Journal for History of Mathematics
Vol. 35 No. 2 (Apr. 2022), 59–70
http://dx.doi.org/10.14477/jhm.2022.35.2.059

On the Geometry of Charged Rotating Black Holes

Yun Jong Gug 윤종국

In this paper, we review some aspects of geometry for charged rotating black holes which are formed from the gravitational collapse of a massive spinning star with electric charge. We also introduce the computation of entropy for black holes from loop quantum gravity which is a quantum theory of gravity based on Einstein's theory of general relativity.

Keywords: Black Holes, Ergosphere, Loop Quantum Gravity, Entropy.

MSC: 53C50, 53Z05

1 Introduction with a Brief History

Roger Penrose won the 2020 Nobel Prize in physics for showing how black holes could form, which is one of the most striking predictions of Einstein's general theory of relativity. It is known that Einstein himself did not believe that black hole exists but nowadays, there are convincing evidences for black hole's existence at the center of galaxies including our own Milky way and researchers estimate that there are as many as ten million to a billion black holes in the Milky way alone.

Black hole is very difficult to observe directly but it is very important to understand the origin of galaxy and the Universe itself.

From the historical viewpoint, K. Schwarzschild [10] found, in 1916, the first modern solution of Einstein's field equation of general relativity which characterizes a black hole. Schwarzschild's solution is the unique vacuum solution of the field equation with spherical symmetry and it is 'static' according to Birkhoff's theorem stating that all spherically symmetic and vacuum spacetimes are static(Here,

Yun Jong Gug: Dept. of Math. Edu., Korea National Univ. of Edu. E-mail: jgyun69@knue.ac.kr
Received on Mar. 17, 2022, revised on Apr. 19, 2022, accepted on Apr. 21, 2022.

static spacetime means that it admits a one-parameter group of isometries whose orbits are both timelike and orthogonal to a spacelike hypersurface.)

In 1916 and in 1918 respectively, Reissner [8] and Nordström [7] found independently a *charged vacuum* generalization(that is, vacuum except for the electromagnetic fields due to the charge of the black hole) of the Schwarzschild's solution.

It was not until 1963 that Kerr [4] discovered a *stationary vacuum* generalization (that is, vacuum with a one-parameter group of isometries whose orbits are timelike) of the Schwarzschild's solution which corresponds the *rotating* black hole.

Shortly after, in 1965, Newmann *et al* [6] obtained a charged generalization of Kerr solution which are the unique *stationary electro-vacuum*(that is, stationary and vacuum except for the electromagnetic fields due to the charge of the black hole) spacetime.

These charged Kerr solutions have the spacetime metric in the coordinates (t, r, θ, ϕ) as follows:

$$\begin{aligned} ds^2 =& g_{\mu\nu} dx^\mu dx^\nu = -\left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\right) dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma} dt d\phi \\ &+ \left[\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \end{aligned}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta,$$
$$\Delta = r^2 + a^2 + Q^2 - 2Mr,$$

and Q, a, M are the three parameters which are interpreted as total charge, total angular momentum per mass and total mass of the black hole respectively.

When $Q = 0$ the above metric reduces to the Kerr's stationary vacuum solution. When $a = 0$ we obtain the Reissner-Nordström solution, and when $Q = a = 0$ the above metric reduces to the Schwarzschild's solution.

We also note that in the charged Kerr solution, $\partial_t \left(\equiv \frac{\partial}{\partial t}\right), \partial_\phi \left(\equiv \frac{\partial}{\partial \phi}\right)$ are Killing fields, since $g_{\mu\nu}$ is independent of t and ϕ and so $a\partial_t + b\partial_\phi$ is also a Killing field for any two constants a, b.

It is well known that black hole has thermodynamical features like entropy and the calculation of the entropy for a black hole was indeed performed by Bekenstein [1] and Hawking [2]. Recalling that, in statistical mechanics, entropy is understood as counting the number of microscopic configurations of a system that have the same macroscopic qualities (such as mass, charge, pressure, etc.), we may expect such counting would be possible for black hole too. But since we have not yet fully succeeded in developing the quantum theory of gravity, [1] and [2] do not present such a counting of microscopic configurations for black holes. Instead, some progress has been made in various approaches to quantum gravity like loop quantum gravity. In this paper, we review some aspects of geometry of the charged Kerr balck hole and introduce the calculation of its entropy using the theory of loop quantum gravity.

