frac{1}{2}}) = \boldsymbol{\sigma}(\boldsymbol{x}, t^{\frac{1}{2}})$, $\bar{u}^1 = u(\bar{\boldsymbol{x}}, t^1)$, $\bar{\bar{u}}^0 = u(\bar{\bar{\boldsymbol{x}}}, t^0)$, $u^1 = u(\boldsymbol{x}, t^1)$, $u^0 = u(\boldsymbol{x}, t^0)$, $u^{\frac{1}{2}} = \frac{1}{2}(u^1 + u^0)$, $E_1 = c(\boldsymbol{x})\frac{\bar{u}^1 - \bar{\bar{u}}^0}{\Delta t} - \psi(\boldsymbol{x})\frac{\partial u}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, t^{\frac{1}{2}})$, $u(t^{\frac{1}{2}}) = u(\boldsymbol{x}, t^{\frac{1}{2}})$, $E_2 = f(u(t^{\frac{1}{2}})) - f(u^{\frac{1}{2}})$, $E_3 = (a(u^{\frac{1}{2}}) - a(u(t^{\frac{1}{2}})))\frac{\nabla u^1 - \nabla u^0}{\Delta t}$, $E_4 = a(u(t^{\frac{1}{2}}))(\frac{\nabla u^1 - \nabla u^0}{\Delta t} - \nabla u_t(t^{\frac{1}{2}}))$, $E_5 = (b(u^{\frac{1}{2}}) - b(u(t^{\frac{1}{2}})))\nabla u^{\frac{1}{2}}$, and $E_6 = b(u(t^{\frac{1}{2}}))(\nabla u^{\frac{1}{2}} - \nabla u(t^{\frac{1}{2}}))$.

Therefore we define $(u_h^1, \boldsymbol{\sigma}_h^{\frac{1}{2}}) \in V_h \times \boldsymbol{W}_h$ as follows:

$$\begin{aligned}
&(c(\boldsymbol{x})\frac{\bar{u}_h^1 - \bar{\bar{u}}_h^0}{\Delta t}, v_h) + (\nabla \cdot \boldsymbol{\sigma}_h^{\frac{1}{2}}, v_h) = (f(u_h^{\frac{1}{2}}), v_h), \quad v_h \in V_h, \\
&(\boldsymbol{\sigma}_h^{\frac{1}{2}}, \boldsymbol{\tau}_h) + (a(u_h^{\frac{1}{2}})\frac{\nabla u_h^1 - \nabla u_h^0}{\Delta t}, \boldsymbol{\tau}_h) + (b(u_h^{\frac{1}{2}})\nabla u_h^{\frac{1}{2}}, \boldsymbol{\tau}_h) = 0, \ \boldsymbol{\tau}_h \in \boldsymbol{W}_h, \\
&(u_h^0, v_h) = (u_0, v_h), \quad v_h \in V_h,
\end{aligned} \tag{4.13}$$

where $u_h^{\frac{1}{2}} = \frac{1}{2}(u_h^1 + u_h^0)$, $\bar{u}_h^1 = u_h^1(\bar{\boldsymbol{x}})$, and $\bar{\bar{u}}_h^0 = u_h^0(\bar{\bar{\boldsymbol{x}}})$.

As in [15, 16], we define a projection $\tilde{u}(x,t)$ of $u(x,t)$ onto V_h satisfying

$$\begin{cases} (a(u)\nabla(u-\tilde{u})_t, \nabla v_h) + (b(u)\nabla(u-\tilde{u}), \nabla v_h) = 0, \ \ \forall v_h \in V_h, \\ (\tilde{u}(0), v) = (u_0, v), \ \ \forall v_h \in V_h. \end{cases} \tag{4.14}$$

Then, by the assumption (A2), it is obvious that there exists unique projection $\tilde{u}(x,t) \in V_h$. Let $\eta = u - \tilde{u}$ and $\xi = u_h - \tilde{u}$ and state the estimates of η below. Hereafter a constant K denotes a generic positive constant depending on Ω and u, but independent of h and Δt, and also any two $K's$ in different places don't need to be the same.

Lemma 4.2 ([18])**.** *Let $u_0 \in H^s(\Omega)$, $u_t, u_{tt}, u_{ttt} \in H^s(\Omega)$, $u_t \in L^2(H^s(\Omega))$, and $s \geq 2$. If $u_t, u_{tt} \in L^\infty(\Omega \times [0,T])$, then there exists a constant K, independent of h, such that*

(**i**) $\|\eta\| + h\|\eta\|_1 \leq Kh^\mu(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s)$,

(**ii**) $\|\eta_t\| + h\|\eta_t\|_1 \leq Kh^\mu(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s)$,

(**iii**) $\|\eta_{tt}\|_1 \leq Kh^{\mu-1}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s + \|u_{tt}\|_s)$,

(**iv**) $\|\eta_{ttt}\|_1 \leq Kh^{\mu-1}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s + \|u_{tt}\|_s + \|u_{ttt}\|_s)$,

where $\mu = \min(r+1, s)$.

Proof. The results except (**iv**) are given in [18] and the proof of (**iv**) is similar to one of (**iii**). □

Lemma 4.3 ([18])**.** *Let $u_0 \in H^s(\Omega)$, $u, u_t, u_{tt}, u_{ttt} \in L^\infty(H^s(\Omega)) \cap L^\infty(W^{1,\infty}(\Omega))$, $u_t \in L^2(H^s(\Omega))$, and $s \geq 2$. If $\mu = \min(r+1, s) \geq 1 + \frac{m}{2}$, then the following statements hold:*

$$\max\{\|\eta\|_\infty, \ \|\nabla\eta\|_\infty, \ \|\nabla\eta_t\|_\infty, \ \|\nabla\eta_{tt}\|_\infty, \ \|\nabla\eta_{ttt}\|_\infty\} \leq \tilde{K}.$$

Proof. The results except $\|\nabla\eta_{ttt}\|_\infty$ are given in [18] and the proof of $\|\nabla\eta_{ttt}\|_\infty$ is similar to one of $\|\nabla\eta_{tt}\|_\infty$. □

Lemma 4.4. *If $u \in W^{1,\infty}(W^{3,\infty}(\Omega)) \cap W^{3,\infty}(W^{1,\infty}(\Omega))$, then*

$$\|Q_i^n\| \leq K(\Delta t)^2, i = 1,2,3,4$$

and

$$\|E_i\| \leq K(\Delta t)^2, i = 1,2,\cdots,6.$$

Proof. By applying Taylor's expansion, these results can be obtained. □

For $\boldsymbol{\sigma} = (\sigma_1, \sigma_2) \in \boldsymbol{W}$, let $\tilde{\boldsymbol{\sigma}} = (\tilde{\sigma}_1, \tilde{\sigma}_2)$ be a projection of $\boldsymbol{\sigma}$ onto $\boldsymbol{W}_h$ satisfying

$$(c(\boldsymbol{x})^{-1}\nabla \cdot (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}), \nabla \cdot \boldsymbol{\tau}) + \lambda(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}, \boldsymbol{\tau}) = 0, \;\; \forall \boldsymbol{\tau} \in \boldsymbol{W}_h, \tag{4.15}$$

where λ is a positive real number. The existence of $\tilde{\boldsymbol{\sigma}}$ can be obtained from the Lax-Milgram lemma.

Lemma 4.5 ([18])**.** *Let $\boldsymbol{\sigma} \in \boldsymbol{W} \cap \boldsymbol{H}^s(\Omega)$. Then there exists a constant $K > 0$ such that*

$$\|\nabla \cdot (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}})\| + \|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}\| \leq K h^{\mu - 1} \|\boldsymbol{\sigma}\|_s,$$

where $\mu = \min(k+1, s)$.

5. Error analysis for u_h^1 and $\boldsymbol{\sigma}_h^{\frac{1}{2}}$

In this section we will obtain the error estimates for u_h^1 and $\boldsymbol{\sigma}_h^{\frac{1}{2}}$ in (4.13) which will be used in the next section to obtain the error estimates for u_h^n and $\boldsymbol{\sigma}_h^n$.

Theorem 5.1. *Assume that the hypotheses of Lemma 4.2, Lemma 4.3, and Lemma 4.4 hold. If $\Delta t = O(h)$, then*

$$\|u^1 - u_h^1\|_l \leq K(h^{\mu - l} + (\Delta t)^2), \quad l = 0, 1,$$

where $\mu = \min(k+1, s)$.

