ZEROS OF NEW BERGMAN KERNELS

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Abstract. In this paper we determine explicitly the kernels $K_{\alpha,\beta}$ associated with new Bergman spaces $A^2_{\alpha,\beta}(D)$ considered recently by the first author and M. Zaway. Then we study the distribution of the zeros of these kernels essentially when $\alpha \in \mathbb{N}$ where the zeros are given by the zeros of a real polynomial $Q_{\alpha,\beta}$. Some numerical results are given throughout the paper.

1. Introduction

The notion of Bergman kernels has several applications and represents an essential tool in complex analysis and geometry. This notion was introduced first by Bergman [1], then it has been greatly developed by finding the relationship with other notions as in [6]. Sometimes it is necessary to determine these kernels explicitly. However, this is not simple in general. In fact if an orthonormal basis of a Hilbert space is given, then the Bergman kernel of this space can be obtained as a series using the basis elements. For example, the Bergman kernel of the space $A^2_{\alpha}(D)$ of holomorphic functions on the unit disk $D$ of $\mathbb{C}$ that are square integrable with respect to the positive measure $d\mu_{\alpha}(z) = (\alpha + 1)(1 - |z|^2)^{\alpha}dA(z)$ is given by $K_{\alpha}(z,w) = \frac{1}{1 - \overline{z}w}$. Hence this kernel has no zero in $D$. For more details about Bergman spaces one can see [4]. In order to obtain kernels with zeros in $D$, Krantz consider in his book [5] some subspaces of $A^2(D)$. In our statement, instead of considering subspaces, we modify slightly the measure $d\mu_{\alpha}$ to obtain a Bergman kernel that is comparable in some sense with the previous one with some zeros in $D$. These spaces are considered recently by N. Ghiloufi and M. Zaway in [3]. We recall the main background of this paper:

Throughout the paper, $D$ will be the unit disk of the complex plane $\mathbb{C}$ as it was mentioned before and $D^* = D \setminus \{0\}$. We let $\mathbb{N} := \{0, 1, 2, \ldots \}$ be the set of positive integers and $\mathbb{R}$ be the set of real numbers. We use the convention that a real number $x$ is said to be positive (resp. negative) if $x \geq 0$ (resp. $x \leq 0$).
For every $-1 < \alpha, \beta < +\infty$, we consider the positive measure $\mu_{\alpha, \beta}$ on $\mathbb{D}$ defined by
\[
d\mu_{\alpha, \beta}(z) := \frac{1}{\mathcal{B}(\alpha + 1, \beta + 1)} |z|^2 \mathcal{B}(1 - |z|^2) \alpha dA(z),
\]
where $\mathcal{B}$ is the beta function defined by
\[
\mathcal{B}(s, t) = \int_0^1 x^{s-1}(1-x)^{t-1} dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}, \quad \forall s, t > 0
\]
and
\[
dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta, \quad z = x + iy = re^{i\theta}
\]
the normalized area measure on $\mathbb{D}$.

We denote by $A^2_{\alpha, \beta}(\mathbb{D})$ the set of holomorphic functions on $\mathbb{D}^*$ that belongs to the space:
\[
\mathbf{L}^2(\mathbb{D}, d\mu_{\alpha, \beta}) = \{ f : \mathbb{D} \to \mathbb{C} : \text{measurable function such that } \| f \|_{\alpha, \beta, 2} < +\infty \},
\]
where
\[
\| f \|_{\alpha, \beta, 2}^2 := \int_{\mathbb{D}} |f(z)|^2 d\mu_{\alpha, \beta}(z).
\]
The set $A^2_{\alpha, \beta}(\mathbb{D})$ is a Hilbert space and $A^2_{\alpha, \beta}(\mathbb{D}) = A^2_{\alpha, m}(\mathbb{D})$ if $\beta = \beta_0 + m$ with $m \in \mathbb{N}$ and $-1 < \beta_0 \leq 0$ (see [3] for more details). We claim here that $A^2_{\alpha, \beta_0}(\mathbb{D}) = A^2_{\alpha, \beta}(\mathbb{D})$ if $\beta_0 > -1$, if we set
\[
e_n(z) = \sqrt{\frac{\mathcal{B}(\alpha + 1, \beta + 1)}{\mathcal{B}(\alpha + 1, n + \beta + 1)}} z^n
\]
for every $n \geq -m$, then the sequence $(e_n)_{n \geq -m}$ is an orthonormal basis of $A^2_{\alpha, \beta}(\mathbb{D})$. Furthermore, if $f, g \in A^2_{\alpha, \beta}(\mathbb{D})$ with
\[
f(z) = \sum_{n=-m}^{+\infty} a_n z^n, \quad g(z) = \sum_{n=-m}^{+\infty} b_n z^n,
\]
then
\[
\langle f, g \rangle_{\alpha, \beta} = \sum_{n=-m}^{+\infty} a_n b_n \frac{\mathcal{B}(\alpha + 1, n + \beta + 1)}{\mathcal{B}(\alpha + 1, \beta + 1)},
\]
where $\langle \cdot, \cdot \rangle_{\alpha, \beta}$ is the inner product in $A^2_{\alpha, \beta}(\mathbb{D})$ inherited from $\mathbf{L}^2(\mathbb{D}, d\mu_{\alpha, \beta})$.

The following main result determines the reproducing kernel of $A^2_{\alpha, \beta}(\mathbb{D})$.

**Theorem 1.1.** Let $-1 < \alpha, \beta < +\infty$ and $\mathcal{K}_{\alpha, \beta}$ be the reproducing Bergman kernel of $A^2_{\alpha, \beta}(\mathbb{D})$. Then $\mathcal{K}_{\alpha, \beta}(w, z) = \mathcal{K}_{\alpha, \beta}(w \overline{z})$, where
\[
\mathcal{K}_{\alpha, \beta}(\xi) = \frac{Q_{\alpha, \beta}(\xi)}{(\xi)^m(1 - \xi)^{2+\alpha}}
\]
with
\[ Q_{\alpha,\beta}(\xi) = \begin{cases} 
(\alpha+1) & \text{if } \beta \in \mathbb{N}, \\
\beta_0 \sum_{n=0}^{+\infty} \frac{(-\xi)^n}{n + \beta_0} & \text{if } \beta \not\in \mathbb{N},
\end{cases} \]

with \( \beta_0 = \beta - \lfloor \beta \rfloor - 1 = \beta - m. \)

As a consequence of this main result, the study can be reduced to the case \( \beta = \beta_0 \in ]-1,0[. \) Indeed if we set
\[ M : A^2_{\alpha,\beta}(\mathbb{D}) \longrightarrow A^2_{\alpha,\beta_0}(\mathbb{D}) \]
\[ f \longrightarrow \mathcal{B}(\alpha+1,\beta_0+1) \Rightarrow \mathcal{B}(\alpha+1,\beta_0+1) \]
then the linear operator \( M \) is invertible and bi-continuous and \( K_{\alpha,\beta} = M^{-1} \circ K_{\alpha,\beta_0}. \) Thus we can assume that \( m = 0, \) i.e., \( \beta = \beta_0. \)