2 Static Limit and Ergosphere

We first introduce the following definition of the stationary observers.

Definition 2.1. *Stationary observers are the observers who move along a worldline of constant* (r, θ) *with uniform angular velocity* $\Omega = \frac{d\phi}{dt}$.

Stationary observers see an unchanging spacetime geometry in his neighborhood and 4-velocity vector u^μ of stationary observers is of the form

$$\begin{aligned} u^\mu &= \frac{dt}{d\tau}\partial_t + \frac{d\phi}{d\tau}\partial_\phi = \frac{dt}{d\tau}(\partial_t + \frac{d\phi}{d\tau}/\frac{dt}{d\tau}\partial_\phi) \\ &= \frac{dt}{d\tau}(\partial_t + \frac{d\phi}{dt}\partial_\phi) = u^t(\partial_t + \Omega\partial_\phi) \\ &= \frac{\partial_t + \Omega\partial_\phi}{||\partial_t + \Omega\partial_\phi||} = \frac{\partial_t + \Omega\partial_\phi}{\sqrt{-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi}}}, \end{aligned}$$

where τ denotes the proper time.

Here, note that $||\partial_t + \Omega\partial_\phi|| = \sqrt{-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi}}$ is constant along the worldline, because $g_{tt}, g_{t\phi}, g_{\phi\phi}$ depend only on r, θ, Ω. (Alternatively, note that $\partial_t + \Omega\partial_\phi$ is a Killing field, so it is an infinitesimal isometry and its norm along the integral curves is unchanged)

The stationary observers at given r, θ with $g_{\phi\phi} > 0$ cannot have every angular velocity, since its velocity 4-vector should be timelike:

$$\begin{aligned} \langle u, u\rangle &= \frac{1}{||\partial_t + \Omega\partial_\phi||^2}\langle \partial_t + \Omega\partial_\phi, \partial_t + \Omega\partial_\phi\rangle < 0 \\ \Leftrightarrow \langle \partial_t + \Omega\partial_\phi, \partial_t + \Omega\partial_\phi\rangle &= g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} < 0 \\ \Leftrightarrow \Omega_{\text{min}} := \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} &< \Omega < \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} =: \Omega_{\text{max}}, \end{aligned}$$

where

$$\omega := -\frac{g_{\phi t}}{g_{\phi\phi}} \left(= \frac{1}{2}(\Omega_{\text{max}} + \Omega_{\text{min}}) \right)$$

Note that $\Omega_{\text{min}}, \Omega_{\text{max}}$ correspond to the allowed angular velocity of lightlike particles. Since Kerr spacetime is asymptotically flat, we have

$$\omega = -\frac{g_{\phi t}}{g_{\phi\phi}} \to 0, \qquad \frac{g_{tt}}{g_{\phi\phi}} \to -\frac{1}{r^2}$$

as $r \to \infty, \theta = \pi/2$. Thus we can see that

$$\lim_{r\to\infty} r\Omega_{\text{min}} = -1, \qquad \lim_{r\to\infty} r\Omega_{\text{max}} = 1$$

Note that ± 1 corresponds the standard limits imposed by the speed of light in flat spacetime.

We now introduce the following definition of static observers.

Definition 2.2. *Static observers are the observers moving along a worldline of constant (r, θ, ϕ). He is 'static' relative to the black hole's asymptotic Lorentz frame (i.e. relative to the distant stars). Equivalently, static observers are the stationary observers with $\Omega = 0$ or equivalently, observers whose velocity 4-vectors are proportional to ∂_t.*

From now on, we assume that $M^2 \geq Q^2 + a^2$ for some physical reasons[1)]and note that for $r_0(\theta) = M + \sqrt{M^2 - Q^2 - a^2\cos^2\theta}$ at which g_{tt} and Ω_{min} become zero, it is easy to see that $r \searrow r_0(\theta)$ implies $\Omega_{\text{min}} \nearrow 0$(note that if $r \geq r_0(\theta)$, then $\omega > 0$).