Proof. Using $\eta = u - \tilde{u}$ and $\xi = u_h - \tilde{u}$, we get from (4.3) and (4.4)

$$\begin{aligned}
&(c(\boldsymbol{x})\frac{\bar{\eta}^1 - \bar{\xi}^1 - \bar{\bar{\eta}}^0 + \bar{\bar{\xi}}^0}{\Delta t}, v_h) + (\nabla \cdot (\boldsymbol{\sigma}(t^{\frac{1}{2}}) - \boldsymbol{\sigma}_h^{\frac{1}{2}}), v_h) \\
=&(f(u^{\frac{1}{2}}) - f(u_h^{\frac{1}{2}}), v_h) + (E_1 + E_2, v_h), \qquad v_h \in V_h,
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&(\boldsymbol{\sigma}(t^{\frac{1}{2}}) - \boldsymbol{\sigma}_h^{\frac{1}{2}}, \boldsymbol{\tau}_h) + (a(u_h^{\frac{1}{2}})\frac{\nabla\eta^1 - \nabla\xi^1 - \nabla\eta^0 + \nabla\xi^0}{\Delta t}, \boldsymbol{\tau}_h) \\
&\qquad + (b(u_h^{\frac{1}{2}})(\nabla\eta^{\frac{1}{2}} - \nabla\xi^{\frac{1}{2}}), \boldsymbol{\tau}_h) \\
=&((a(u_h^{\frac{1}{2}}) - a(u^{\frac{1}{2}}))\frac{\nabla u^1 - \nabla u^0}{\Delta t}, \boldsymbol{\tau}_h) + ((b(u_h^{\frac{1}{2}}) - b(u^{\frac{1}{2}}))\nabla u^{\frac{1}{2}}, \boldsymbol{\tau}_h) \\
&\qquad + (E_3 + E_4 + E_5 + E_6, \boldsymbol{\tau}_h), \qquad \boldsymbol{\tau}_h \in \boldsymbol{W}_h.
\end{aligned} \tag{5.2}$$

And so, we have

$$
\begin{aligned}
&(c(\boldsymbol{x})\frac{\xi^1-\xi^0}{\Delta t}, v_h) - (\nabla\cdot(\boldsymbol{\sigma}(t^{\frac{1}{2}}) - \boldsymbol{\sigma}_h^{\frac{1}{2}}), v_h)\\
=&(c(\boldsymbol{x})\frac{\bar{\eta}^1-\eta^1}{\Delta t}, v_h) + (c(\boldsymbol{x})\frac{\eta^1-\eta^0}{\Delta t}, v_h) + (c(\boldsymbol{x})\frac{\eta^0-\bar{\bar{\eta}}^0}{\Delta t}, v_h)\\
&+(c(\boldsymbol{x})\frac{\xi^1-\bar{\xi}^1}{\Delta t}, v_h) + (c(\boldsymbol{x})\frac{\bar{\bar{\xi}}^0-\xi^0}{\Delta t}, v_h)\\
&-(f(u^{\frac{1}{2}}) - f(u_h^{\frac{1}{2}}), v_h) - (E_1+E_2, v_h), \qquad v_h\in V_h.
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
&-(\boldsymbol{\sigma}(t^{\frac{1}{2}}) - \boldsymbol{\sigma}_h^{\frac{1}{2}}, \boldsymbol{\tau}_h) + (a(u_h^{\frac{1}{2}})\frac{\nabla\xi^1-\nabla\xi^0}{\Delta t}, \boldsymbol{\tau}_h) + (b(u_h^{\frac{1}{2}})\nabla\xi^{\frac{1}{2}}, \boldsymbol{\tau}_h)\\
=&(a(u_h^{\frac{1}{2}})\frac{\nabla\eta^1-\nabla\eta^0}{\Delta t}, \boldsymbol{\tau}_h) + (b(u_h^{\frac{1}{2}})\nabla\eta^{\frac{1}{2}}, \boldsymbol{\tau}_h)\\
&-((a(u_h^{\frac{1}{2}}) - a(u^{\frac{1}{2}}))\frac{\nabla u^1-\nabla u^0}{\Delta t}, \boldsymbol{\tau}_h) - ((b(u_h^{\frac{1}{2}}) - b(u^{\frac{1}{2}}))\nabla u^{\frac{1}{2}}, \boldsymbol{\tau}_h)\\
&\qquad -(E_3+E_4+E_5+E_6, \boldsymbol{\tau}_h), \qquad \boldsymbol{\tau}_h\in \boldsymbol{W}_h.
\end{aligned}
\tag{5.4}
$$

Letting $v_h = \xi^1$ in (5.3) and $\boldsymbol{\tau}_h = \nabla\xi^1$ in (5.4) and adding both sides together, we get

$$
\begin{aligned}
&(c(\boldsymbol{x})\frac{\xi^1-\xi_0}{\Delta t}, \xi^1) + (a(u_h^{\frac{1}{2}})\frac{\nabla\xi^1-\nabla\xi_0}{\Delta t}, \nabla\xi^1) + (b(u_h^{\frac{1}{2}})\nabla\xi^{\frac{1}{2}}, \nabla\xi^1)\\
=&(c(\boldsymbol{x})\frac{\bar{\eta}^1-\eta^1}{\Delta t}, \xi^1) + (c(\boldsymbol{x})\frac{\eta^1-\eta^0}{\Delta t}, \xi^1) + (c(\boldsymbol{x})\frac{\eta^0-\bar{\bar{\eta}}^0}{\Delta t}, \xi^1)\\
&+(c(\boldsymbol{x})\frac{\xi^1-\bar{\xi}^1}{\Delta t}, \xi^1) + (c(\boldsymbol{x})\frac{\bar{\bar{\xi}}^0-\xi^0}{\Delta t}, \xi^1) - (f(u^{\frac{1}{2}}) - f(u_h^{\frac{1}{2}}), \xi^1)\\
&-(E_1, \xi^1) - (E_2, \xi^1) + (a(u_h^{\frac{1}{2}})\frac{\nabla\eta^1-\nabla\eta^0}{\Delta t}, \nabla\xi^1) + (b(u_h^{\frac{1}{2}})\nabla\eta^{\frac{1}{2}}, \nabla\xi^1)\\
&-((a(u_h^{\frac{1}{2}}) - a(u^{\frac{1}{2}}))\frac{\nabla u^1-\nabla u^0}{\Delta t}, \nabla\xi^1) - ((b(u_h^{\frac{1}{2}}) - b(u^{\frac{1}{2}}))\nabla u^{\frac{1}{2}}, \nabla\xi^1)\\
&-(E_3, \nabla\xi^1) - (E_4, \nabla\xi^1) - (E_5, \nabla\xi^1) - (E_6, \nabla\xi^1)\\
=&\sum_{i=1}^{16} R_i.
\end{aligned}
$$

Since $\xi^0 \equiv 0$, we have

$$
c_*\|\xi^1\|^2 + a_*\|\nabla\xi^1\|^2 + \frac{b_*}{2}\Delta t\|\nabla\xi^1\|^2 \le \Delta t\sum_{i=1}^{16} R_i. \tag{5.5}
$$

For $R_1 \sim R_8$ and $R_{13} \sim R_{16}$, we obtain the following bounds:

$$\begin{aligned}
R_1 &\le K(\|\eta^1\|^2 + \|\xi^1\|^2 + \|\nabla\xi^1\|^2),\\
R_2 &\le K(\|\eta_t^1\|^2 + \|\xi^1\|^2),\\
R_3 &\le K(\|\eta^0\|^2 + \|\xi^1\|^2 + \|\nabla\xi^1\|^2),\\
R_4 &\le K(\|\nabla\xi^1\|^2 + \|\xi^1\|^2),\\
R_5 &= 0,\\
R_6 &\le K(\|\xi^1\|^2 + \|\eta^1\|^2 + \|\eta^0\|^2),\\
R_7 &\le K((\Delta t)^4 + \|\xi^1\|^2),\\
R_8 &\le K((\Delta t)^4 + \|\xi^1\|^2),\\
R_{13} &\le K((\Delta t)^4 + \|\nabla\xi^1\|^2),\\
R_{14} &\le K((\Delta t)^4 + \|\nabla\xi^1\|^2),\\
R_{15} &\le K((\Delta t)^4 + \|\nabla\xi^1\|^2),\\
R_{16} &\le K((\Delta t)^4 + \|\nabla\xi^1\|^2).
\end{aligned}$$

For $R_9 + R_{10}$, we can split into four terms:

$$\begin{aligned}
R_9 + R_{10} = &((a(u_h^{\frac{1}{2}}) - a(u(t^{\frac{1}{2}})))\frac{\nabla\eta^1 - \nabla\eta^0}{\Delta t}, \nabla\xi^1)\\
&+ (a(u(t^{\frac{1}{2}}))(\frac{\nabla\eta^1 - \nabla\eta^0}{\Delta t} - \nabla\eta_t(t^{\frac{1}{2}})), \nabla\xi^1)\\
&+ (b(u(t^{\frac{1}{2}}))(\nabla\eta^{\frac{1}{2}} - \nabla\eta(t^{\frac{1}{2}})), \nabla\xi^1)\\
&+ ((b(u_h^{\frac{1}{2}}) - b(u(t^{\frac{1}{2}})))\nabla\eta^{\frac{1}{2}}, \nabla\xi^1)\\
= &J_1 + J_2 + J_3 + J_4
\end{aligned}$$

For $J_1 \sim J_4$, we obtain the following bounds:

$$\begin{aligned}
J_1 &\le K\|\nabla\partial_t\eta^1\|_\infty(\|\xi^{\frac{1}{2}}\| + \|\eta^{\frac{1}{2}}\| + (\Delta t)^2)\|\nabla\xi^1\|\\
&\le K(\|\xi^1\|^2 + \|\nabla\xi^1\|^2 + \|\eta^1\|^2 + \|\eta^0\|^2 + (\Delta t)^4),\\
J_2 &\le K((\Delta t)^4 + \|\nabla\xi^1\|^2),\\
J_3 &\le K((\Delta t)^4 + \|\nabla\xi^1\|^2),\\
J_4 &\le K\|\nabla\eta^{\frac{1}{2}}\|_\infty(\|\xi^{\frac{1}{2}}\| + \|\eta^{\frac{1}{2}}\| + (\Delta t)^2)\|\nabla\xi^1\|\\
&\le K(\|\xi^1\|^2 + \|\nabla\xi^1\|^2 + \|\eta^1\|^2 + \|\eta^0\|^2 + (\Delta t)^4),
\end{aligned}$$

Therefore we have

$$R_9 + R_{10} \le K(\|\xi^1\|^2 + \|\nabla\xi^1\|^2 + \|\eta^1\|^2 + \|\eta^0\|^2 + (\Delta t)^4).$$