The proof of the main result is the aim of the following section. Then as a consequence, we will prove that for \( \alpha \in \mathbb{N} \) and \( \beta \in ]-1,0[, \) the zeros set of \( \mathcal{K}_{\alpha,\beta} \) is a totally real submanifold of \( \mathbb{D}^* \times \mathbb{D}^* \) with real dimension one formed by at most \( (\alpha+1) \) connected components. This set is reduced to one connected component for \( \beta \) closed to \( -1 \) \( (\beta \to (-1)^+) \) and it is empty for \( \beta \) near \( 0 \) \( (\beta \to 0^-) \). These zeros are related to the zeros set \( Z_{Q_{\alpha,\beta}} \) of \( Q_{\alpha,\beta} \) in \( \mathbb{C}. \) Hence we will concentrate essentially on the distribution of \( Z_{Q_{\alpha,\beta}}. \) This will be the aim of the third section of the paper where we start by a general study and we conclude that \( Z_{Q_{\alpha,\beta}} \) is formed by exactly \( (\alpha+1) \) connected regular curves when \( \beta \) varies in the interval \( ]-1,0[. \)

We finish the paper with some open problems. Using Python software, some numerical results are investigated in the annex of the paper where we confirm numerically some asymptotic results.

2. Proof of the main result

The proof of the first case is simple (it was done in [3]) however, the proof of the second one is more delicate and it will be done by steps. Using the sequence \( (e_n)_{n \geq -m} \) given by (1.1), we deduce that the reproducing kernel of \( A^2_{\alpha,\beta}(\mathbb{D}) \) can be written as follows:

\[ K_{\alpha,\beta}(w, z) = \sum_{n=-m}^{+\infty} e_n(w) \overline{e_n(z)} = \sum_{n=-m}^{+\infty} \mathcal{B}(\alpha+1,\beta+1) \mathcal{B}(\alpha+1,n+\beta+1) w^n \overline{z}^n = \frac{\mathcal{R}_{\alpha,\beta}(w\overline{z})}{(w\overline{z})^m} =: \mathcal{K}_{\alpha,\beta}(w\overline{z}), \]
where
\[ R_{\alpha,\beta}(\xi) = \mathcal{B}(\alpha + 1, \beta + 1) \sum_{n=0}^{+\infty} \frac{\xi^n}{\mathcal{B}(\alpha + 1, n + \beta - m + 1)}. \]

This series is well-defined as a consequence of Stirling formula. If \( \beta = m \in \mathbb{N} \), then
\[ R_{\alpha,m}(\xi) = \mathcal{B}(\alpha + 1, m + 1) \sum_{n=0}^{+\infty} \frac{\xi^n}{\mathcal{B}(\alpha + 1, n + 1)} = \frac{(\alpha + 1)\mathcal{B}(\alpha + 1, m + 1)}{(1 - \xi)^{2+\alpha}}. \]

We consider now the case \( \beta \in [m-1, m] \) with \( m \in \mathbb{N} \) and we prove the result in two steps:
- **First step:** The case \( \alpha \in \mathbb{N} \).

We start by proving the following preliminary lemma.

**Lemma 2.1.** We have
\[ R_{\alpha,\beta}(\xi) = \frac{Q_{\alpha,\beta}(\xi)}{(1 - \xi)^{2+\alpha}}, \]
where \( Q_{\alpha,\beta} \) is a polynomial of degree \( \alpha + 1 \) with \( Q_{\alpha,\beta}(1) \neq 0 \) that satisfies the recurrence formula:
\[ Q_{\alpha+1,\beta}(\xi) = \frac{1}{\alpha+\beta+2} \left[ \xi (1-\xi) Q'_{\alpha,\beta}(\xi) + (\alpha + \beta - m + 2 + (m - \beta)\xi) Q_{\alpha,\beta}(\xi) \right]. \]

**Proof.** If \( \alpha = 0 \), then we have
\[ R_{0,\beta}(\xi) = \mathcal{B}(1, \beta + 1) \sum_{n=0}^{+\infty} \frac{\xi^n}{\mathcal{B}(1, n + \beta - m + 1)} = \frac{1}{\beta + 1} \sum_{n=0}^{+\infty} (n + \beta - m + 1)\xi^n = \frac{Q_{0,\beta}(\xi)}{(1 - \xi)^{2}} \]
with
\[ Q_{0,\beta}(\xi) = \frac{1}{\beta + 1} ((m - \beta)\xi + \beta - m + 1). \]
Assume that the result is proved for \( \alpha \in \mathbb{N} \), i.e.,
\[ R_{\alpha,\beta}(\xi) = \frac{Q_{\alpha,\beta}(\xi)}{(1 - \xi)^{2+\alpha}}, \]
where \( Q_{\alpha,\beta} \) is a polynomial of degree \( \alpha + 1 \) with \( Q_{\alpha,\beta}(1) \neq 0 \) and we will prove that it is true for \( \alpha + 1 \). Indeed, we have
\[ R_{\alpha+1,\beta}(\xi) = \mathcal{B}(\alpha + 2, \beta + 1) \sum_{n=0}^{+\infty} \frac{\xi^n}{\mathcal{B}(\alpha + 2, n + \beta - m + 1)}. \]
\[
\begin{align*}
&= B(\alpha + 2, \beta + 1) \sum_{n=0}^{+\infty} \frac{\Gamma(\alpha + 3 + n + \beta - m) \xi^n}{\Gamma(\alpha + 2) \Gamma(n + \beta - m + 1)} \\
&= B(\alpha + 1, \beta + 1) \frac{1}{\alpha + \beta + 2} \sum_{n=0}^{+\infty} \frac{(\alpha + 2 + n + \beta - m) \xi^n}{B(\alpha + 1, n + \beta - m + 1)} \\
&= \left( \frac{1}{\alpha + \beta + 2} \left( \xi R_{\alpha, \beta}^\prime (\xi) + (\alpha + \beta - m + 2) R_{\alpha, \beta} (\xi) \right) \right) \\
&\quad + \left( \frac{1}{\alpha + \beta + 2} \left( \frac{Q_{\alpha, \beta} (\xi)}{(1 - \xi)^{2 + \alpha}} \right) \right) \\
&= Q_{\alpha + 1, \beta} (\xi) \left( \frac{(1 - \xi) Q_{\alpha, \beta} (\xi)}{\alpha + \beta + 2} \right) \\
\end{align*}
\]

with

\[ Q_{\alpha + 1, \beta} (\xi) = \frac{\xi (1 - \xi) Q_{\alpha, \beta} (\xi) + (\alpha + \beta - m + 2 + (m - \beta) \xi) Q_{\alpha, \beta} (\xi)}{\alpha + \beta + 2} \]

Thus \( Q_{\alpha + 1, \beta} \) is a polynomial of degree \( \alpha + 2 \) and

\[ Q_{\alpha + 1, \beta} (1) = \frac{\alpha + 2}{\alpha + \beta + 2} Q_{\alpha, \beta} (1) \neq 0. \]

\textbf{Proof of Theorem 1.1.} Now, we can deduce the proof of Theorem 1.1 in the case \( \alpha \in \mathbb{N} \). This will be done by induction on \( \alpha \). The result is true for \( \alpha = 0 \). Indeed, we have

\[ Q_{0, \beta} (\xi) = \frac{1}{\beta + 1} (1 + \beta_0 - \beta_0 \xi) = \beta_0 B(1, \beta + 1) \frac{1}{\beta_0} \left( \frac{1}{\beta_0} - \frac{\xi}{1 + \beta_0} \right). \]