Consequently, we know that at and inside the surface $r = r_0(\theta)$, all stationary observers must orbit the black hole with positive angular velocity. In other words, static observers can exist outside and only outside $r = r_0(\theta)$. For this reason, the region $r = r_0(\theta)$ is called the 'static limit' [5].

As one moves through the static limit into the 'ergosphere' which is defined as a region $r_+ < r < r_0(\theta)$, where $r_+ = M + \sqrt{M^2 - Q^2 - a^2}$ (the region defined by $r = r_+$ is called the 'outer event horizon') one sees the allowed range of angular velocities become ever more positive (i.e. ever more 'frame dragging'). At the same time, one sees the allowed range becomes narrow down, until finally the limits Ω_{min} and Ω_{max} meet each other at $r = r_+$, where $\omega^2 = \frac{g_{tt}}{g_{\phi\phi}}$ happens.

Note that $\Omega_{\text{min}} = \Omega_{\text{max}}$ at $r = r_+$ means that $\{\Omega \mid \Omega_{\text{min}} < \Omega < \Omega_{\text{max}}\}$ is empty, so at the horizon, there are no stationary observers.

1) In the case $M^2 < Q^2 + a^2$, the spacetime violates the cosmic censorship conjecture stating that there is no 'naked singularity' and it also leads to a causality violation.

Note also that at $r = r_+$, we have

$$\left(\frac{g_{\phi t}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}} = \left(\frac{1}{g_{\phi\phi}}\right)^2 (g_{\phi t}^2 - g_{tt}g_{\phi\phi}) = 0$$
$$\Leftrightarrow \det \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix}\Bigg|_{r=r_+} = 0,$$

which can be easily checked by putting $g_{tt}|_{r=r_+}$, $g_{t\phi}|_{r=r_+}$, $g_{\phi\phi}|_{r=r_+}$ directly in the $\det \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix}$. Now since $\Omega_H := \Omega_{\text{min}} = \Omega_{\text{max}} = \omega = -\frac{g_{\phi t}}{g_{\phi\phi}}$ at $r = r_+$, we have at $r = r_+$,

$$\begin{aligned}\langle \partial_t + \Omega_H \partial_\phi, \partial_t + \Omega_H \partial_\phi \rangle &= g_{tt} + 2\Omega_H g_{t\phi} + \Omega_H^2 g_{\phi\phi} \\ &= g_{tt} - 2\frac{g_{\phi t}}{g_{\phi\phi}} g_{t\phi} + \left(\frac{g_{\phi t}}{g_{\phi\phi}}\right)^2 g_{\phi\phi} = 0\end{aligned}$$

This means that $\partial_t + \Omega_H \partial_\phi$ is null and tangent to the outer event horizon defined by $r = r_+$ (Recall that $\{\partial_t, \partial_\phi, \partial_\theta\}$ is a basis of tangent space of the hypersurface $r = r_+$.) This implies that the outer event horizon $r = r_+$ is a null hypersurface and the structure of light cones reveal that all timelike worldlines point inward at $r = r_+$. Consequently there is no escape from the inside the outer event horizon $r = r_+$.

We also use $\Omega_H := -\frac{g_{\phi t}}{g_{\phi\phi}}\Big|_{r=r_+}$ to define the angular velocity of the event horizon itself:

$$\Omega_H = -\frac{g_{\phi t}}{g_{\phi\phi}}\Bigg|_{r=r_+} = \frac{d\phi}{dt}\Bigg|_{r=r_+} = \frac{a}{r_+^2 + a^2}$$

3 Redshift of Time

Assume that an observer, far enough from the black hole and at rest in the black hole's asymptotic inertial frame, watches a particle moving along a stationary orbit near the black hole. Let $\Omega = \frac{d\phi}{dt}$ be the particle's angular velocity measured by the observer. The distant observer uses his clock with $dt = \frac{d\phi}{\Omega}$ to measure the time T required for the particle to make one complete circuit around the black hole as follows.

$$T = \int_0^{2\pi} \frac{d\phi}{\Omega} = \frac{2\pi}{\Omega}$$

On the other hand, let an observer u comoving with the particle measure its circuit time $\Delta\tau$ using his eyes and a clock he carries.