Finally, for R_{11} and R_{12}, we obtain the following bounds:

$$R_{11} \leq K(\|\xi^1\|^2 + \|\nabla\xi^1\|^2 + \|\eta^1\|^2 + \|\eta^0\|^2 + (\Delta t)^4),$$
$$R_{12} \leq K(\|\xi^1\|^2 + \|\nabla\xi^1\|^2 + \|\eta^1\|^2 + \|\eta^0\|^2 + (\Delta t)^4).$$

Thus, by using these bounds for $R_1 \sim R_{16}$, we obtain from (5.5)

$$\begin{aligned} &c_*\|\xi^1\|^2 + a_*\|\nabla\xi^1\|^2 + \frac{b_*}{2}\Delta t\|\nabla\xi^1\|^2 \\ &\leq K\Delta t(\|\xi^1\|^2 + \|\nabla\xi^1\|^2 + \|\eta_t^1\|^2 + \|\eta^1\|^2 + \|\eta^0\|^2 + (\Delta t)^4). \end{aligned}$$

Taking Δt sufficiently small and using Lemma 4.1, we get

$$\begin{aligned} \|\xi^1\|^2 + \|\nabla\xi^1\|^2 + \Delta t\|\nabla\xi^1\|^2 &\leq K\Delta t(\|\eta_t^1\|^2 + \|\eta^1\|^2 + \|\eta^0\|^2 + (\Delta t)^4) \\ &\leq K\Delta t(h^{2\mu} + (\Delta t)^4). \end{aligned} \tag{5.6}$$

Thus, by triangular inequality and Lemma 4.1, we obtain the result of this theorem. □

Theorem 5.2. *Assume that the hypotheses of Lemma 4.2, Lemma 4.3, and Lemma 4.4 hold. If $\Delta t = O(h)$, then*

$$\|\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\| + \|\nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})\| \leq K[h^{\mu-1} + (\Delta t)^2],$$

where $\mu = \min(k+1, s)$.

Proof. From (4.12) and (4.13), we get

$$\begin{aligned} (\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\sigma}^{\frac{1}{2}}, \boldsymbol{\tau}_h) =& -(a(u_h^{\frac{1}{2}})\frac{\nabla u_h^1 - \nabla u_h^0}{\Delta t} - a(u^{\frac{1}{2}})\frac{\nabla u^1 - \nabla u^0}{\Delta t}, \boldsymbol{\tau}_h) \\ &- (b(u_h^{\frac{1}{2}})\nabla u_h^{\frac{1}{2}} - b(u^{\frac{1}{2}})\nabla u^{\frac{1}{2}}, \boldsymbol{\tau}_h) - (E_3 + E_4 + E_5 + E_6, \boldsymbol{\tau}_h) \end{aligned}$$

and

$$\begin{aligned} (\nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\sigma}^{\frac{1}{2}}), v_h) =& -(c(\boldsymbol{x})(\frac{\bar{u}_h^1 - \bar{\bar{u}}_h^0}{\Delta t} - \frac{\bar{u}^1 - \bar{\bar{u}}^0}{\Delta t}), v_h) \\ &- (f(u_h^{\frac{1}{2}}) - f(u^{\frac{1}{2}}), v_h) - (E_1 + E_2, v_h). \end{aligned}$$

So, we have

$$\begin{aligned}
&\|\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\|^2 + \|\nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})\|^2 \\
=&(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}, \boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}) + (\nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}), \nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})) \\
=&(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\sigma}^{\frac{1}{2}}, \boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}) + (\boldsymbol{\sigma}^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}, \boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}) \\
&+ (\nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\sigma}^{\frac{1}{2}}), \nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})) \\
&+ (\nabla\cdot(\boldsymbol{\sigma}^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}), \nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})) \\
=&-(a(u_h^{\frac{1}{2}})\frac{\nabla u_h^1 - \nabla u_h^0}{\Delta t} - a(u^{\frac{1}{2}})\frac{\nabla u^1 - \nabla u^0}{\Delta t}, \boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}) \\
&- (b(u_h^{\frac{1}{2}})\nabla u_h^{\frac{1}{2}} - b(u^{\frac{1}{2}})\nabla u^{\frac{1}{2}}, \boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}) \\
&- (E_3 + E_4 + E_5 + E_6, \boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}) + (\boldsymbol{\sigma}^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}, \boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}) \\
&- (c(\boldsymbol{x})(\frac{\bar{u}_h^1 - \bar{\bar{u}}_h^0}{\Delta t} - \frac{\bar{u}^1 - \bar{\bar{u}}^0}{\Delta t}), \nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})) \\
&- (f(u_h^{\frac{1}{2}}) - f(u^{\frac{1}{2}}), \nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})) - (E_1 + E_2, \nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})) \\
&+ (\nabla\cdot(\boldsymbol{\sigma}^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}), \nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})) \\
=&\sum_{i=1}^{8} S_i.
\end{aligned}$$

Notice that

$$\|\nabla u_t\|_\infty \le K, \qquad \|\nabla u^{\frac{1}{2}}\|_\infty \le K,$$

and

$$\|\nabla u_h^{\frac{1}{2}}\|_\infty \le \|\nabla u^{\frac{1}{2}}\|_\infty + \|\nabla\eta^{\frac{1}{2}}\|_\infty + Kh^{-1}\|\nabla\xi^1\| \le K.$$

For S_1 and S_2, we obtain the following bounds:

$$\begin{aligned}
S_1 =&\frac{1}{\Delta t}((a(u_h^{\frac{1}{2}})((u_h^1 - u^1) - (u_h^0 - u^0)), \nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})) \\
&- \frac{1}{\Delta t}(a'(u_h^{\frac{1}{2}})\nabla u_h^{\frac{1}{2}}((u_h^1 - u^1) - (u_h^0 - u^0)), \boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}) \\
&- ((a(u_h^{\frac{1}{2}}) - a(u^{\frac{1}{2}}))\frac{\nabla u^1 - \nabla u^0}{\Delta t}, \boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}) \\
\le& K[\frac{1}{\Delta t}(\|u_h^1 - u^1\| + \|u_h^0 - u^0\|)\|\nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})\| \\
&+ \frac{1}{\Delta t}(\|u_h^1 - u^1\| + \|u_h^0 - u^0\|)\ \|\boldsymbol{\sigma}_h^{\frac{1}{2}} - \mathbf{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\|],
\end{aligned}$$

$$
\begin{aligned}
S_2 =& -(b(u_h^{\frac{1}{2}})\nabla(u_h^{\frac{1}{2}} - u^{\frac{1}{2}}), \boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}) - ((b(u_h^{\frac{1}{2}}) \\
&- b(u^{\frac{1}{2}}))\nabla u^{\frac{1}{2}}, \boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}) \\
\leq& K[(\|\nabla(u_h^1 - u^1)\| + \|\nabla(u_h^0 - u^0)\|)\|\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\| \\
&\quad + (\|u_h^1 - u^1\| + \|u_h^0 - u^0\|)\ \|\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\|].
\end{aligned}
$$

And for $S_3 \sim S_8$, we obtain the following bounds:

$$
\begin{aligned}
&S_3 \leq K(\Delta t)^2\|\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\|, \\
&S_4 \leq K\|\boldsymbol{\sigma}^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\|\ \|\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\|, \\
&S_5 \leq \frac{K}{\Delta t}(\|u_h^1 - u^1\| + \|u_h^0 - u^0\|)\|\nabla\cdot\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\|, \\
&S_6 \leq K(\|u_h^1 - u^1\| + \|u_h^0 - u^0\|)\|\nabla\cdot\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\|, \\
&S_7 \leq K(\Delta t)^2\|\nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})\|, \\
&S_8 \leq K\|\nabla\cdot(\boldsymbol{\sigma}^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})\|\ \|\nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})\|.
\end{aligned}
$$

Using all bounds for $S_1 \sim S_8$, (3.2), and Theorem 5.1, we have

$$
\begin{aligned}
&\|\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\| + \|\nabla\cdot(\boldsymbol{\sigma}_h^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})\| \\
&\leq K[\frac{1}{\Delta t}(\|u_h^1 - u^1\| + \|u_h^0 - u^0\|) \\
&+ (\|u_h^1 - u^1\| + \|u_h^0 - u^0\|) + (\|\nabla(u_h^1 - u^1)\| + \|\nabla(u_h^0 - u^0)\|) \\
&+ \|\boldsymbol{\sigma}^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}}\| + \|\nabla\cdot(\boldsymbol{\sigma}^{\frac{1}{2}} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}^{\frac{1}{2}})\| + (\Delta t)^2] \\
&\leq K[\frac{1}{\Delta t}h^{\mu} + h^{\mu-1} + (\Delta t)^2]
\end{aligned}
$$

and so, by triangular inequality and (3.2), we obtain the result of this theorem. □

6. Error analysis for u_h^n and $\boldsymbol{\sigma}_h^n$

In this section, we will obtain the error estimates for u_h^n and $\boldsymbol{\sigma}_h^n$ which are given in (4.10) and (4.11) as the split least-squares characteristic mixed element scheme.

Theorem 6.1. *Assume that the hypotheses of Lemma 4.2, Lemma 4.3, and Lemma 4.4 hold. If $\Delta t = O(h)$, then*

$$\|u^n - u_h^n\|_l \leq K(h^{\mu-l} + (\Delta t)^2), \quad \textit{for } n \geq 2 \textit{andl} = 0, 1$$

where $\mu = \min(k+1, s)$.