Assume that the result is true until the value \( \alpha \). Thanks to Lemma 2.1, we have

\[ Q_{\alpha + 1, \beta} (\xi) = \frac{1}{\alpha + \beta + 2} \left( \xi (1 - \xi) Q_{\alpha, \beta} (\xi) + (\alpha + 2 + \beta_0 - \beta_0 \xi) Q_{\alpha, \beta} (\xi) \right) \]

\[ \begin{align*}
&= \frac{\beta_0 B(\alpha + 1, \beta + 1)}{(\alpha + \beta + 2) B(\alpha + 1, \beta_0 + 1)} \left[ \sum_{j=1}^{\alpha + 1} \frac{(-1)^{j+1}}{j + \beta_0} \binom{\alpha + 1}{j} \xi^j \right] \\
&\quad + \sum_{j=1}^{\alpha + 2} (j - 1) \frac{(-1)^{j+1}}{j - 1 + \beta_0} \binom{\alpha + 1}{j - 1} \xi^j \\
&\quad + (\alpha + 2 + \beta_0) \sum_{j=0}^{\alpha + 1} \frac{(-1)^j}{j + \beta_0} \binom{\alpha + 1}{j} \xi^j \\
\end{align*} \]
\[ +\beta_0 \sum_{j=1}^{\alpha+2} \frac{(-1)^j}{j-1+\beta_0} \binom{\alpha+1}{j-1} \xi^j \]
\[ = \beta_0 \frac{\mathcal{B}(\alpha+2,\beta+1)}{\mathcal{B}(\alpha+2,\beta_0+1)} \sum_{j=0}^{\alpha+2} \frac{(-\xi)^j}{j+\beta_0} \binom{\alpha+2}{j}. \]

This achieves the first step.

- **Second step:** The general case \((\alpha > -1)\).

We set
\[ S_{\alpha,\beta_0}(\xi) := \frac{\mathcal{B}(\alpha+1,\beta_0+1)}{\beta_0 \mathcal{B}(\alpha+1,\beta+1)} Q_{\alpha,\beta}(\xi) \]
\[ = \frac{(1-\xi)^{\alpha+2}}{\beta_0} \sum_{n=0}^{\infty} \frac{\mathcal{B}(\alpha+1,\beta_0+1)}{\mathcal{B}(\alpha+1,n+\beta_0+1)} \xi^n \]
\[ = \frac{Q_{\alpha,\beta_0}(\xi)}{\beta_0} \]

and
\[ G_{\alpha,\beta_0}(\xi) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n+\beta_0} \binom{\alpha+1}{n} \xi^n. \]

To prove the result it suffices to attest that \(S_{\alpha,\beta_0} = G_{\alpha,\beta_0}\) on \(D\). To show this equality we will prove that both functions \(S_{\alpha,\beta_0}\) and \(G_{\alpha,\beta_0}\) satisfy the following differential equation:

\[ \xi F'(\xi) = -\beta_0 F(\xi) + (1-\xi)^{\alpha+1}, \quad \forall \xi \in D. \]

It follows that \(S_{\alpha,\beta_0} - G_{\alpha,\beta_0}\) satisfies on \(D^*\) the homogenous differential equation: \(\xi F'(\xi) = -\beta_0 F(\xi)\). In particular it satisfies the same homogenous differential equation on \([0,1]\). Thus there exists a constant \(\sigma \in \mathbb{R}\) such that for every \(t \in [0,1]\) we have \(S_{\alpha,\beta_0}(t) - G_{\alpha,\beta_0}(t) = \sigma t^{-\beta_0}\). Since \(S_{\alpha,\beta_0} - G_{\alpha,\beta_0}\) is differentiable at 0, we get \(\sigma = 0\), i.e., \(S_{\alpha,\beta_0} = G_{\alpha,\beta_0}\) on \([0,1]\) and by the analytic extension principle we conclude that \(S_{\alpha,\beta_0} = G_{\alpha,\beta_0}\) on \(D\).

To finish the proof we will show that both functions \(S_{\alpha,\beta_0}\) and \(G_{\alpha,\beta_0}\) satisfy the differential equation (2.2). For \(G_{\alpha,\beta_0}\) the result is obvious. Indeed
\[ \xi G'_{\alpha,\beta_0}(\xi) = \sum_{n=0}^{\infty} \frac{n}{n+\beta_0} \binom{\alpha+1}{n} (-\xi)^n \]
\[ = \sum_{n=0}^{\infty} \left(1-\frac{\beta_0}{n+\beta_0}\right) \binom{\alpha+1}{n} (-\xi)^n \]
\[ = (1-\xi)^{\alpha+1} - \beta_0 G_{\alpha,\beta_0}(\xi). \]
Now for $S_{\alpha,\beta}$, it is not hard to prove that
\[
\xi S'_{\alpha,\beta}(\xi) = -(1 - \xi)^{\alpha + 1} \sum_{n=0}^{+\infty} \frac{(\alpha + 1) B(\alpha + 1, \beta_0 + 1)}{(\alpha + 1 + n + \beta_0) B(\alpha + 1, n + \beta_0 + 1)} \xi^n
\]
\[
= (1 - \xi)^{\alpha + 1} - \beta_0 S_{\alpha,\beta}(\xi).
\]
Thus the proof of Theorem 1.1 is finished. \qed

As a first consequence of Theorem 1.1, we obtain the following identity:

**Corollary 2.2.** Let $-1 < \alpha < +\infty$ and $-1 < \beta < 0$. For every $n \in \mathbb{N}$,
\[
\sum_{k=0}^{n} \binom{\alpha + 2}{k} \frac{(-1)^k}{\beta} = \frac{\beta}{B(\alpha + 1, \beta + 1)} \left[ (\alpha + 1) \binom{\alpha + 1}{n} (-1)^n \right].
\]

**Proof.** Thanks to Theorem 1.1, we have
\[
S_{\alpha,\beta}(\xi) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n + \beta} \binom{\alpha + 1}{n} \xi^n
\]
\[
= \frac{B(\alpha + 1, \beta + 1)}{\beta} (1 - \xi)^{\alpha + 2} \sum_{n=0}^{+\infty} \frac{\xi^n}{B(\alpha + 1, n + \beta + 1)}
\]
\[
= \frac{B(\alpha + 1, \beta + 1)}{\beta} \left[ \sum_{n=0}^{+\infty} \binom{\alpha + 2}{n} (-\xi)^n \right] \left[ \sum_{n=0}^{+\infty} \frac{\xi^n}{B(\alpha + 1, n + \beta + 1)} \right]
\]
\[
= \frac{B(\alpha + 1, \beta + 1)}{\beta} \sum_{n=0}^{+\infty} d_n \xi^n,
\]
where
\[
d_n = \sum_{k=0}^{n} \binom{\alpha + 2}{k} \frac{(-1)^k}{B(\alpha + 1, n - k + \beta + 1)}.
\]
So the result follows. \qed

Using the proof of Theorem 1.1, one can conclude the following corollary:

**Corollary 2.3.** For every $-1 < \alpha$ and $-1 < \beta < 0$, the function $G_{\alpha,\beta}$ defined in (2.1) satisfies:
\[
\xi G'_{\alpha,\beta}(\xi) = (1 - \xi)^{\alpha + 1} - \beta G_{\alpha,\beta}(\xi)
\]
and
\[
G_{\alpha+1,\beta}(\xi) = \frac{1}{\alpha + \beta + 2} \left( \xi (1 - \xi) G'_{\alpha,\beta}(\xi) + (\alpha + \beta + 2 - \beta \xi) G_{\alpha,\beta}(\xi) \right)
\]
= \frac{1}{\alpha + \beta + 2} \left( (\alpha + 2) G_{\alpha,\beta}(\xi) + (1 - \xi)^{\alpha + 2} \right).