This time, the observer u could calculate as follows:

$$\begin{aligned}-1 = \langle u, u\rangle &= \left\langle \frac{dt}{d\tau}\partial_t + \frac{d\phi}{d\tau}\partial_\phi, \frac{dt}{d\tau}\partial_t + \frac{d\phi}{d\tau}\partial_\phi \right\rangle \\ &= g_{tt}\left(\frac{dt}{d\tau}\right)^2 + 2g_{t\phi}\left(\frac{d\phi}{d\tau}\right)\left(\frac{dt}{d\tau}\right) + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 \\ &= g_{tt}\left(\frac{dt}{d\tau}\right)^2 + 2\Omega g_{t\phi}\left(\frac{dt}{d\tau}\right)^2 + \Omega^2 g_{\phi\phi}\left(\frac{dt}{d\tau}\right)^2 \quad \left(\Leftarrow \frac{d\phi}{d\tau} = \frac{d\phi/dt}{d\tau/dt} = \Omega\frac{dt}{d\tau}\right)\end{aligned}.$$

Thus we have $d\tau = \sqrt{-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi}}\ dt$, so $\sqrt{-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi}}$ is the 'redshift factor', which is constant along the circular trajectory whose θ and r coordinates are constant.

Consequently, we have

$$\Delta\tau = \int_0^T d\tau = \frac{2\pi}{\Omega}\sqrt{-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi}}.$$

4 Energy and Angular Momentum Measured at Infinity

We first note that if γ is a geodesic with tangent 4-vector u and ξ is a Killing vector field, then $\langle \xi, u\rangle\,(=\ \xi_a u^a)$ is constant along γ. This can easily be checked as follows:

$$u^b\nabla_b(\xi_a u^a) = u^b u^a \nabla_b \xi_a + \xi_a u^b \nabla_b u^a = 0.$$

Here, we used $u^b u^a \nabla_b \xi_a = \frac{1}{2}(u^a u^b \nabla_a \xi_b + u^b u^a \nabla_b \xi_a) = \frac{1}{2}u^a u^b(\nabla_a \xi_b + \nabla_b \xi_a) = 0$ by the Killing equation $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ and $u^b \nabla_b u^a = 0$ since γ is a geodesic.

In the charged Kerr spacetime, which is stationary, we now consider a geodesic particle of which 4-velocity is u and define the following *conserved* quantities along u.

$$E = -\langle \partial_t, u\rangle, \qquad L = \langle \partial_\phi, u\rangle$$

(Recall that ∂_t and ∂_ϕ are Killing vector field, so E and L is constant along u).

Since the spacetime is asymptotically flat, we know that $g_{tt} \to -1$, and $g_{t\phi} \to 0$ as $r \to \infty$. Thus we have

$$E = -g_{ab}(\partial_t)^a u^b = -g_{tt}u^t - g_{t\phi}u^\phi \to u^t,$$

as $r \to \infty$. But, u^t is the special relativistic total energy per unit rest mass of the particle as measured by a static observer.

Thus we may interpret E for timelike geodesic as representing the total energy (including gravitational potential energy) per unit rest mass following the geodesic, relative to a static observer at infinity, since it is the energy that would be required for such an observers in order to put a unit rest mass particle in the given orbit.

Note also that $E = -\langle \partial_t, u \rangle = -||\partial_t|| \langle \partial_t/||\partial_t||, u \rangle = -|g_{tt}|^{1/2} \langle \xi, u \rangle$, where ξ is the static observer who is passed by the particle. Thus the energy is 'redshifted' by an amount $|g_{tt}|^{1/2}$.

In the Kerr-Newmann spacetime, it turns out [11] that

$$E = -u^a \xi_a = (1 - \frac{2Mr}{\Sigma})\dot{t} + \frac{2Mar\sin^2\theta}{\Sigma}\dot{\phi},$$

$$L = u^a (\partial_\phi)_a = -\frac{2Mar\sin^2\theta}{\Sigma}\dot{t} + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\sin^2\theta\dot{\phi},$$

where $\dot{x^\mu} = \frac{dx^\mu}{d\tau}$.

Thus, even though $\dot{\phi} = \frac{d\phi}{d\tau} = 0$, L needs not be zero. Note also that even if $L = 0$, $\frac{d\phi}{d\tau}$ needs not be zero (in the ergosphere, it is always positive).