Proof. From (4.4), we get

$$
\begin{aligned}
&(c(\boldsymbol{x})u^n, v) + (A(Eu^n)\nabla u^n, \nabla v) \\
=&\frac{4}{3}(c(\boldsymbol{x})\check{u}^{n-1}, v) - \frac{1}{3}(c(\boldsymbol{x})\hat{u}^{n-2}, v) + \frac{2}{3}\Delta t\ (f(Eu^n) + Q_1^n + Q_2^n, v) \\
&+ \frac{4}{3}(a(Eu^n)\nabla u^{n-1}, \nabla v) - \frac{1}{3}(a(Eu^n)\nabla u^{n-2}, \nabla v) \\
&+ \frac{2}{3}\Delta t(Q_3^n + Q_4^n, \nabla v)
\end{aligned}
\tag{6.1}
$$

for any $(v, \boldsymbol{\tau}) \in V \times \boldsymbol{W}$. So, from (4.10) and (4.13), we get

$$
\begin{aligned}
&(c(\boldsymbol{x})(u^n - u_h^n), v_h) + (A(Eu^n)\nabla u^n - A(Eu_h^n)\nabla u_h^n, \nabla v_h) \\
=&\frac{4}{3}(c(\boldsymbol{x})(\check{u}^{n-1} - \check{u}_h^{n-1}), v_h) - \frac{1}{3}(c(\boldsymbol{x})(\hat{u}^{n-2} - \hat{u}_h^{n-2}), v_h) \\
&+ \frac{2}{3}\Delta t\ (f(Eu^n) - f(Eu_h^n), v_h) + \frac{2}{3}\Delta t\ (Q_1^n + Q_2^n, v_h) \\
&+ \frac{4}{3}(a(Eu^n)\nabla u^{n-1} - a(Eu_h^n)\nabla u_h^{n-1}, \nabla v_h) \\
&- \frac{1}{3}(a(Eu^n)\nabla u^{n-2} - a(Eu_h^n)\nabla u_h^{n-2}, \nabla v_h) \\
&+ \frac{2}{3}\Delta t\ (Q_3^n + Q_4^n, \nabla v_h).
\end{aligned}
\tag{6.2}
$$

Since $A(Eu^n) = a(Eu^n) + \frac{2}{3}\Delta t\ b(Eu^n)$, $A(Eu_h^n) = a(Eu_h^n) + \frac{2}{3}\Delta t\ b(Eu_h^n)$, and $-\nabla u_h^n = -\nabla u^n + \nabla\eta^n - \nabla\xi^n$, we have

$$
\begin{aligned}
&(c(\boldsymbol{x})(u^n - u_h^n), v_h) + (A(Eu^n)\nabla u^n - A(Eu_h^n)\nabla u_h^n, \nabla v_h) \\
=&(c(\boldsymbol{x})(\eta^n - \xi^n), v_h) + ((a(Eu^n) - a(Eu_h^n))\nabla u^n, \nabla v_h) \\
&+ (a(Eu_h^n)(\nabla\eta^n - \nabla\xi^n), \nabla v_h) + \frac{2}{3}\Delta t\ ((b(Eu^n) - b(Eu_h^n))\nabla u^n, \nabla v_h) \\
&+ \frac{2}{3}\Delta t\ (b(Eu_h^n)(\nabla\eta^n - \nabla\xi^n), \nabla v_h)
\end{aligned}
$$

and from (6.2), we get

$$
\begin{aligned}
&(c(\boldsymbol{x})(\eta^n-\xi^n),v_h)+((a(Eu^n)-a(Eu_h^n))\nabla u^n,\nabla v_h)\\
&+(a(Eu_h^n)(\nabla\eta^n-\nabla\xi^n),\nabla v_h)\\
&+\frac{2}{3}\Delta t\ ((b(Eu^n)-b(Eu_h^n))\nabla u^n,\nabla v_h)\\
&+\frac{2}{3}\Delta t\ (b(Eu_h^n)(\nabla\eta^n-\nabla\xi^n),\nabla v_h)\\
=&\frac{4}{3}(c(\boldsymbol{x})(\check{\eta}^{n-1}-\check{\xi}^{n-1}),v_h)-\frac{1}{3}(c(\boldsymbol{x})(\hat{\eta}^{n-2}-\hat{\xi}^{n-2}),v_h)\\
&+\frac{2}{3}\Delta t\ (f(Eu^n)-f(Eu_h^n),v_h)+\frac{2}{3}\Delta t\ (Q_1^n+Q_2^n,v_h)\\
&+\frac{4}{3}(a(Eu^n)\nabla u^{n-1}-a(Eu_h^n)\nabla u_h^{n-1},\nabla v_h)\\
&-\frac{1}{3}(a(Eu^n)\nabla u^{n-2}-a(Eu_h^n)\nabla u_h^{n-2},\nabla v_h)+\frac{2}{3}\Delta t\ (Q_3^n+Q_4^n,\nabla v_h).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&(c(\boldsymbol{x})(\xi^n-\xi^{n-1}),v_h)+(a(Eu_h^n)(\nabla\xi^n-\nabla\xi^{n-1}),\nabla v_h)\\
&\quad+\frac{2}{3}\Delta t\ (b(Eu_h^n)\nabla\xi^n,\nabla v_h)\\
=&(c(\boldsymbol{x})(\eta^n-\eta^{n-1}),v_h)+\frac{4}{3}(c(\boldsymbol{x})(\eta^{n-1}-\check{\eta}^{n-1}),v_h)\\
&-\frac{1}{3}(c(\boldsymbol{x})(\eta^{n-1}-\eta^{n-2}),v_h)-\frac{1}{3}(c(\boldsymbol{x})(\eta^{n-2}-\hat{\eta}^{n-2}),v_h)\\
&+\frac{4}{3}(c(\boldsymbol{x})(\check{\xi}^{n-1}-\xi^{n-1}),v_h)+\frac{1}{3}(c(\boldsymbol{x})(\xi^{n-2}-\hat{\xi}^{n-2}),v_h)\\
&+\frac{1}{3}(c(\boldsymbol{x})(\xi^{n-1}-\xi^{n-2}),v_h)\\
&+((a(Eu^n)-a(Eu_h^n))(\nabla u^n-\nabla u^{n-1}),\nabla v_h)\\
&-\frac{1}{3}((a(Eu^n)-a(Eu_h^n))(\nabla u^{n-1}-\nabla u^{n-2}),\nabla v_h)\\
&+\frac{1}{3}(a(Eu_h^n)(\nabla\xi^{n-1}-\nabla\xi^{n-2}),\nabla v_h)\\
&+((a(Eu_h^n)-a(u^n))(\nabla\eta^n-\frac{4}{3}\nabla\eta^{n-1}+\frac{1}{3}\nabla\eta^{n-2}),\nabla v_h)\\
&+(a(u^n)(\nabla\eta^n-\frac{4}{3}\nabla\eta^{n-1}+\frac{1}{3}\nabla\eta^{n-2}-\frac{2}{3}\Delta t\ \nabla\eta_t^n),\nabla v_h)\\
&-\frac{2}{3}\Delta t\ (b(u^n)\nabla\eta^n,\nabla v_h)+\frac{2}{3}\Delta t\ (b(Eu_h^n)\nabla\eta^n,\nabla v_h)\\
&+\frac{2}{3}\Delta t\ ((b(Eu^n)-b(Eu_h^n))\nabla u^n,\nabla v_h)-\frac{2}{3}\Delta t\ (f(Eu^n)-f(Eu_h^n),v_h)\\
&-\frac{2}{3}\Delta t\ (Q_1^n+Q_2^n,v_h)-\frac{2}{3}\Delta t\ (Q_3^n+Q_4^n,\nabla v_h).
\end{aligned}
\tag{6.3}
$$

Letting $v_h = \partial_t \xi^n$ in (6.3), we have

$$\begin{aligned}
&(c(\boldsymbol{x})[(\xi^n - \xi^{n-1}) - \frac{1}{3}(\xi^{n-1} - \xi^{n-2})], \partial_t \xi^n) \\
&+ (a(Eu_h^n)[(\nabla\xi^n - \nabla\xi^{n-1}) - \frac{1}{3}(\nabla\xi^{n-1} - \nabla\xi^{n-2})], \nabla\partial_t\xi^n) \\
&+ \frac{2}{3}\Delta t\ (b(Eu_h^n)\nabla\xi^n, \nabla\partial_t\xi^n) \\
=&(c(\boldsymbol{x})(\eta^n - \eta^{n-1}), \partial_t\xi^n) + \frac{4}{3}(c(\boldsymbol{x})(\eta^{n-1} - \check{\eta}^{n-1}), \partial_t\xi^n) \\
&- \frac{1}{3}(c(\boldsymbol{x})(\eta^{n-1} - \eta^{n-2}), \partial_t\xi^n) - \frac{1}{3}(c(\boldsymbol{x})(\eta^{n-2} - \hat{\eta}^{n-2}), \partial_t\xi^n) \\
&+ \frac{4}{3}(c(\boldsymbol{x})(\check{\xi}^{n-1} - \xi^{n-1}), \partial_t\xi^n) + \frac{1}{3}(c(\boldsymbol{x})(\xi^{n-2} - \hat{\xi}^{n-2}), \partial_t\xi^n) \\
&+ ((a(Eu^n) - a(Eu_h^n))(\nabla u^n - \nabla u^{n-1}), \nabla\partial_t\xi^n) \qquad (6.4) \\
&- \frac{1}{3}((a(Eu^n) - a(Eu_h^n))(\nabla u^{n-1} - \nabla u^{n-2}), \nabla\partial_t\xi^n) \\
&+ ((a(Eu_h^n) - a(u^n))(\nabla\eta^n - \frac{4}{3}\nabla\eta^{n-1} + \frac{1}{3}\nabla\eta^{n-2}), \nabla\partial_t\xi^n) \\
&+ (a(u^n)(\nabla\eta^n - \frac{4}{3}\nabla\eta^{n-1} + \frac{1}{3}\nabla\eta^{n-2} - \frac{2}{3}\Delta t\ \nabla\eta_t^n), \nabla\partial_t\xi^n) \\
&- \frac{2}{3}\Delta t\ (b(u^n)\nabla\eta^n, \nabla\partial_t\xi^n) + \frac{2}{3}\Delta t\ (b(Eu_h^n)\nabla\eta^n, \nabla\partial_t\xi^n) \\
&+ \frac{2}{3}\Delta t\ ((b(Eu^n) - b(Eu_h^n))\nabla u^n, \nabla\partial_t\xi^n) - \frac{2}{3}\Delta t\ (Q_1^n + Q_2^n, \partial_t\xi^n) \\
&- \frac{2}{3}\Delta t\ (f(Eu^n) - f(Eu_h^n), \partial_t\xi^n) - \frac{2}{3}\Delta t\ (Q_3^n + Q_4^n, \nabla\partial_t\xi^n).
\end{aligned}$$

