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SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY LOMMEL OPERATOR

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Abstract. We use convolution techniques to define certain classes of starlike functions which are associated with Lommel operator. Some inclusion results are investigated. It is also shown that these classes are invariant under Bernardi integral operator.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

analytic in the open unit disc $E = \{z : |z| < 1\}$.

Let $f(z)$ be given by (1) and $g(z)$ defined as:

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

The Hadamard product (or convolution) of f and g is defined as:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

For f and g analytic in E we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwartz function $\omega(z)$ analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in E$ such that $f(z) = g(\omega(z))$. If g is univalent in E, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$. Using subordination concept we generalize the class $P(\delta)$ as: Let $p(z)$ be analytic in E with $p(0) = 1$. Then $p \in P(\delta)$, $0 \le \delta < 1$ if and only if $Rep(z) > \delta$, $z \in E$. The class $P(0) = P$ is well known class of Carathedory functions of positive real part. Let p be analytic in E with $p(0) = 1$. Then $p \in P(h)$ if $p(z) \prec h(z)$. We note that $p \in P(\delta)$ if

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$p(z) \prec \frac{1+(1-2\delta)z}{1-z} = h(z)$. From [5], if $m \geq 2$, $p \in P_m(h)$ if and only if there exist p_1, p_2 such that

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z),\ p_i \prec h,\ i = 1, 2.$$

It is obvious that $P_2\left(\frac{1+z}{1-z}\right) = P$.

Let $\omega_{\mu,\nu}(z)$ be a particular solution of the inhomogeneous Bessel differential equation:

$$z^2\omega^{''}(z) + z\omega^{'}(z) + (z^2 - \upsilon^2)\omega(z) = z^{\mu+1},$$

which can be expressed as:

$$\omega_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu - \upsilon + 1)(\mu - \upsilon + 1)}\ {}_2F_1\left(\frac{\mu - \nu + 3}{2}, \frac{\mu - \nu + 3}{2}; \frac{-z^2}{4}\right), \tag{2}$$

where ${}_2F_1$ denotes hypergeometric function and $\mu \pm \nu$ are not negative odd integers. The function represented by (2) is called the Lommel function of the first kind.

Let $f_{\mu,\nu}(z)$ be the normalized form of $\omega_{\mu,\nu}(z)$ which is expressed in [1,8] as:

$$\begin{aligned} f_{\mu,\nu}(z) &= (\mu - \upsilon + 1)(\mu - \upsilon + 1)z^{\left(\frac{1+\mu}{2}\right)}\omega_{\mu,\nu}(\sqrt{z}) \\ &= \sum_{n=1}^{\infty} \frac{(-\frac{1}{4})^{n-1}}{(\frac{\mu-\nu+3}{2})_{n-1}(\frac{\mu+\nu+3}{2})_{n-1}} z^n,\ \left(\frac{\mu \pm \nu + 3}{2} \notin \mathbb{N}\right), \end{aligned} \tag{3}$$

where $(\gamma)_n$ is Pochhamer symbol defined as:

$$(\gamma)_n = (\gamma)(\gamma + 1) \cdots (\gamma + n - 1).$$

Using (3) and convolution a linear operator $L_{\mu,\nu} : A \to A$ is defined in [7] by:

$$L_{\mu,\nu}f(z) = f_{\mu,\nu}(z) * f(z),\ \left(\frac{\mu \pm \nu + 3}{2} \notin \mathbb{N},\ z \in E,\ f \in A\right). \tag{4}$$

From (3) and (4) the following identity holds:

$$z\left(L_{\mu+1,\nu+1}(f(z))\right)' = \left(\frac{\mu+\nu+3}{2}\right) L_{\mu,\nu}f(z) - \left(\frac{\mu+\nu+1}{2}\right) L_{\mu+1,\nu+1}f(z). \tag{5}$$