**Remarks 2.4.**
(1) Using the Stirling formula, one can prove that $G_{\alpha,\beta}$ is bounded on the closed unit disk $\mathbb{D}$. This fact will be used frequently in the hole of the paper.
(2) Thanks to Lemma 2.1, for \( \alpha \in \mathbb{N} \), one has \( G_{\alpha,0}(1) \neq 0 \). For the general case, if \( G_{\alpha,0}(1) \neq 0 \) for some \(-1 < \alpha_0 \leq 0\), then \( G_{\alpha+n,0}(1) \neq 0 \) for every \( n \in \mathbb{N} \).

In the rest of the paper, we assume that \( G_{\alpha,0}(1) \neq 0 \). This may be true for any \(-1 < \alpha \) and \(-1 < \beta < 0\).

3. Zeros of Bergman kernels

Using Theorem 1.1, the function \( K_{\alpha,0} \) has no zero in the unit disk \( D \). However if \(-1 < \beta < 0\), then \( K_{\alpha,\beta} \) may have some zeros in \( D \). We claim that if \( \xi \in D^* \) is a zero of \( K_{\alpha,\beta} \), then the sets \( \{(z,w) \in D^2; w = \xi \} \) and \( \{(z,w) \in D^2; z = \xi \} \) define two totally real algebraic surfaces (of real dimension equal to 2) of \( \mathbb{C}^2 \) that are contained in the zeros set of the Bergman kernel \( K_{\alpha,\beta} \). Thus it suffices to study the zeros set of \( K_{\alpha,\beta} \).

Due to an algebraic problem, we focus sometimes on the case \( \alpha \in \mathbb{N} \), because in this case the zeros of \( K_{\alpha,0} \) are given by the zeros of the polynomial \( G_{\alpha,0} \) contained in \( D \). Thus for \( \alpha \in \mathbb{N} \), we will study the zeros set of \( G_{\alpha,0} \) in the whole complex plane \( \mathbb{C} \). It is interesting to discuss the variations of these sets in terms of the parameter \( \beta \). All results on \( G_{\alpha,0} \) can be viewed as particular cases of those of the following linear transformation.

3.1. The linear transformation \( T_\beta \)

If \( \mathcal{O}(D(0,R)) \) is the space of holomorphic function on the disk \( D(0,R) \) and \(-1 < \beta < 0\), then we define \( T_\beta \) on \( \mathcal{O}(D(0,R)) \) by

\[
T_\beta f(z) = \sum_{n=0}^{+\infty} \frac{a_n}{n + \beta} z^n
\]

for any \( f(z) = \sum_{n=0}^{+\infty} a_n z^n \). The transformation \( T_\beta \) is linear and bijective from \( \mathcal{O}(D(0,R)) \) onto itself. It transforms any polynomial to a polynomial with the same degree. We start by the study of zeros of \( T_\beta f \) in general then we specialize the study to the case \( f(z) = P_\alpha(z) = (1 - z)^{\alpha+1} \), where \( T_\beta P_\alpha \) is exactly \( G_{\alpha,\beta} \).

**Theorem 3.1.** Let \( 0 < R \leq +\infty \) and \( f \) be a holomorphic function on \( \mathbb{D}(0,R) \) such that \( (f(0),f'(0)) \neq (0,0) \). Then for every \( 0 < r_0 < R \), there exist \( \beta_1(f,r_0) \) and \( \beta_2(f,r_0) \), with

\[
-1 < \beta_1(f,r_0) \leq -\frac{|f(0)|}{|f(0)| + |r_0| |f'(0)|} \leq \beta_2(f,r_0) < 0,
\]

depending on \( f \) and \( r_0 \) such that the function \( T_\beta f \) has no zero in \( \mathbb{D}(0,r_0) \) for every \( \beta_2(f,r_0) < \beta < \beta_1(f,r_0) \) and has exactly one simple zero in \( \mathbb{D}(0,r_0) \) for every \(-1 < \beta < \beta_1(f,r_0) \).

When \( f(0) = 0 \) and \( f'(0) \neq 0 \) the result is reduced to “0 is the unique zero (simple) of the function \( T_\beta f \) in \( \mathbb{D}(0,r_0) \) for every \(-1 < \beta < 0\)”.

However,
when \( f'(0) = 0 \) and \( f(0) \neq 0 \) then “the function \( T_\beta f \) has no zero in \( \mathbb{D}(0, r_0) \) for every \(-1 < \beta < 0\).”

**Proof of Theorem 3.1.** If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) for every \( z \in \mathbb{D}(0, R) \) with \((a_0, a_1) \neq (0, 0)\), then we set
\[
F_{\beta, f}(z) = \frac{a_0}{\beta} + \frac{a_1}{1 + \beta} z.
\]
If \( |z| = r_0 \) we have
\[
|T_{\beta} f(z) - F_{\beta, f}(z)| \leq \sum_{n=2}^{\infty} \frac{|a_n|}{n + \beta} r_0^n.
\]
Moreover, if we set
\[
\psi(\beta) = \left| \frac{a_0}{\beta} + \frac{|a_1|r_0}{1 + \beta} \right| - \sum_{n=2}^{\infty} \frac{|a_n|}{n + \beta} r_0^n,
\]
then
\[
\psi \left( \frac{|a_0|}{|a_0| + r_0|a_1|} \right) < 0 \quad \text{and} \quad \lim_{\beta \to 0^+} \psi(\beta) = +\infty \quad (\text{resp.} \quad \lim_{\beta \to -(1)^+} \psi(\beta) = +\infty)
\]
when \( a_0 \neq 0 \) (resp. \( a_1 \neq 0 \)). It follows that there exist \( \beta_1 \) and \( \beta_2 \), with
\[-1 < \beta_1 \leq -\frac{|a_0|}{|a_0| + r_0|a_1|} \leq \beta_2 < 0,
\]
depending on \( f \) and \( r_0 \) such that for every \( \beta \in ]-1, \beta_1]\cup[\beta_2, 0[ \) one has \( \psi(\beta) > 0 \). Hence, for every \( \beta \in ]-1, \beta_1]\cup[\beta_2, 0[ \) and \( |z| = r_0 \), we have \(|T_{\beta} f(z) - F_{\beta, f}(z)| < |F_{\beta, f}(z)|\). Thus by Rouché Theorem, \( T_{\beta} f \) and \( F_{\beta, f} \) have the same number of zeros counted with their multiplicities in the disk \( \mathbb{D}(0, r_0) \). □

In the following lemma we collect some useful properties of \( T_{\beta} f \) that will be used frequently in the sequel.