Defining $Eu_h^1 = 0$, we obtain lower bounds for terms in the left-hand side in (6.4) as follows:

$$\begin{aligned}
L_1 \geq& \frac{\Delta t}{6}\left[\|\sqrt{c(\boldsymbol{x})}\partial_t\xi^n\|^2 - \|\sqrt{c(\boldsymbol{x})}\partial_t\xi^{n-1}\|^2\right] + \frac{2c_*\Delta t}{3}\ \|\partial_t\xi^n\|^2, \\
L_2 \geq& \frac{\Delta t}{6}\Big(\|\sqrt{a(Eu_h^n)}\nabla\partial_t\xi^n\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla\partial_t\xi^{n-1}\|^2\Big) \\
&+ \frac{\Delta t}{6}\Big(\|\sqrt{a(Eu_h^{n-1})}\nabla\partial_t\xi^{n-1}\|^2 - \|\sqrt{a(Eu_h^n)}\nabla\partial_t\xi^{n-1}\|^2\Big) \\
&+ \frac{2a_*\Delta t}{3}\ \|\nabla\partial_t\xi^n\|^2, \\
L_3 \geq& \frac{1}{3}\Big(\|\sqrt{b(Eu_h^n)}\nabla\xi^n\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla\xi^{n-1}\|^2\Big) \\
&\qquad + \frac{1}{3}\Big(\|\sqrt{b(Eu_h^{n-1})}\nabla\xi^{n-1}\|^2 - \|\sqrt{b(Eu_h^n)}\nabla\xi^{n-1}\|^2\Big).
\end{aligned}$$

Thus, from (6.4), we have

$$
\begin{aligned}
&\frac{\Delta t}{6}\left(\|\sqrt{c(\boldsymbol{x})}\partial_t\xi^n\|^2-\|\sqrt{c(\boldsymbol{x})}\partial_t\xi^{n-1}\|^2\right)+\frac{2c_*\Delta t}{3}\ \|\partial_t\xi^n\|^2\\
&+\frac{\Delta t}{6}\Big(\|\sqrt{a(Eu_h^n)}\nabla\partial_t\xi^n\|^2-\|\sqrt{a(Eu_h^{n-1})}\nabla\partial_t\xi^{n-1}\|^2\Big)\\
&+\frac{2a_*\Delta t}{3}\ \|\nabla\partial_t\xi^n\|^2+\frac{1}{3}\Big(\|\sqrt{b(Eu_h^n)}\nabla\xi^n\|^2-\|\sqrt{b(Eu_h^{n-1})}\nabla\xi^{n-1}\|^2\Big)\\
&\le\frac{\Delta t}{6}\Big(\|\sqrt{a(Eu_h^n)}\nabla\partial_t\xi^{n-1}\|^2-\|\sqrt{a(Eu_h^{n-1})}\nabla\partial_t\xi^{n-1}\|^2\Big)\\
&+\frac{1}{3}\Big(\|\sqrt{b(Eu_h^n)}\nabla\xi^{n-1}\|^2-\|\sqrt{b(Eu_h^{n-1})}\nabla\xi^{n-1}\|^2\Big)\\
&+(c(\boldsymbol{x})(\eta^n-\eta^{n-1}),\partial_t\xi^n)+\frac{4}{3}(c(\boldsymbol{x})(\eta^{n-1}-\check{\eta}^{n-1}),\partial_t\xi^n)\\
&-\frac{1}{3}(c(\boldsymbol{x})(\eta^{n-1}-\eta^{n-2}),\partial_t\xi^n)-\frac{1}{3}(c(\boldsymbol{x})(\eta^{n-2}-\hat{\eta}^{n-2}),\partial_t\xi^n)\\
&+\frac{4}{3}(c(\boldsymbol{x})(\check{\xi}^{n-1}-\xi^{n-1}),\partial_t\xi^n)+\frac{1}{3}(c(\boldsymbol{x})(\xi^{n-2}-\hat{\xi}^{n-2}),\partial_t\xi^n)\\
&+((a(Eu^n)-a(Eu_h^n))(\nabla u^n-\nabla u^{n-1}),\nabla\partial_t\xi^n)\\
&-\frac{1}{3}((a(Eu^n)-a(Eu_h^n))(\nabla u^{n-1}-\nabla u^{n-2}),\nabla\partial_t\xi^n)\\
&+((a(Eu_h^n)-a(u^n))(\nabla\eta^n-\frac{4}{3}\nabla\eta^{n-1}+\frac{1}{3}\nabla\eta^{n-2}),\nabla\partial_t\xi^n)\\
&+(a(u^n)(\nabla\eta^n-\frac{4}{3}\nabla\eta^{n-1}+\frac{1}{3}\nabla\eta^{n-2}-\frac{2}{3}\Delta t\ \nabla\eta_t^n),\nabla\partial_t\xi^n)\\
&-\frac{2}{3}\Delta t\ (b(u^n)\nabla\eta^n,\nabla\partial_t\xi^n)+\frac{2}{3}\Delta t\ (b(Eu_h^n)\nabla\eta^n,\nabla\partial_t\xi^n)\\
&+\frac{2}{3}\Delta t\ ((b(Eu^n)-b(Eu_h^n))\nabla u^n,\nabla\partial_t\xi^n)-\frac{2}{3}\Delta t\ (Q_1^n+Q_2^n,\partial_t\xi^n)\\
&-\frac{2}{3}\Delta t\ (f(Eu^n)-f(Eu_h^n),\partial_t\xi^n)-\frac{2}{3}\Delta t\ (Q_3^n+Q_4^n,\nabla\partial_t\xi^n).\\
&=\sum_{i=1}^{18}R_i.
\end{aligned}
\tag{6.5}
$$

Notice that

$$
\begin{aligned}
&|Eu_h^n-Eu_h^{n-1}|=\Delta t\ |2\partial_t\xi^{n-1}-\partial_t\xi^{n-2}+2\partial_t\tilde{u}^{n-1}-\partial_t\tilde{u}^{n-2}|,\\
&Eu^n-Eu_h^n=2\eta^{n-1}-\eta^{n-2}-2\xi^{n-1}+\xi^{n-2},\\
&\|\partial_t\tilde{u}^l\|_\infty\le\|\partial_t u^l\|_\infty+\|\partial_t\eta^l\|_\infty\le K,\qquad l=1,2,\cdots,n-1,
\end{aligned}
$$

and

$$
\|\partial_t\xi^l\|_\infty\le Kh^{-\frac{m}{2}}\|\partial_t\xi^l\|\le K,\qquad l=1,2,\cdots,n-1.
$$

Therefore we can estimate bounds for $R_1 \sim R_{18}$ as follow:

$$
\begin{aligned}
R_1 &\le K(\Delta t)^2\, \|\nabla\partial_t\xi^{n-1}\|^2,\\
R_2 &\le K\Delta t\, \|\nabla\xi^{n-1}\|^2,\\
R_3 &\le K\Delta t\, \|\eta_t^{n-1}\|^2 + \epsilon\Delta t\, \|\partial_t\xi^n\|^2,\\
R_4 &\le K\Delta t\, \|\eta_t^{n-1}\|(\|\partial_t\xi^n\| + \|\nabla\partial_t\xi^n\|)\\
&\le K\Delta t\, \|\eta_t^{n-1}\|^2 + \epsilon\Delta t\|\partial_t\xi^n\|^2 + \epsilon\Delta t\, \|\nabla\partial_t\xi^n\|^2,\\
R_5 &\le K\Delta t\, \|\eta_t^{n-1}\|\|\partial_t\xi^n\| \le K\Delta t\, \|\eta_t^{n-1}\|^2 + \epsilon\Delta t\|\partial_t\xi^n\|^2,\\
R_6 &\le K\Delta t\, \|\eta_t^{n-2}\|(\|\partial_t\xi^n\| + \|\nabla\partial_t\xi^n\|)\\
&\le K\Delta t\, \|\eta_t^{n-2}\|^2 + \epsilon\Delta t\|\partial_t\xi^n\|^2 + \epsilon\Delta t\, \|\nabla\partial_t\xi^n\|^2,\\
R_7 &\le K\Delta t\, \|\nabla\xi^{n-1}\|\|\partial_t\xi^n\| \le K\Delta t\, \|\nabla\xi^{n-1}\|^2 + \epsilon\Delta t\|\partial_t\xi^n\|^2,\\
R_8 &\le K\Delta t\, \|\nabla\xi^{n-2}\|\|\partial_t\xi^n\| \le K\Delta t\, \|\nabla\xi^{n-2}\|^2 + \epsilon\Delta t\|\partial_t\xi^n\|^2,\\
R_9 &\le K\Delta t\, \|\nabla u_t^{n-1}\|_\infty\|Eu^n - Eu_h^n\|\|\nabla\partial_t\xi^n\|\\
&\le K\Delta t\, (\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + \|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2) + \epsilon\Delta t\|\partial_t\xi^n\|^2,\\
R_{10} &\le K\Delta t\, \|\nabla u_t^{n-2}\|_\infty\|Eu^n - Eu_h^n\|\|\nabla\partial_t\xi^n\|\\
&\le K\Delta t\, (\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + \|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2) + \epsilon\Delta t\|\partial_t\xi^n\|^2,\\
R_{11} &\le K\Delta t\, (\|\nabla\partial_t\eta^n\| + \|\nabla\partial_t\eta^{n-1}\|)\|\nabla\partial_t\xi^n\|\\
&\le K\Delta t\, (\|\nabla\partial_t\eta^n\|^2 + \|\nabla\partial_t\eta^{n-1}\|^2) + \epsilon\Delta t\|\partial_t\xi^n\|^2,\\
R_{12} &\le K(\Delta t)^5\, \|\nabla\eta_{ttt}^n\|^2 + \epsilon\Delta t\|\nabla\partial_t\xi^n\|^2,\\
R_{13} + R_{14} &\le K\Delta t\, \|\nabla\eta^n\|\, \|\nabla\partial_t\xi^n\| \le K\Delta t\, \|\nabla\eta^n\|^2 + \epsilon\Delta t\|\nabla\partial_t\xi^n\|^2,\\
R_{15} &\le K\Delta t\, \|\nabla u^n\|_\infty\|Eu^n - Eu_h^n\|\|\nabla\partial_t\xi^n\|\\
&\le K\Delta t\, (\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + \|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2) + \epsilon\Delta t\|\nabla\partial_t\xi^n\|^2,\\
R_{16} &\le K\Delta t\, (\Delta t)^2\|\partial_t\xi^n\| \le K(\Delta t)^5\; + \epsilon\Delta t\|\partial_t\xi^n\|^2,\\
R_{17} &\le K\Delta t\, \|Eu^n - Eu_h^n\|\|\nabla\partial_t\xi^n\|\\
&\le K\Delta t\, (\|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + \|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2) + \epsilon\Delta t\|\partial_t\xi^n\|^2,\\
R_{18} &\le K\Delta t\, (\Delta t)^2\|\partial_t\xi^n\| \le K(\Delta t)^5\; + \epsilon\Delta t\|\nabla\partial_t\xi^n\|^2.
\end{aligned}
$$

Thus, using all bounds for $R_1 \sim R_{18}$, we obtain from (6.5)

$$
\begin{aligned}
&\frac{\Delta t}{6}\left(\|\sqrt{c(\boldsymbol{x})}\partial_t\xi^n\|^2 - \|\sqrt{c(\boldsymbol{x})}\partial_t\xi^{n-1}\|^2\right) + \frac{2c_*\Delta t}{3}\, \|\partial_t\xi^n\|^2\\
&+ \frac{\Delta t}{6}\Big(\|\sqrt{a(Eu_h^n)}\nabla\partial_t\xi^n\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla\partial_t\xi^{n-1}\|^2\Big) + \frac{2a_*\Delta t}{3}\, \|\nabla\partial_t\xi^n\|^2\\
&+ \frac{1}{3}\Big(\|\sqrt{b(Eu_h^n)}\nabla\xi^n\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla\xi^{n-1}\|^2\Big)
\end{aligned}
$$

$$\leq K\Delta t[\|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2 + \|\eta_t^{n-1}\|^2 + \|\eta_t^{n-2}\|^2 + \|\xi^{n-1}\|^2 + \|\xi^n\|^2 + \|\nabla\xi^{n-1}\|^2 + \|\nabla\xi^n\|^2 + (\Delta t)^4] + 15\epsilon\Delta t\,\|\nabla\partial_t\xi^n\|^2$$

and so, for sufficiently small $\epsilon > 0$, we have

$$\begin{aligned}
&\Delta t\,\Big(\|\sqrt{c(\boldsymbol{x})}\partial_t\xi^n\|^2 - \|\sqrt{c(\boldsymbol{x})}\partial_t\xi^{n-1}\|^2\Big) + \Delta t\,\|\partial_t\xi^n\|^2 \\
&+ \Delta t\Big(\|\sqrt{a(Eu_h^n)}\nabla\partial_t\xi^n\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla\partial_t\xi^{n-1}\|^2\Big) \\
&+ \Delta t\,\|\nabla\partial_t\xi^n\|^2 + \Big(\|\sqrt{b(Eu_h^n)}\nabla\xi^n\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla\xi^{n-1}\|^2\Big) \\
&\leq K\Delta t[\|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2 + \|\eta_t^{n-1}\|^2 + \|\eta_t^{n-2}\|^2 + \|\xi^{n-1}\|^2 \\
&\qquad + \|\xi^n\|^2 + \|\nabla\xi^{n-1}\|^2 + \|\nabla\xi^n\|^2 + (\Delta t)^4].
\end{aligned} \tag{6.6}$$

Now, summing both sides of (6.6) from $n = 2$ to k and using the assumptions on a and b, we get

$$\begin{aligned}
&\Delta t\Big(\|\partial_t\xi^k\|^2 + \|\nabla\partial_t\xi^k\|^2\Big) + \Delta t\sum_{n=2}^{k}\Big(\|\partial_t\xi^n\|^2 + \|\nabla\partial_t\xi^n\|^2\Big) + \|\nabla\xi^k\|^2 \\
&\leq K\Big[\Delta t\Big(\|\partial_t\xi^1\|^2 + \|\nabla\partial_t\xi^1\|^2 + \|\xi^1\|^2 + \|\nabla\xi^1\|^2\Big) \\
&\qquad + \Delta t\sum_{n=1}^{k}\Big(\|\eta^n\|^2 + \|\eta_t^n\|^2 + (\Delta t)^4\Big) \\
&\qquad + \Delta t\sum_{n=2}^{k}\Big(\|\xi^n\|^2 + \|\nabla\xi^n\|^2\Big)\Big] + K\|\nabla\xi^1\|^2.
\end{aligned} \tag{6.7}$$

Since $\|\xi^k\|^2 \leq K\|\nabla\xi^k\|^2$, by applying Gronwall's inequality, Lemma 4.2, and (5.6), we have

$$\|\xi^k\|^2 + \|\nabla\xi^k\|^2 \leq K[h^{2\mu} + (\Delta t)^2]. \tag{6.8}$$

Thus, by the triangular inequality and Lemma 4.2, we obtain the result of this theorem.

□

Remark 6.1. From Theorem 6.1, we know that the approximations u_h^n has higher order of convergence in a time increment which improves our previous result in [18].

For our error analysis, we let $\boldsymbol{\pi} = \boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}$ and $\boldsymbol{\rho} = \tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h$. Then $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h = \boldsymbol{\pi} + \boldsymbol{\rho}$.

Theorem 6.2. *Assume that the hypotheses of Lemma 4.2, Lemma 4.3, and Lemma 4.4 hold. Let $\boldsymbol{\sigma} \in \boldsymbol{W} \cap \boldsymbol{H}^s(\Omega)$. Then we have*

$$\|\nabla\cdot(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n)\| + \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\| \leq K(h^{\mu-1} + (\Delta t)^2), \qquad n \geq 2.$$

where $\mu = \min(k+1, s)$.

Proof. First, we will prove that

$$\|\nabla\cdot\boldsymbol{\rho}^n\| + \|\boldsymbol{\rho}^n\| \le K(h^{\mu-1} + (\Delta t)^2), \qquad n \ge 2.$$

By applying Lemma 4.1 to (4.8) with $v = 0$, we get

$$\begin{aligned}
&(\Delta t)^2\ (c(\boldsymbol{x})^{-1}\nabla\cdot\boldsymbol{\sigma^n},\ \nabla\cdot\boldsymbol{\tau}) + (\Delta t)^2\ (A(Eu^n)^{-1}\boldsymbol{\sigma^n},\ \boldsymbol{\tau})\\
=&\Big(c(\boldsymbol{x})^{-1}\big(2c(\boldsymbol{x})\check{u}^{n-1} - \frac{1}{2}c(\boldsymbol{x})\hat{u}^{n-2} + \Delta t(f(Eu^n) + Q_1^n + Q_2^n)\big),\ \Delta t\ \nabla\cdot\boldsymbol{\tau}\Big)\\
&+\Big(A(Eu^n)^{-1}\big(a(Eu^n)(2\nabla u^{n-1} - \frac{1}{2}\nabla u^{n-2}) + \Delta t(Q_3^n + Q_4^n)\big),\ \Delta t\boldsymbol{\tau}\Big)
\end{aligned}$$

and so, we get

$$\begin{aligned}
&(c(\boldsymbol{x})^{-1}\nabla\cdot\boldsymbol{\sigma^n}, \nabla\cdot\boldsymbol{\tau}) + (A(Eu^n)^{-1}\boldsymbol{\sigma^n}, \boldsymbol{\tau})\\
=&\frac{1}{\Delta t}\ \Big(2\check{u}^{n-1} - \frac{1}{2}\hat{u}^{n-2}, \nabla\cdot\boldsymbol{\tau}\Big) + \Big(c(\boldsymbol{x})^{-1}(f(Eu^n) + Q_1^n + Q_2^n), \nabla\cdot\boldsymbol{\tau}\Big)\\
&+\frac{1}{\Delta t}\ \Big(A(Eu^n)^{-1}a(Eu^n)(2\nabla u^{n-1} - \frac{1}{2}\nabla u^{n-2}), \boldsymbol{\tau}\Big)\\
&+\Big(A(Eu^n)^{-1}(Q_3^n + Q_4^n), \boldsymbol{\tau}\Big).
\end{aligned} \tag{6.9}$$