Consequently, we have

$$\frac{d\phi}{d\tau} = 0 \not\Leftrightarrow L = 0 \Leftrightarrow \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}},$$

where the second '$\Leftrightarrow$' is clear from $\langle \partial_\phi, u \rangle = \left\langle \partial_\phi, \frac{dt}{d\tau}\partial_t + \frac{d\phi}{d\tau}\partial_\phi \right\rangle = \frac{dt}{d\tau} g_{t\phi} + \frac{d\phi}{d\tau} g_{\phi\phi}$.

By the same argument as above, we have

$$L = g_{ab}(\partial_\phi)^a u^b = g_{\phi t} u^t + g_{\phi\phi} u^\phi \to r^2 \dot{\phi}$$

as $r \to \infty,\ \theta = \frac{\pi}{2}$. So we may interpret L as the angular momentum per unit rest mass of the particle measured by a static observer at infinity.

5 Surface Gravity and Black Hole Thermodynamics

Let (M, g_{ab}) be a static spacetime with timelike Killing field $\xi^a = \partial_t$. Then the acceleration $a^b = u^a \nabla_a u^b$ of a static observer $u^a = \xi^a/V$ with $V = (-\xi^a \xi_a)^{1/2}$ is given by $a^b = \nabla^b \ln V$, which can be checked as follows:

$$\begin{aligned}
a_b = u^a \nabla_a u_b &= \frac{\xi^a}{V}\left(\frac{\nabla_a \xi_b}{V} - \frac{\xi_b \nabla_a V}{V^2}\right) \\
&= \frac{\xi^a}{V}\left(\frac{\nabla_a \xi_b}{V} - \frac{\xi_b \nabla_a V^2}{2V^3}\right) = -\frac{\xi^a \nabla_b \xi_a}{V^2} + \frac{2\xi^a \xi_b \xi^c \nabla_a \xi_c}{2V^4} \\
&= -\frac{\nabla_b(\xi_a \xi^a)}{2V^2} + \frac{\xi_b \xi^a \xi^c (\nabla_a \xi_c + \nabla_c \xi_a)}{2V^4} = \frac{\nabla_b V^2}{2V^2} + 0 \\
&= \frac{\nabla_b V}{V} = \nabla_b \ln V.
\end{aligned}$$

Here, we used the Killing equation $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$.

Now suppose in addition that (M, g_{ab}) is asymptotically flat (i.e. for the coordinates t, x, y, z with $\xi^a = \left(\frac{\partial}{\partial t}\right)^a$, we have that the components of g_{ab} approach to diag$(-1,1,1,1)$ as $r \to \infty$, where $r^2 = x^2 + y^2 + z^2$) and let m and u^a be the rest mass and 4-velocity of a particle which is held in the static state with respect to the static observer at infinity along a massless string attached to the index finger of the static observer at infinity.

Then, even though u does not follow a geodesic (in order to stay at fixed spatial coordinates), we can see that the quantity $E = -m\xi^a a$ is conserved as follows:

$$\begin{aligned} u^b\nabla_b(\xi^a u_a) &= u^b\xi^a\nabla_b u_a + u^b u_a\nabla_b\xi^a \\ &= u^0\xi^0\nabla_0 u_0 + g_{00}(u^0)^2\nabla_0\xi^0 \\ &= u^0 g_{00}\nabla_0 u^0 + 0 = \frac{g_{00}}{2}\nabla_0(u^0)^2 \\ &= \frac{g_{00}}{2}\nabla_0\left(\frac{-1}{g_{00}^2}\right) = 0 \end{aligned}$$

Here, we used that $u^i = \xi^i = g_{0i} = 0$ for $i = 1, 2, 3$ and $\xi^0 = 1$, since u is a static observer and (M, g_{ab}) is a static spacetime with timelike Killing field $\xi^a = \left(\frac{\partial}{\partial t}\right)^a$. We also used that g_{00} is independent of t and that $-1 = g_{ab}u^a u_b = g_{00}^2(u^0)^2$ with $u_b = g_{ba}u^a = g_{b0}u^0 = 0$ if $b \neq 0$.