We now define the following classes:

$$S^*(\mu, \upsilon; h) = \left\{ f \in A : F = L_{\mu,\nu}(f); \frac{zF^{'}}{F} \prec h,\ z \in E \right\},$$

$$C(\mu, \upsilon; h) = \{ f \in A : zf' \in S^*(\mu, \upsilon; h),\ z \in E \}$$

and

$$M_\sigma(\mu, \upsilon; h) = \left\{ f \in A : F = L_{\mu,\nu}(f); \left[(1 - \sigma)\frac{zF^{'}(z)}{F(z)} + \sigma\frac{\left(zF^{'}(z)\right)^{'}}{F^{'}(z)}\right] \prec h(z) \right\},$$

where $\sigma \geq 0$ and $F(z) \cdot F^{'}(z) \neq 0$. It is clear that $M_0(\mu, \upsilon; h) = S^*(\mu, \upsilon; h)$ and $M_1(\mu, \upsilon; h) = C(\mu, \upsilon; h)$.

Let $f \in A$ and let $L_{\mu,\nu} : A \to A$ be defined by (3) and (4). Then $f \in K_m(\mu, \upsilon; h)$ ($m \geq 2$, $h \in P$) if and only if there exists $g \in S^*(\mu, \upsilon; h)$ such that $\frac{z(L_{\mu,\nu} f(z))'}{L_{\mu,\nu} g(z)} \in P_m(h)$. We note that $F = L_{\mu,\nu} f$ is close-to-convex univalent in E for $m = 2$.

2. Preliminary results

We need the following lemmas to prove our results:

Lemma 2.1 ([4])**.** *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ with $u_1, u_2, v_1, v_2 \in \mathbb{R}$ and $D \subset \mathbb{C}^2$. Suppose $\psi : D \to \mathbb{C}$ satisfies the following conditions*:

(i) *$\psi(u, v)$ is continuous in D.*

(ii) *$(1, 0) \in D$ and $\Re\{\psi(1, 0)\} > 0$.*

(iii) *$\Re\{\psi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{(1+u_2^2)}{2}$.*

Let p be an analytic function in E with $p(0) = 1$ and $(p(z), zp'(z)) \in D$ for all $z \in E$. If $\Re\{\psi(p(z), zp'(z))\} > 0$ in E, then $\Re(p(z)) > 0$.

Lemma 2.2 ([3])**.** *Let h be convex univalent in E and h_0 be analytic in E with $\Re\{h_0(z)\} > 0$ in E. If p is analytic in E and $p(0) = h(0)$, then the subordination*

$$p(z) + h_0(z) \cdot zp'(z) \prec h(z), \;\; z \in E$$

implies that $p(z) \prec h(z)$ in E.

Lemma 2.3 ([7])**.** *Let μ and ν be real numbers such that $\mu \pm \upsilon$ are not integers with $\mu > 2$,*

$$(\mu + 1)\left[(\mu + 1)(\mu + 1) - \upsilon^2\right] \geq \frac{1}{8}$$

and

$$\begin{cases} \frac{1}{96}(\mu - 2)^{-1} + (\mu - 2) - \frac{1}{4}\left((\mu - 1)^2 - \upsilon^2\right) \leq 0 & \text{if } \mu \leq \frac{25}{12}, \\ -\frac{1}{4}\mu + \frac{5}{4} - \frac{1}{4}\left((\mu - 1)^2 - \upsilon^2\right) \leq 0 & \text{if } \mu > \frac{25}{12}. \end{cases}$$

Then the function $f_{\mu,\nu}(z)$ defined by (3) is convex in E.

Lemma 2.4 ([4])**.** *Let h be convex in E with $\Re[\beta h(z) + \gamma] > 0$. If p is analytic in E with $p(0) = h(0)$, then $p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z)$.*

3. Main results

Theorem 3.1. *Let μ and υ be real numbers such that $\mu \pm \upsilon$ are not negative odd integers. If h is convex in E with $\Re(h(z) + \gamma) > 0$, $\gamma = \frac{\mu + \upsilon + 1}{2}$, then*