Since

$$\begin{aligned}
A(Eu^n)^{-1}a(Eu^n) &= A(Eu^n)^{-1}(A(Eu^n) - \frac{2}{3}\Delta t b(Eu^n)))\\
&= 1 - \frac{2}{3}\Delta t A(Eu^n)^{-1}b(Eu^n),
\end{aligned}$$

we have from (6.9)

$$\begin{aligned}
&(c(\boldsymbol{x})^{-1}\nabla\cdot\boldsymbol{\sigma^n}, \nabla\cdot\boldsymbol{\tau}) + (A(Eu^n)^{-1}\boldsymbol{\sigma^n}, \boldsymbol{\tau})\\
=&\frac{1}{\Delta t}\ \Big(2\nabla u^{n-1} - \frac{1}{2}\nabla u^{n-2}, \boldsymbol{\tau}\Big) + \frac{1}{\Delta t}\ \Big(2\check{u}^{n-1} - \frac{1}{2}\hat{u}^{n-2}, \nabla\cdot\boldsymbol{\tau}\Big)\\
&+\Big(c(\boldsymbol{x})^{-1}(f(Eu^n) + Q_1^n + Q_2^n), \nabla\cdot\boldsymbol{\tau}\Big)\\
&-\frac{2}{3}\ \Big(A(Eu^n)^{-1}b(Eu^n)(2\nabla u^{n-1} - \frac{1}{2}\nabla u^{n-2}), \boldsymbol{\tau}\Big)\\
&+\Big(A(Eu^n)^{-1}(Q_3^n + Q_4^n), \boldsymbol{\tau}\Big)\\
=&\frac{1}{\Delta t}\ \Big(2\check{u}^{n-1} - \frac{1}{2}\hat{u}^{n-2} - 2u^{n-1} + \frac{1}{2}u^{n-2}, \nabla\cdot\boldsymbol{\tau}\Big)\\
&+\Big(c(\boldsymbol{x})^{-1}(f(Eu^n) + Q_1^n + Q_2^n), \nabla\cdot\boldsymbol{\tau}\Big)\\
&-\frac{2}{3}\ \Big(A(Eu^n)^{-1}b(Eu^n)(2\nabla u^{n-1} - \frac{1}{2}\nabla u^{n-2}), \boldsymbol{\tau}\Big)\\
&+\Big(A(Eu^n)^{-1}(Q_3^n + Q_4^n), \boldsymbol{\tau}\Big).
\end{aligned} \tag{6.10}$$

Similarly, we have

$$
\begin{aligned}
&(c(\boldsymbol{x})^{-1}\nabla\cdot\boldsymbol{\sigma}_h^n,\nabla\cdot\boldsymbol{\tau}_h)+(A(Eu_h^n)^{-1}\boldsymbol{\sigma}_h^n,\boldsymbol{\tau}_h)\\
=&\frac{1}{\Delta t}\left(2\check{u}_h^{n-1}-\frac{1}{2}\hat{u}_h^{n-2}-2u_h^{n-1}+\frac{1}{2}u_h^{n-2},\nabla\cdot\boldsymbol{\tau}_h\right)\\
&+\left(c(\boldsymbol{x})^{-1}f(Eu_h^n),\nabla\cdot\boldsymbol{\tau}_h\right)\\
&-\frac{2}{3}\left(A(Eu_h^n)^{-1}b(Eu_h^n)(2\nabla u_h^{n-1}-\frac{1}{2}\nabla u_h^{n-2}),\boldsymbol{\tau}_h\right).
\end{aligned}
\tag{6.11}
$$

Therefore, from (6.10) and (6.11), we have

$$
\begin{aligned}
&(c(\boldsymbol{x})^{-1}\nabla\cdot(\boldsymbol{\sigma}^n-\boldsymbol{\sigma}_h^n),\nabla\cdot\boldsymbol{\tau}_h)+(A(Eu^n)^{-1}\boldsymbol{\sigma}^n-A(Eu_h^n)^{-1}\boldsymbol{\sigma}_h^n,\boldsymbol{\tau}_h)\\
=&\frac{1}{\Delta t}\left(2\check{u}^{n-1}-\frac{1}{2}\hat{u}^{n-2}-2u^{n-1}+\frac{1}{2}u^{n-2},\nabla\cdot\boldsymbol{\tau}_h\right)\\
&-\frac{1}{\Delta t}\left(2\check{u}_h^{n-1}-\frac{1}{2}\hat{u}_h^{n-2}-2u_h^{n-1}+\frac{1}{2}u_h^{n-2},\nabla\cdot\boldsymbol{\tau}_h\right)\\
&+\left(c(\boldsymbol{x})^{-1}(f(Eu^n)-f(Eu_h^n)),\nabla\cdot\boldsymbol{\tau}_h\right)\\
&+\left(c(\boldsymbol{x})^{-1}Q_1^n,\nabla\cdot\boldsymbol{\tau}_h\right)+\left(c(\boldsymbol{x})^{-1}Q_2^n,\nabla\cdot\boldsymbol{\tau}_h\right)\\
&-\frac{2}{3}\left(A(Eu^n)^{-1}b(Eu^n)(2\nabla u^{n-1}-\frac{1}{2}\nabla u^{n-2}),\boldsymbol{\tau}\right)\\
&+\frac{2}{3}\left(A(Eu_h^n)^{-1}b(Eu_h^n)(2\nabla u_h^{n-1}-\frac{1}{2}\nabla u_h^{n-2}),\boldsymbol{\tau}_h\right)\\
&+\left(A(Eu^n)^{-1}Q_3^n,\boldsymbol{\tau}_h\right)+\left(A(Eu^n)^{-1}Q_4^n,\boldsymbol{\tau}_h\right)
\end{aligned}
$$

and hence

$$
\begin{aligned}
&(c(\boldsymbol{x})^{-1}\nabla\cdot(\boldsymbol{\sigma}^n-\boldsymbol{\sigma}_h^n),\nabla\cdot\boldsymbol{\tau}_h)+(A(Eu_h^n)^{-1}(\boldsymbol{\sigma}^n-\boldsymbol{\sigma}_h^n),\boldsymbol{\tau}_h)\\
=&((A(Eu_h^n)^{-1}-A(Eu^n)^{-1})\boldsymbol{\sigma}^n,\boldsymbol{\tau}_h)\\
&-\frac{2}{\Delta t}\left(\eta^{n-1}-\check{\eta}^{n-1},\nabla\cdot\boldsymbol{\tau}_h\right)+\frac{1}{2\Delta t}\left(\eta^{n-2}-\hat{\eta}^{n-2},\nabla\cdot\boldsymbol{\tau}_h\right)\\
&+\frac{2}{\Delta t}\left(\xi^{n-1}-\check{\xi}^{n-1},\nabla\cdot\boldsymbol{\tau}_h\right)-\frac{1}{2\Delta t}\left(\xi^{n-2}-\hat{\xi}^{n-2},\nabla\cdot\boldsymbol{\tau}_h\right)\\
&+\left(c(\boldsymbol{x})^{-1}(f(Eu^n)-f(Eu_h^n)),\nabla\cdot\boldsymbol{\tau}_h\right)\\
&+\left(c(\boldsymbol{x})^{-1}Q_1^n,\nabla\cdot\boldsymbol{\tau}_h\right)+\left(c(\boldsymbol{x})^{-1}Q_2^n,\nabla\cdot\boldsymbol{\tau}_h\right)\\
&-\frac{4}{3}\left(A(Eu^n)^{-1}b(Eu^n)\nabla u^{n-1}-A(Eu_h^n)^{-1}b(Eu_h^n)\nabla u_h^{n-1},\boldsymbol{\tau}_h\right)\\
&+\frac{1}{3}\left(A(Eu^n)^{-1}b(Eu^n)\nabla u^{n-2}-A(Eu_h^n)^{-1}b(Eu_h^n)\nabla u_h^{n-2},\boldsymbol{\tau}_h\right)\\
&+\left(A(Eu^n)^{-1}Q_3^n,\boldsymbol{\tau}_h\right)+\left(A(Eu^n)^{-1}Q_4^n,\boldsymbol{\tau}_h\right).
\end{aligned}
\tag{6.12}
$$