So we may still interpret E as the (potential) energy of the particle as measured by the static observer at infinity.

Let F denote the magnitude of the force exerted by the string on the particle. According to $a^b = \nabla^b \ln V$, we have $F = mV^{-1}[\nabla_a V\nabla^a V]^{1/2}$.

On the other hand, the magnitude F_∞ of the force exerted by the observer at infinity on the other end of the string can be calculated, using $E = -m\xi^a u_a = -mV$, as follows:

$$F_\infty = [\nabla_a E\nabla^a E]^{1/2} = m[\nabla_a V\nabla^a V]^{1/2}.$$

Thus, the magnitude of the force exerted at infinity differs from the force exerted on the particle locally by the redshift factor V.

Now we let κ denote the magnitude of acceleration of static observer near the horizon of the black hole, as measured by a static observer at infiniy (or alternatively, by the magnitude of the force that must be exerted at infinity to hold a unit test mass at fixed spatial coordinate).

Then by the above argument, we have

$$\kappa = \lim[\nabla_a V\nabla^a V]^{1/2},$$

where $V = (-\xi^a\xi_a)^{1/2}$ with $\xi^a = \left(\frac{\partial}{\partial t}\right)^a$ and 'lim' stands for the limit as one approaches the horizon. The quantity κ is typically finite and called *surface gravity* of the black hole.

In the special case of Schwarzschild spacetime with mass M, we can calculate κ by direct calculation to obtain

$$\kappa = \frac{1}{4M}.$$

Thus, we know that although the local acceleration of the particle required to stay at fixed r diverges as we approach the horizon, but the force acting at infinity tends to the value $\frac{1}{4M}$.

For more general black holes such as charged Kerr black holes with $\Omega_H \neq 0$, instead of ξ^a, we can use the Killing field $\chi^a = \xi^a + \Omega_H \partial_\phi$ which is null on the horizon to define κ in a similar way as follows:

$$\kappa = \lim(Va),$$

where $a = (a^c a_c)^{1/2}$, $V = (-\chi^a \chi_a)^{1/2}$ and $a^c = (\chi^b/V)\nabla_b(\chi^c/V))$ is the acceleration of the stationary observer χ^a/V.

For a charged Kerr black hole, the value of κ is calculated as follows[2)] [11]:

$$\kappa = \frac{(M^2 - a^2 - Q^2)^{1/2}}{2M[M + (M^2 - a^2 - Q^2)^{1/2}] - Q^2}$$

For simplicity, we go to the case of Schwarzschild spacetime and calculate the area A of the horizon $r = 2M$ as follows:

$$A = \int_0^{\pi} \int_0^{2\pi} (2M)^2 \sin\theta d\theta d\phi = 16\pi M^2.$$

Thus we obtain $dA = 32\pi M dM$ and recalling $\kappa = \frac{1}{4M}$ we can write

$$dM = \frac{1}{8\pi}\kappa dA.$$

This formula for the black hole is reminiscent of the first law of thermodynamics,

$$dE = TdS,$$

where E, T, S are the energy, temperature and entropy of the system respectively.

The two formulas coincide under the identifications

$$E = M,\ S = A/4,\ T = \kappa/2\pi,$$

which indeed turned out to be the energy, entropy and temperature of a quantum black hole.

Specifically, in 1973 and 1974, J. Bekenstein and S. Hawking suggested that black hole has an entropy given by $A/4$, where A is the area of the horizon of the black hole

2) Since a test mass can't be held static with respect to infinity near the rotating black hole, the above quantity κ cannot be interpreted as the gravitational acceleration of a static observer as seen at infinity.

[1, 2]. S. Hawking also showed that black hole emits radiation with a temperature given by $\kappa/2\pi$ in 1975 [3].

In the general case of a charged Kerr black hole, the right hand side of the above formula for a black hole contains another term which corresponds to the work we do on the black hole by, for example, throwing a rock into it. Thus the formula corresponds to the first law of thermodynamics,

$$dE = TdS - pdV,$$

where p, V is the pressure and volume of the system respectively, so pdV is the work we do to the system.