(i) *$S^*(\mu, \upsilon; h) \subset S^*(\mu + 1, \upsilon + 1; h)$.*

(ii) *In particular, if* $f \in S^*\left(\mu, \upsilon; \frac{1+(1-2\beta)z}{1-z}\right)$ *and* $2\beta+\mu+\upsilon+1 \geq 0$, $\beta \in [0,1)$, *then* $f \in S^*\left(\mu+1, \upsilon+1; \frac{1+(1-2\alpha)z}{1-z}\right)$, *where*

$$\alpha = \frac{2(1+2\beta\gamma)}{\sigma + \sqrt{\sigma^2 + 8(1+2\beta\gamma)}}, \sigma = 2\gamma - 2\beta + 1. \tag{6}$$

Proof. (i) Let $f \in S^*(\mu, \upsilon; h)$. Define

$$\frac{z\left(L_{\mu+1,\upsilon+1}f(z)\right)'}{L_{\mu+1,\upsilon+1}f(z)} = H(z), \tag{7}$$

where $H(z)$ is analytic in E with $H(0) = 1$.

Using identity (5) in (7) we get:

$$\frac{L_{\mu,\upsilon}f(z)}{L_{\mu+1,\upsilon+1}f(z)} = \frac{2}{\mu+\upsilon+3}\left[H(z) + \frac{\mu+\upsilon+1}{2}\right]. \tag{8}$$

After applying logarithmic differentiation on (8) and with some simple computations and the supposition $f \in S^*(\mu, \upsilon; h)$ it follows that,

$$H(z) + \frac{zH'(z)}{H(z)+\gamma} \in P(h), \ \gamma = \frac{\mu+\upsilon+1}{2}. \tag{9}$$

Applying Lemma 2.4 to (9), it follows that $H(z) \prec h(z)$. Consequently by virtue of (7), $f \in S^*(\mu+1, \upsilon+1; h)$ in E and (i) is established.

(ii) Let $h(z) = \frac{1+(1-2\beta)z}{1-z}$. Then

$$H(z) + \frac{zH'(z)}{H(z)+\gamma} \in P(\beta).$$

With $H(z) = (1-\alpha)p(z) + \alpha$, we have

$$\Re\left\{(1-\alpha)p(z) + (\alpha-\beta) + \frac{(1-\alpha)zp'(z)}{(1-\alpha)p(z)+\alpha+\gamma}\right\} > 0, \ z \in E.$$

We now construct the functional $\psi(u,v)$ in Lemma 2.1 by taking $u = p(z)$, $v = zp'(z)$ as:

$$\psi(u,v) = (1-\alpha)u + (\alpha-\beta) + \frac{v}{u + \frac{\alpha+\gamma}{1-\alpha}}.$$

The first two conditions of Lemma 2.1 are easily verified. We check condition (iii) as follows:

$$\begin{aligned}
\Re\left\{\psi(iu_2, v_1)\right\} &= (\alpha-\beta) + \Re\left(\frac{v_1}{iu_2 + \frac{\alpha+\gamma}{1-\alpha}}\right) \\
&\leq (\alpha-\beta) - \frac{1}{2}\frac{\frac{\alpha+\gamma}{1-\alpha}(1+u_2^2)}{u_2^2 + (\frac{\alpha+\gamma}{1-\alpha})^2} \text{ as } v_1 \leq -\frac{1}{2}(1+u_2^2) \\
&= \frac{2(\alpha-\beta)\left[u_2^2+\gamma_1^2\right] - \gamma_1(1+u_2^2)}{2\left[u_2^2+\gamma_1^2\right]}, \ \gamma_1 = \frac{\alpha+\gamma}{1-\alpha}.
\end{aligned} \tag{10}$$