**Lemma 3.2.** If \( f \) is a holomorphic function on \( \mathbb{D}(0, R) \) and \(-1 < \beta < 0\), then the following assertions hold:

1. The number 0 is a zero of \( f \) if and only if it is a zero of \( T_\beta f \) (with the same multiplicity).
2. The derivative of \( T_\beta f \) satisfies
   \[
z(T_\beta f)'(z) = f(z) - \beta T_\beta f(z), \quad \forall \, z \in \mathbb{D}(0, R).
   \]
3. The functions \( f \) and \( T_\beta f \) have a common zero in \( \mathbb{D}^*(0, R) \) if and only if the function \( T_\beta f \) has a zero in \( \mathbb{D}^*(0, R) \) with multiplicity greater than or equal to 2.

Now we consider a fixed holomorphic function \( f \) on \( \mathbb{D}(0, R) \) without common zero with \( T_\beta \) for any \( \beta \in ]-1, 0[ \). We set
\[
H_f(\beta, z) := T_\beta f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n + \beta} z^n.
\]
for \((\beta, z) \in ]-1,0[\times \mathbb{D}(0, R)\) and
\[
\mathcal{Y}_f := \{(\beta, z) \in ]-1,0[\times \mathbb{D}(0, R); \ H_f(\beta, z) = 0\}.
\]

We assume that the set \(\mathcal{Y}_f\) is not empty. Indeed if \(f \equiv c\) is a constant function, then \(T_\beta f \equiv \frac{c}{\beta}\), thus \(\mathcal{Y}_c = \emptyset\) if \(c \neq 0\) and \(\mathcal{Y}_c = ]-1,0[\times \mathbb{C}\) if \(c = 0\). Moreover it is easy to find some examples of non constant holomorphic functions \(g\) where \(T_\beta g\) has no zero for some value of \(\beta\). But we don’t know if there exists a non constant function \(g\) such that \(\mathcal{Y}_g\) is empty.

**Proposition 3.3.** The set \(\mathcal{Y}_f\) is a submanifold of (real) dimension one in \(\mathbb{R}^3\) formed by at most countable connected components \((\mathcal{Y}_{f,k})_k\).

If \(\mathcal{Y}\) is a connected component of \(\mathcal{Y}_f\), then there exist \(-1 \leq a_\mathcal{Y} < b_\mathcal{Y} \leq 0\) and a \(C^\infty\)-function \(\mathcal{X} : ]a_\mathcal{Y}, b_\mathcal{Y}[ \rightarrow \mathbb{D}(0, R)\) such that
\[
\mathcal{Y} = \text{Graph}(\mathcal{X}) := \{(\beta, \mathcal{X}(\beta)); \ \beta \in ]a_\mathcal{Y}, b_\mathcal{Y}[\}.
\]

Moreover for every \(\beta \in ]a_\mathcal{Y}, b_\mathcal{Y}[\), one has
\[
(\beta, z) \in \mathcal{Y}_{f,k}\) \Rightarrow \mathcal{X}'(\beta) = \mathcal{X}(\beta) \frac{\mathcal{X}(\beta)}{f(\mathcal{X}(\beta))} \sum_{n=0}^{+\infty} \frac{a_n}{(n+\beta)^2} (\mathcal{X}(\beta))^n.
\]

**Proof.** For every \((\beta, z) \in \mathcal{Y}_f\) we have
\[
\frac{\partial H_f}{\partial z}(\beta, z) = (T_\beta f)'(z) = \frac{1}{z}(f(z) - \beta T_\beta(z)) = \frac{1}{z} f(z) \neq 0.
\]

The result follows using the implicit functions theorem. \(\square\)

It is easy to see that if \(0 < R < +\infty\), then \(a_\mathcal{Y} > -1\) for all connected components \(\mathcal{Y}\) of \(\mathcal{Y}_f\) except the unique component \(\mathcal{Y}_{f,0}\) given by Theorem 3.1 where \(a_\mathcal{Y}_{f,0} = -1\). However, \(b_\mathcal{Y} = 0\) if and only if \(R = +\infty\), i.e., \(f\) is an entire function. In this case, of entire functions, all functions \(\mathcal{X}_{f,k}\) are defined on \([-1,0]\).

**Remark 3.4.** Using the same proof, the previous result can be improved to the complex case as follows: If we set \(\Omega := \{\beta \in \mathbb{C}; -1 < \Re(\beta) < 0\}\) and \(\mathcal{D}_f := \{(\beta, z) \in \Omega \times \mathbb{D}(0, R); \ H_f(\beta, z) = 0\}\), then \(\mathcal{D}_f\) is a submanifold of (complex) dimension one in \(\Omega \times \mathbb{D}(0, R)\) formed by connected components. Thus, the Lelong number of the current \([\mathcal{D}_f]\) of integration over \(\mathcal{D}_f\) is equal to one at every point of \(\mathcal{D}_f\). (This is due to the fact that all zeros of \(H_f\) are simple.) For more details about currents and Lelong numbers, one can refer to [2].

The following theorem gives the asymptotic behaviors of functions \(\mathcal{X}_f\) near \(-1\) and \(0\) when \(f\) is a polynomial. We use the notation \(\sim\) to indicate the classical equivalence, i.e., two functions \(h_1(t) \sim h_2(t)\) if we have \(\lim_{t \to t_0} \frac{h_1(t)}{h_2(t)} = 1\) whenever \(h_2(t) \neq 0\). We claim that if \(f(z) = a_0 + a_1 z\), then the solution is explicitly determined by
\[
\mathcal{X}_f(\beta) = \frac{a_0}{a_1} \beta + 1, \beta.
\]
Hence we will consider the case when \( \deg(f) \geq 2 \).

**Theorem 3.5.** Let \( f(z) = \sum_{n=0}^{p} a_n z^n \) be a polynomial of degree \( p \geq 2 \) with \( f(0) \neq 0 \). We set \( a_n = |a_n|e^{i\theta_n} \) for any \( 0 \leq n \leq p \). The set \( \mathcal{D}_f \) is formed exactly by \( p \) connected components \( (\mathcal{X}_{f,k})_{0 \leq k \leq p-1} \) with the corresponding functions \( \mathcal{X}_{f,k} : [-1, 0[ \to \mathbb{C} \). Again we keep \( \mathcal{X}_{f,k} \) to indicate the function related to the unique component given by Theorem 3.1.

1. For every \( 0 \leq k \leq p-1 \), we have \( \lim_{\beta \to 0^{-}} |\mathcal{X}_{f,k}(\beta)| = +\infty \) and

\[
\mathcal{X}_{f,k}(\beta) \sim_{\beta \to 0^{-}} \left( -\frac{p|a_0|}{|a_p|} \right) \frac{\beta^{a_0-a_p+2j_k}}{p} e^{\frac{\theta_0-a_p}{p}2j_k},
\]

where \( j_k \in \mathbb{Z} \) that depends on \( k \).