6 Entropy from Loop Quantum Gravity

From the statistical mechanical viewpoint, entropy of a physical system is understood as counting the number of microscopic configurations of a system that have the same macroscopic qualities. In Bekenstein-Hawking's derivation of entropy, since quantum properties of gravitational field are neglected, we don't still fully understand what kinds of the microscopic degrees of freedom are responsible for the entropy. Indeed, the capability of answering this question has become a standard benchmark against which a quantum theory of gravity can be tested.

In the quest for quantum gravity, the leading alternative to string theory is the theory of loop quantum gravity, which tries to merge the classical general relativity into quantum mechanics in a canonical way without any assumptions on supersymmetry, extradimensions, etc. In this section, we present a brief idea of how loop quantum gravity explains the microscopic degrees of freedom responsible for the entropy from the Plank scale.

In loop quantum gravity, the quantum states of the gravitational field are represented by 's-knots'. An *s-knot* is an equivalence class under diffeomorphisms of a graph(called a 'spin network') which is composed of several links and nodes where different links intersect each other. Each link carries a quantum number corresponding to an irreducible representation of $SU(2)$, and each node also carries a quantum number corresponding to the invariant coupling between such representations. These quantum numbers on each link and node are called the 'colors' of the s-knot.

If a surface S in space is given, we can define an 'area operator' A_S acting on the quantum states of a gravitational field and it turns out that spin networks are the eigen states of this operator.

Given a quantum eigen state of gravitational field represented by an s-knot and an event horizon S of a black hole, then the (quantized) area of S is determined by the colors on the intersections of the s-knot with the surface. Indeed, let $i = 1, 2, \ldots, n$ label such intersections, and $p_i \in \frac{\mathbb{Z}}{2}$ be the color of the link corresponding to i. Assuming generically that no node of the s-knot be on the surface, it turns out that the area of S is the eigen value of the operator A_S of the form

$$A_S = 8\pi\gamma \sum_i \sqrt{p_i(p_i+1)},$$

where the sum runs over all intersections of the links of the s-knot and the surface S.(The parameter γ is called Barbero-Immirzi parameter and has the value of $\frac{\ln 2}{\pi\sqrt{3}}$).

Thus for the macroscopic quantity A of the area of the horizon of a black hole, we may regard the corresponding number of microscopic(quantum) states as the number of n-tuple of n colors $\overrightarrow{p} = (p_1, \ldots, p_n)$(with arbitrary n), which satisfies

$$\frac{A}{8\pi\gamma} = \sum_i \sqrt{p_i(p_i+1)}.$$

The estimating of this number is purely algebraic problem of combinatorics and the resulting value $N(A)$ can be written as [9]

$$\ln N(A) = \frac{\ln 2}{\sqrt{3}} \frac{A}{4\pi\gamma}.$$

Consequently, we obtain the statistical entropy(in the Plank unit) $S(A)$ which coincides the Bekenstein-Hawking's entropy:

$$S(A) = \ln N(A) = \frac{A}{4}.$$

References

1. J. D. BEKENSTEIN, Black Holes and Entropy, *Phys. Rev.* D7 (1973), 2333–2346.
2. S. W. HAWKING, Black Hole Explosions?, *Nature* 248(1974), 30–31.
3. S. W. HAWKING, Particle Creation by Black Holes, *Commun. Math. Phys.* 43 (1975), 199–220.
4. R. P. KERR, Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics, *Physical Review Letters* 11(5) (1963), 237–238.
5. C. MISNER, K. S. THORNE, J. WHEELER, *Gravitation*, W. H. Freeman and Company, 1973.
6. E. T. NEWMAN et al, Metric of a Rotating Charged Mass, *J. Math. Phys.* 6 (1965), 918–919.
7. G. NORDSTRÖM, On the Energy of the Gravitational Field in Einstein's Theory, *Verhandl. Koninkl. Ned. Akad. Wetenschap., Afdel. Natuurk* Amsterdam, 26 (1918), 1201–1208.
8. H. REISSNER, Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie, *Annalen der Physik* 50(9) (1916), 106–120.

9. C. Rovelli, *Quantum Gravity*, Cambridge University Press, 2004.
10. K. Schwarzschild, Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie, *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* 7 (1916), 189–196.
11. R. M. Wald, *General Relativity*, The Univ. of Chicago Press, 1984.