Letting $\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n = \boldsymbol{\pi}^n - \boldsymbol{\rho}^n$ in (6.12) with $\boldsymbol{\tau}_h = \boldsymbol{\rho}^n$ and using (4.15), we obtain

$$
\begin{aligned}
&(c(\boldsymbol{x})^{-1}\nabla\cdot\boldsymbol{\rho}^n, \nabla\cdot\boldsymbol{\rho}^n) + (A(Eu_h^n)^{-1}\boldsymbol{\rho}^n, \boldsymbol{\rho}^n)\\
=&\lambda(\boldsymbol{\pi}^n, \boldsymbol{\rho}^n) - (A(Eu_h^n)^{-1}\boldsymbol{\pi}^n, \boldsymbol{\rho}^n)\\
&+ ((A(Eu_h^n)^{-1} - A(Eu^n)^{-1})\boldsymbol{\sigma}^n, \boldsymbol{\rho}^n)\\
&- \frac{2}{\Delta t}\left(\eta^{n-1} - \check{\eta}^{n-1}, \nabla\cdot\boldsymbol{\rho}^n\right) + \frac{1}{2\Delta t}\left(\eta^{n-2} - \hat{\eta}^{n-2}, \nabla\cdot\boldsymbol{\rho}^n\right)\\
&+ \frac{2}{\Delta t}\left(\xi^{n-1} - \check{\xi}^{n-1}, \nabla\cdot\boldsymbol{\rho}^n\right) - \frac{1}{2\Delta t}\left(\xi^{n-2} - \hat{\xi}^{n-2}, \nabla\cdot\boldsymbol{\rho}^n\right)\\
&+ \left(c(\boldsymbol{x})^{-1}(f(Eu^n) - f(Eu_h^n)), \nabla\cdot\boldsymbol{\rho}^n\right)\\
&+ \left(c(\boldsymbol{x})^{-1}Q_1^n, \nabla\cdot\boldsymbol{\rho}^n\right) + \left(c(\boldsymbol{x})^{-1}Q_2^n, \nabla\cdot\boldsymbol{\rho}^n\right)\\
&- \frac{4}{3}\left((A(Eu^n)^{-1}b(Eu^n) - A(Eu_h^n)^{-1}b(Eu_h^n))\nabla(u^{n-1} - u^{n-2}), \boldsymbol{\rho}^n\right)\\
&+ \frac{4}{3}\left(A(Eu_h^n)^{-1}b(Eu_h^n)(\nabla\eta^{n-1} - \nabla\eta^{n-2}), \boldsymbol{\rho}^n\right)\\
&- \frac{4}{3}\left(A(Eu_h^n)^{-1}b(Eu_h^n)(\nabla\xi^{n-1} - \nabla\xi^{n-2}), \boldsymbol{\rho}^n\right)\\
&- \left((A(Eu^n)^{-1}b(Eu^n) - A(Eu_h^n)^{-1}b(Eu_h^n))\nabla u^{n-2}, \boldsymbol{\rho}^n\right)\\
&+ \left(A(Eu_h^n)^{-1}b(Eu_h^n)\nabla\eta^{n-2}, \boldsymbol{\rho}^n\right)\\
&- \left(A(Eu_h^n)^{-1}b(Eu_h^n)\nabla\xi^{n-2}, \boldsymbol{\rho}^n\right)\\
&+ \left(A(Eu^n)^{-1}Q_3^n, \boldsymbol{\rho}^n\right) + \left(A(Eu^n)^{-1}Q_4^n, \boldsymbol{\rho}^n\right)\\
&= \sum_{n=1}^{18} T_i.
\end{aligned}
\tag{6.13}
$$

Notice that

$$\frac{1}{c^*} \le c(\cdot)^{-1} \le \frac{1}{c_*},$$

$$A(\cdot)^{-1} = \frac{1}{a(\cdot) + \frac{2}{3}\Delta t b(\cdot)} \ge \frac{1}{a^* + b^*},$$

$$A(\cdot)^{-1} = \frac{1}{a(\cdot) + \frac{2}{3}\Delta t b(\cdot)} \le \frac{1}{a_*},$$

and

$$
\begin{aligned}
A(Eu_h^n)^{-1} - A(Eu^n)^{-1} &= \frac{A(Eu^n) - A(Eu_h^n)}{A(Eu_h^n)A(Eu^n)}\\
&\le K\left[|\xi^{n-1}| + |\eta^{n-1}| + |\xi^{n-2}| + |\eta^{n-2}|\right].
\end{aligned}
$$

For $T_1 \sim T_{10}$, we obtain the following bounds

$$
\begin{aligned}
T_1 &\le K\|\boldsymbol{\pi}^n\|^2 + \epsilon\|\boldsymbol{\rho}^n\|^2,\\
T_2 &\le K\|\boldsymbol{\pi}^n\|^2 + \epsilon\|\boldsymbol{\rho}^n\|^2,\\
T_3 &\le K(\|\xi^{n-1}\|^2 + \|\eta^{n-1}\|^2 + \|\xi^{n-2}\|^2 + \|\eta^{n-2}\|^2) + \epsilon\|\boldsymbol{\rho}^n\|^2,\\
T_4 &\le K\|\nabla\eta^{n-1}\|^2 + \epsilon\|\nabla\cdot\boldsymbol{\rho}^n\|^2,\\
T_5 &\le K\|\nabla\eta^{n-2}\|^2 + \epsilon\|\nabla\cdot\boldsymbol{\rho}^n\|^2,\\
T_6 &\le K\|\nabla\xi^{n-1}\|^2 + \epsilon\|\nabla\cdot\boldsymbol{\rho}^n\|^2,\\
T_7 &\le K\|\nabla\xi^{n-2}\|^2 + \epsilon\|\nabla\cdot\boldsymbol{\rho}^n\|^2,\\
T_8 &\le K(\|\xi^{n-1}\|^2 + \|\eta^{n-1}\|^2 + \|\xi^{n-2}\|^2 + \|\eta^{n-2}\|^2) + \epsilon\|\nabla\cdot\boldsymbol{\rho}^n\|^2,\\
T_9 &\le K(\triangle t)^4 + \epsilon\|\nabla\cdot\boldsymbol{\rho}^n\|^2,\\
T_{10} &\le K(\triangle t)^4 + \epsilon\|\boldsymbol{\rho}^n\|^2.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left|A(Eu^n)^{-1}b(Eu_n) - A(Eu_h^n)^{-1}b(Eu_h^n)\right|\\
&= \Big|\frac{A(Eu_h^n)b(Eu_n) - A(Eu^n)b(Eu_h^n)}{A(Eu_h^n)A(Eu^n)}\Big|\\
&\le K\left[|\xi^{n-1}| + |\eta^{n-1}| + |\xi^{n-2}| + |\eta^{n-2}|\right],
\end{aligned}
$$

bounds for T_{11} and T_{14} are given as follows:

$$
\begin{aligned}
T_{11} &\le K(\|\xi^{n-1}\|^2 + \|\eta^{n-1}\|^2 + \|\xi^{n-2}\|^2 + \|\eta^{n-2}\|^2) + \epsilon\|\boldsymbol{\rho}^n\|^2,\\
T_{14} &\le K(\|\xi^{n-1}\|^2 + \|\eta^{n-1}\|^2 + \|\xi^{n-2}\|^2 + \|\eta^{n-2}\|^2) + \epsilon\|\boldsymbol{\rho}^n\|^2.
\end{aligned}
$$

And bounds for T_{12}, T_{13}, and $T_{15} \sim T_{18}$ are given as follows:

$$
\begin{aligned}
T_{12} &\le K(\|\nabla\eta^{n-1}\|^2 + \|\nabla\eta^{n-2}\|^2) + \epsilon\|\boldsymbol{\rho}^n\|^2,\\
T_{13} &\le K(\|\nabla\xi^{n-1}\|^2 + \|\nabla\xi^{n-2}\|^2) + \epsilon\|\boldsymbol{\rho}^n\|^2,\\
T_{15} &\le K\|\eta^{n-2}\|^2 + \epsilon\|\boldsymbol{\rho}^n\|^2,\\
T_{16} &\le K\|\xi^{n-2}\|^2 + \epsilon\|\boldsymbol{\rho}^n\|^2,\\
T_{17} &\le K(\triangle t)^4 + \epsilon\|\boldsymbol{\rho}^n\|^2,\\
T_{18} &\le K(\triangle t)^4 + \epsilon\|\boldsymbol{\rho}^n\|^2.
\end{aligned}
$$

Thus, by using estimates for $T_1 \sim T_{18}$ and taking $\epsilon > 0$ sufficiently small, we get from (6.13)

$$
\begin{aligned}
&\|\nabla\cdot\boldsymbol{\rho}^n\|^2 + \|\boldsymbol{\rho}^n\|^2\\
\le &K\Big(\|\boldsymbol{\pi}^n\|^2 + \|\eta^{n-1}\|^2 + \|\eta^{n-2}\|^2 + \|\nabla\eta^{n-1}\|^2 + \|\nabla\eta^{n-2}\|^2\\
&\qquad + \|\xi^{n-1}\|^2 + \|\xi^{n-2}\|^2 + \|\nabla\xi^{n-1}\|^2 + \|\nabla\xi^{n-2}\|^2 + (\Delta t)^4\Big).
\end{aligned}
$$

Therefore, by Lemma 4.2, Lemma 4.5, (5.6), and (6.8), we get

$$\|\nabla\cdot\boldsymbol{\rho}^n\|^2+\|\boldsymbol{\rho}^n\|^2\le K\Big(h^{2(\mu-1)}+(\Delta t)^4\Big)$$

and so

$$\|\nabla\cdot\boldsymbol{\rho}^n\|+\|\boldsymbol{\rho}^n\|\le K(h^{\mu-1}+(\Delta t)^2).$$

Thus by the triangular inequality and Lemma 4.5, we obtain the result of this theorem.

□

Remark 6.2. From Theorem 6.2, we know that the approximations $\boldsymbol{\sigma}_h^n$ has higher order of convergence in a time increment which improves our previous result in [18].

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