2. If \( f'(0) \neq 0 \), then for every \( 1 \leq k \leq p-1 \)
\[
\lim_{\beta \to (-1)^+} |\mathcal{X}_{f,k}(\beta)| = +\infty \quad \text{and} \quad \lim_{\beta \to (-1)^+} \mathcal{X}_{f,0}(\beta) = 0.
\]

Moreover, we have

\[
\mathcal{X}_{f,0}(\beta) \sim_{\beta \to (-1)^+} \frac{a_0}{a_1} (1 + \beta)
\]

and

\[
\mathcal{X}_{f,k}(\beta) \sim_{\beta \to (-1)^+} \left( \frac{(p-1)|a_1|}{(1+\beta)|a_p|} \right)^{\frac{1}{p}} e^{\frac{\theta_0-a_p+2sk+1}{p-1}}, \quad \forall 1 \leq k \leq p-1
\]

for some \( s_k \in \mathbb{Z} \) that depends on \( k \).

If \( f(0) \neq 0 \) and \( f'(0) = 0 \), then all functions \( \mathcal{X}_{f,k} \) are bounded near \(-1\).

**Proof.** Let \( 0 \leq k \leq p-1 \). As \( a_0 \neq 0 \) then using the equality

\[
\frac{a_0}{\beta} + \sum_{n=1}^{p} \frac{a_n}{n+\beta}(\mathcal{X}_{f,k}(\beta))^n = 0
\]

we obtain

\[
\lim_{\beta \to 0^{-}} \mathcal{X}_{f,k}(\beta) = \infty
\]

and

\[
-\frac{a_0}{\beta} \sim_{\beta \to 0^{-}} \frac{a_p}{p+\beta}(\mathcal{X}_{f,k}(\beta))^p.
\]

That means

\[
(\mathcal{X}_{f,k}(\beta))^p \sim_{\beta \to 0^{-}} -\frac{pa_0}{a_p}\beta
\]

so we get Equation (3.2).

With the same way if \( a_1 \neq 0 \), then for every \( 0 \leq k \leq p-1 \) we have

\[
\lim_{\beta \to (-1)^+} \mathcal{X}_{f,k}(\beta) \in \{0, \infty\}.
\]

Thanks to Theorem 3.1,

\[
\lim_{\beta \to (-1)^+} \mathcal{X}_{f,0}(\beta) = 0 \quad \text{and} \quad \lim_{\beta \to (-1)^+} \mathcal{X}_{f,k}(\beta) = \infty, \quad \forall 1 \leq k \leq p-1.
\]
Thus, we obtain
\[ \chi_{f,0}(\beta) \sim \frac{a_0}{a_1} \frac{1 + \beta}{\beta}. \]
Therefore, Equation (3.3) follows.

For \(1 \leq k \leq p - 1\) we obtain
\[ \frac{a_1}{1 + \beta} \sim \frac{a_p}{p + \beta} \left(\chi_{f,k}(\beta)^{p-1}\right) \]
thus,
\[ \left(\chi_{f,k}(\beta)^{p-1}\right) \sim \frac{(p - 1)a_1}{a_p(1 + \beta)} \]
and Equation (3.4) follows. \(\square\)

3.2. Application on the Bergman kernels

As mentioned before, for any \(\alpha \in \mathbb{N}\), \(T_\beta P_\alpha = G_{\alpha,\beta}\), where \(P_\alpha(z) = (1 - z)^{\alpha+1}\). Hence all previous results are valid and more precision are needed to accomplish the study of \(X_{\alpha,k} := X_{P_\alpha,k}\), \(0 \leq k \leq \alpha\). We start by claiming that if \(x < 0\), then there exists \(\beta_x \in ]-1,0[\) such that \(G_{\alpha,\beta_x}(x) = 0\). It follows that \((\beta_x, x)\) is in a component (say \(S_{\alpha,0}\)) of \(D_\alpha := D_{P_\alpha}\). Hence, the corresponding function \(X_{\alpha,0}\) maps \([-1,0[\) onto \([-\infty,0[\). Indeed we have \(X'_{\alpha,0}(\beta) < 0\) for every \(\beta \in ]-1,0[\) thus it is a decreasing function and
\[ \lim_{\beta \to -1^+} X_{\alpha,0}(\beta) = -\infty, \quad \lim_{\beta \to 0^-} X_{\alpha,0}(r) = 0. \]

Using Corollary 2.3, we can deduce that \(X_{\alpha,0}(\beta) \geq X_{\alpha+1,0}(\beta)\) for every \(\beta \in ]-1,0[\) (See Figure 1).

Remark 3.6. For any \(\alpha \in \mathbb{N}\), we set \(-1 < s_\alpha < 0\) the unique solution of \(X_{\alpha,0}(\beta) = -1\). The polynomial \(G_{\alpha,\beta}\) has no zero in \([-1,0]\) for every \(s_\alpha < \beta < 0\) and has exactly one simple zero in \([-1,0[\) for every \(-1 < \beta < s_\alpha\).

We claim that \((s_\alpha)\alpha\) is an increasing sequence (See again Figure 1).

The following lemma explain differently the conclusion of Theorem 3.1 in the current statement (See Table 1 for numerical values of \(\beta_1\) and \(\beta_2\) given by Theorem 3.1 for this example).

Lemma 3.7. For every \(\alpha > -1\), the family of functions \((\beta(1 + \beta)G_{\alpha,\beta})_{\beta \in ]-1,0[}\) converges uniformly on \(\mathbb{D}\) to the constant 1 (resp. to the polynomial \((\alpha + 1)\xi\) as \(\beta \to 0^-\) (resp. as \(\beta \to -1^+\)).

In particular, for every \(m \in \mathbb{N}\) (resp. \(m \in \mathbb{N}^*\)) the family of kernels \((K_{\alpha,\beta})_{\beta \in ]m-1,m[}\) converges uniformly on every compact subset of \(\mathbb{D}^*\) to \(K_{\alpha,m}\) (resp. to \(K_{\alpha,m-1}\)) as \(\beta \to m^-\) (resp. as \(\beta \to (m-1)^+\)).

Proof. The lemma is a simple consequence of the following equality:
\[ \beta(1 + \beta)G_{\alpha,\beta}(\xi) = (1 + \beta) - \beta(1 + \alpha)\xi + \beta(1 + \beta) \sum_{n=2}^{+\infty} \frac{(-\xi)^n}{n + \beta} \binom{\alpha + 1}{n} \]

and the fact that the series converges normally on $D$ (obtained using the Stirling formula).

□

Table 1. Numerical values of $\beta_1(P_\alpha, 1)$ and $\beta_2(P_\alpha, 1)$ given by Theorem 3.1.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta_1(P_\alpha, 1)$</th>
<th>$\beta_2(P_\alpha, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-0.381966</td>
<td>-0.177124</td>
</tr>
<tr>
<td>3</td>
<td>-0.493058</td>
<td>-0.107610</td>
</tr>
<tr>
<td>4</td>
<td>-0.667086</td>
<td>-0.0649539</td>
</tr>
<tr>
<td>5</td>
<td>-0.793482</td>
<td>-0.0387481</td>
</tr>
<tr>
<td>6</td>
<td>-0.870294</td>
<td>-0.0227925</td>
</tr>
<tr>
<td>7</td>
<td>-0.917737</td>
<td>-0.0132128</td>
</tr>
<tr>
<td>8</td>
<td>-0.947843</td>
<td>-0.00755239</td>
</tr>
<tr>
<td>9</td>
<td>-0.967185</td>
<td>-0.0042614</td>
</tr>
</tbody>
</table>

The most important conclusion of this lemma is the continuity of the Bergman kernels $K_{\alpha, \beta}$ in terms of the parameter $\beta$. Essentially the fact that the Bergman kernel $K_{\alpha, \beta}$ converges uniformly on every compact subset of $D^2$ to the classical Bergman kernel $K_{\alpha, 0}$ when $\beta \to 0^-$.

Now we will focus on the other components of $D_{\alpha}$. We use $X_{\alpha, k}$, $0 \leq k \leq \alpha$ to indicate the corresponding functions such that $\Im m(X_{\alpha, k}(\beta)) \leq 0$ for every
\[
0 \leq k \leq \left\lfloor \frac{\alpha+1}{2} \right\rfloor \quad \text{and} \quad X_{\alpha, \alpha+1-k}(\beta) = \overline{X}_{\alpha,k}(\beta) \quad \text{for every} \quad 1 \leq k \leq \alpha.
\]

Theorem 3.5 can be written as follows:

**Proposition 3.8.** For every \( \alpha \in \mathbb{N} \), we have

\[
\begin{cases}
X_{\alpha,k}(\beta) \sim \left( -\frac{\alpha+1}{\beta} \right)^{\frac{1}{2}} e^{i \frac{(2k-\alpha-1)\pi}{\alpha+1}}, & \forall \ 0 \leq k \leq \alpha, \\
X_{\alpha,k}(\beta) \sim \left( \frac{\alpha(\alpha+1)}{1+\beta} \right)^{\frac{1}{2}} e^{i \frac{(2k-\alpha-1)\pi}{\alpha+1}}, & \forall \ 1 \leq k \leq \alpha, \\
X_{\alpha,0}(\beta) \sim \frac{1}{\alpha+1}.
\end{cases}
\]

The following figures (Figures 2 and 3) explain numerically the result of Proposition 3.8.

**Figure 2.** Graphs of \( X_{3,0} \) (in red) with asymptotic curves (to \( C_{X_{3,0}} \) in blue and to \( C_{X_{3,2}} \) in green)

### 3.3. Even and odd Bergman kernels

Following the idea of Krantz developed in [5], we consider the subspaces \( E_{\alpha,\beta}^2(D) \) and \( L_{\alpha,\beta}^2(D) \) of \( A_{\alpha,\beta}^2(D) \) generated respectively by the even \( (e_{2n})_{n \geq 0} \) and the odd \( (e_{2n+1})_{n \geq 0} \) sequences. Hence \( E_{\alpha,\beta}^2(D) \) and \( L_{\alpha,\beta}^2(D) \) are Hilbert subspaces of \( A_{\alpha,\beta}^2(D) \) formed respectively by even and odd functions. The reproducing Bergman kernels of these spaces are given by \( E_{\alpha,\beta}(z,w) = \mathcal{E}_{\alpha,\beta}(z\overline{w}) \) and \( L_{\alpha,\beta}(z,w) = \mathcal{L}_{\alpha,\beta}(z\overline{w}) \), where

\[
\mathcal{E}_{\alpha,\beta}(\xi) = \frac{1}{2} (K_{\alpha,\beta}(\xi) + K_{\alpha,\beta}(-\xi))
\]

\[
= \frac{1}{2(1-\xi^2)^{\alpha+2}} \left( (1+\xi)^{\alpha+2}Q_{\alpha,\beta}(\xi) + (1-\xi)^{\alpha+2}Q_{\alpha,\beta}(-\xi) \right)
\]
Again, to study the zeros of even and odd Bergman kernels, it suffices to study the zeros of the corresponding functions $I_{\alpha,\beta}$ and $J_{\alpha,\beta}$. Let $\varepsilon_{\alpha,\beta}$ (resp. $\Theta_{\alpha,\beta}$) be the number of zeros of the function $I_{\alpha,\beta}$ (resp. $J_{\alpha,\beta}$) in the unit disk $D$ counted with their multiplicities. To determine $\varepsilon_{\alpha,\beta}$ and $\Theta_{\alpha,\beta}$ in the case when $\alpha \in \mathbb{N}$, we start by the case $\beta = 0$. In this case it is easy to check that the zeros of $I_{\alpha,0}$ are given by $z_k := -i \tan \left( \frac{(2k+1)\pi}{2(\alpha+2)} \right)$ where $0 \leq k \leq \alpha + 1$ (we omit the value $k$ for which $\cos \left( \frac{(2k+1)\pi}{2(\alpha+2)} \right) = 0$ whenever $\alpha$ is odd). Similarly to the even case, the zeros of $J_{\alpha,0}$ are given by $w_k := -i \tan \left( \frac{k\pi}{\alpha+2} \right)$, $0 \leq k \leq \alpha + 1$. It follows that if $\alpha = 4\tau + r$ with $\tau \in \mathbb{N}$ and $0 \leq r \leq 3$, then

$$
\varepsilon_{\alpha,0} = \begin{cases} 2\tau & \text{if } r = 0, \\
2\tau + 2 & \text{if } 1 \leq r \leq 3, \end{cases} \quad \Theta_{\alpha,0} = \begin{cases} 2\tau + 1 & \text{if } 0 \leq r \leq 2, \\
2\tau + 3 & \text{if } r = 3. \end{cases}
$$

Proposition 3.9. Let $\alpha = 4\tau + r \in \mathbb{N}$ with $\tau \in \mathbb{N}$ and $0 \leq r \leq 3$.

(1) If $r \neq 0$, then
(a) There exists $-1 < \beta_4 < 0$ such that for every $\beta_4 < \beta \leq 0$ we have $\varepsilon_{\alpha,\beta} = \varepsilon_{\alpha,0}$. 

**Figure 3.** Graphs of $X_{\alpha,\bullet}$ (in red) with asymptotic curve to $C_{X_{\alpha,0}}$ (in green)
We claim that

\[
\alpha = 1
\]

\[
\beta \rightarrow 0
\]

\[
\eta \in \alpha
\]

\[
2
\]

\[
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\]

Proof. We claim that \( i \) are zeros of \( I_{\alpha,0} \) (resp. \( J_{\alpha,0} \)) when \( \alpha = 4\tau \) (resp. \( \alpha = 4\tau + 2 \)). For this reason we omit the corresponding values of \( \alpha \) in the proposition in order to use the Rouché theorem. Hence it suffices to study the convergence in terms of the parameter \( \beta \).

Thanks to Lemma 3.7, the family of polynomials \( (1+\beta)Q_{\alpha,\beta}(\xi)_{1<\xi<0} \) converges to 1 on \( \mathbb{D} \) when \( \beta \rightarrow 0^+ \) and to the polynomial \( (\alpha + 1)\xi \) when \( \beta \rightarrow (-1)^+ \). It follows that \( (1+\beta)I_{\alpha,\beta}(\xi) \) converges to \( I_{\alpha,0}(\xi) \) on \( \mathbb{D} \) as \( \beta \rightarrow 0^+ \) and to \( (\alpha + 1)\xi J_{\alpha,0}(\xi) \) on \( \mathbb{D} \) as \( \beta \rightarrow (-1)^+ \).

For the odd case, the family \( (1+\beta)J_{\alpha,\beta}(\xi) \) converges to \( J_{\alpha,0}(\xi) \) when \( \beta \rightarrow 0^+ \) and to \( (\alpha + 1)\xi J_{\alpha,0}(\xi) \) when \( \beta \rightarrow (-1)^+ \). Using the Rouché theorem the result follows.

To improve the previous result, we consider the number of zeros \( \tilde{\varepsilon}_{\alpha,0} \) (resp. \( \tilde{\Theta}_{\alpha,0} \)) of the function \( I_{\alpha,0} \) (resp. \( J_{\alpha,0} \)) in the closed unit disk \( \mathbb{D} \) given by

\[
\tilde{\varepsilon}_{\alpha,0} = 2\tau + 2
\]

\[
\tilde{\Theta}_{\alpha,0} = \begin{cases} 
2\tau + 1 & \text{if } 0 \leq r \leq 1, \\
2\tau + 3 & \text{if } 2 \leq r \leq 3,
\end{cases}
\]

where \( \alpha = 4\tau + r \). Using the same idea, one can prove the following corollary:

**Corollary 3.10.** Let \( \alpha \in \mathbb{N} \) and \( \eta_0 = \tan \left( \frac{\pi}{4} + \frac{\pi}{\alpha + 2} \right) \).

1. There exist \( -1 < \beta_3 < 0 \) such that for every \( \eta < \eta_0 \), the polynomial \( I_{\alpha,\beta}(\eta\xi) \) has exactly \( \tilde{\varepsilon}_{\alpha,0} \) zeros in \( \mathbb{D} \) for every \( \beta \in [\beta_1,0] \) and \( \tilde{\Theta}_{\alpha,0} + 1 \) zeros in \( \mathbb{D} \) for every \( \beta \in ]-1,\beta_4[ \).

2. There exist \( -1 < \beta_5 < 0 \) such that for every \( 1 < \eta < \eta_0 \), the polynomial \( J_{\alpha,\beta}(\eta\xi) \) has exactly \( \tilde{\Theta}_{\alpha,0} \) zeros in \( \mathbb{D} \) for every \( \beta \in [\beta_6,0] \) and \( \tilde{\varepsilon}_{\alpha,0} + 1 \) zeros in \( \mathbb{D} \) for every \( \beta \in ]-1,\beta_5[ \).

If the conditions of the previous proposition are satisfied, then one can take \( \eta = 1 \) in the corollary to obtain the same result given by the proposition.

### 4. Open problems

It is interesting to study the asymptotic distribution of zeros of \( G_{\alpha,\beta} \) when \( \alpha \in \mathbb{N} \) and goes to infinity. In other words, can we find a positive measure \( \mu \) such that the sequence of measures

\[
\mu_{\alpha,\beta} := \frac{1}{\alpha + 1} \sum_{j=0}^{\alpha+1} \delta_{X_{\alpha,j}(\beta)}
\]
converges weakly to the measure $\mu$ as $\alpha \to +\infty$? Geometrically, the distribution of the set $\{X_{\alpha,j}(\beta), 0 \leq j \leq \alpha + 1\}$ may depend on $\alpha$ in some non trivial way. For example, the distribution of sets $X_{\alpha,\bullet}(-10^{-4})$ for $\alpha = 31, 32, 33, 34, 36$ are similar (see Figure 4) however these are different to the one that correspond to $\alpha = 35$ (see Figure 5).

Can we find explicitly the equation of the parametric curve that describe the set $X_{\alpha,\bullet}(\beta)$? (This curve may be a circle in Figure 4 for $\alpha = 31, 32, 33, 34, 36$.) See also Figures 5 and 6.

![Figure 4](image)

**Figure 4.** The sets $X_{\alpha,\bullet}(-10^{-4})$ for $\alpha \in \{31, 32, 33, 34, 36\}$

It is simple to prove that for every $1 \leq k \leq \alpha$, there exists $t_{\alpha,k} \in ]-1,0[$ such that

$$|X_{\alpha,k}(t_{\alpha,k})| = \min_{-1<\beta<0} |X_{\alpha,k}(\beta)|$$

and satisfies

$$\sum_{j,k=0}^{\alpha+1} \binom{\alpha+1}{j} \binom{\alpha+1}{k} \frac{(-1)^{j+k}}{(j+t_{\alpha,k})^2} R_{\alpha,k}^{j+k} \cos(\theta_{\alpha,k}(j-k)) = 0$$

with $X_{\alpha,k}(t_{\alpha,k}) = R_{\alpha,k} e^{i\theta_{\alpha,k}}$.

One of the most important questions is to see if the critical value $t_{\alpha,k}$ of $\beta$ that realizes the minimum of $|X_{\alpha,k}(\beta)|$ doesn’t depend on $k$. It means that all functions attempt their minimums at the same “time”.

For even and odd kernels, can we prove Corollary 3.10 with $\eta = 1$? Indeed, if we show that the zeros of $I_{\alpha,\beta}(\xi)$ and $J_{\alpha,\beta}(\xi)$ that converges to $\pm i$ are in $\mathbb{D}$, then we conclude the result. We note that this fact is confirmed numerically for some values of $\alpha$. 
Figure 5. The sets $X_{\alpha}(\cdot(-10^{-4}))$ for $\alpha = 35$ (in violet) at left and $\alpha = 49$ (in red) and $\alpha = 51$ (in blue) at right.

Figure 6. The set $X_{101}(\cdot(-10^{-4}))$

Annex: Numerical results

All figures of this paper were produced using Python software. We give here the used code.
from scipy import special
from sympy.abc import x, y, z
def A(beta, alpha):
    s=0
    for j in range(0, alpha+2):
        s=s+ (((1)/(j+beta))*special.binom(alpha+1,j)*(-x)**j)
    return s
import numpy
from cmath import *
from sympy.solvers import solve
import csv
DATA_PATH = '/content/drive/My Drive/graphes data/alpha7_7.csv'
i=1
with open(DATA_PATH, mode='w', newline='') as points_file:
    points_writer = csv.writer(points_file, delimiter=',')
    for beta in numpy.arange(10**(-6), 1, 10**(-2)):
        row = []
        for s in solve(A(beta, 6), x):
            sol = complex(s)
            row.append(-beta)
            row.append(sol.real)
            row.append(sol.imag)
        points_writer.writerow(row)
        print(i)
i=i+1
i=1
for beta in numpy.arange(1-10**(-2), 1, 10**(-3)):
    row = []
    for s in solve(A(beta, 6), x):
        sol = complex(s)
        row.append(-beta)
        row.append(sol.real)
        row.append(sol.imag)
    points_writer.writerow(row)
    print(i)
i=i+1

**********************************************

In Figures 4, 5, 6, we present some zeros sets of \( G_{\alpha,\beta} \) for \( \beta = -10^{-4} \) and \( \alpha \in \{31, 32, 33, 34, 35, 36, 49, 51, 101\} \). The values of \( \alpha \) and \( \beta \) are chosen arbitrary just to see that there is no geometric stability of these zeros. It is possible that the geometric distribution of zeros of \( G_{\alpha,\beta} \) depends on both \( \alpha \) and \( \beta \) in a complicate manner. It is also possible that if we see numerically the geometric
distribution of zeros of $G_{\alpha,\beta}$ for $\alpha$ large enough, then some new limit curve appear. However as a material problem, it was not possible for us to exceed the value $\alpha = 101$.

References


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