Let $A = 2(\alpha-\beta)\gamma_1^2 - \gamma_1$, $B = 2(\alpha-\beta) - \gamma_1$ and $C = u_2^2 + \gamma_1^2$. So from (10) we have

$$\Re\{\psi(iu_2, v_1)\} \leq \frac{A + Bu_2^2}{2C}. \tag{11}$$

Left hand side of inequality (11) is negative if both $A \leq 0$, $B \leq 0$. From $A \leq 0$ we get α as given by (6) and $B \leq 0$ ensures that $\alpha \in [0,1)$. We now apply Lemma 2.1 to conclude that $H \in P(\alpha)$ and hence

$$f \in S^*\left(\mu+1, \upsilon+1; \frac{1+(1-2\alpha)z}{1-z}\right).$$

□

Corollary 3.2. *$S^*\left(\mu, \upsilon; \frac{1+z}{1-z}\right) \subset S^*\left(\mu+1, \upsilon+1; \frac{1+(1-2\alpha_1)z}{1-z}\right)$, where*

$$\alpha_1 = \frac{2}{(2\gamma+1) + \sqrt{(2\gamma+1)^2 + 8}}, \gamma = \frac{\mu+\upsilon+1}{2}.$$

For $\mu = 3$, $\upsilon = 1$ we have

$$S^*\left(3, 1; \frac{1+z}{1-z}\right) \subset S^*\left(4, 2; \frac{1+(1-2\alpha_2)z}{1-z}\right),$$

where $\alpha_2 = \frac{1}{3+\sqrt{11}}$.

Corollary 3.3. *$S^*\left(1, 0; \frac{1+z}{1-z}\right) \subset S^*\left(2, 1; \frac{1+(1-2\alpha_3)z}{1-z}\right)$, where $\alpha_3 = \frac{2}{3+\sqrt{17}}$.*

By virtue of the linearity of the operator $L_{\mu,\upsilon}$ it follows $f \in C(\mu,\nu;h) \Leftrightarrow zf' \in S^*(\mu,\nu;h)$. Using this fact and Theorem 3.1 we have:

Theorem 3.4. *Suppose μ, υ are real numbers and $h(z)$ be convex in E with $h(0) = 1$, $\mu+\upsilon+1 \geq 0$ and $\mu \pm \upsilon$ are not negative odd integers. Then*

$$C(\mu,\nu;h) \subset C(\mu+1,\nu+1;h).$$

Theorem 3.5. *For $\sigma \geq 0$, $M_\sigma(\mu,\nu;h) \subset S^*(\mu,\nu;h)$.*

Proof. The case $\sigma = 0$ is trivial. Let us consider $\sigma > 0$ and suppose $L_{\mu,\upsilon}f = F$. Then $f \in M_\sigma(\mu,\nu;h)$ implies

$$\left\{(1-\sigma)\frac{zF'(z)}{F(z)} + \sigma\frac{(zF'(z))'}{F'(z)}\right\} \prec h(z).$$

Let

$$\frac{zF'(z)}{F(z)} = H(z).$$

Then

$$\frac{(zF'(z))'}{F'(z)} = H(z) + \frac{zH'(z)}{H(z)}$$

and therefore

$$(1-\sigma)\frac{zF'(z)}{F(z)}+\sigma\frac{(zF'(z))'}{F'(z)}=H(z)+\sigma\frac{zH'(z)}{H(z)}.$$

Now using a well known result due to Miller and Mocanu [4, p. 103], it follows that

$$H(z)=\frac{zF'(z)}{F(z)}\prec q(z)\prec h(z),$$

where $q(z)$ is the best dominant and is given by $q(z)=\left[\frac{k(z)}{K(z)}\right]^{\frac{1}{\sigma}}$, where

$$k(z)=z\exp\int_0^z\frac{h(t)-1}{t}dt$$

and

$$K(z)=\frac{1}{\sigma}\left(\int_0^z k^{\frac{1}{\sigma}}(t)t^{-1}dt\right)^{\sigma},\ \sigma>0.$$

□

Theorem 3.6. *For* $\sigma\geq 1$, $M_\sigma(\mu,\nu;h)\subset C(\mu,\nu;h)$.

Proof. Let $L_{\mu,\upsilon}f=F$. Then we can write

$$(1-\sigma)\frac{zF'(z)}{F(z)}+\sigma\frac{(zF'(z))'}{F'(z)}=p(z),\ p\prec h.$$

That is,

$$\frac{(zF'(z))'}{F'(z)}=\left(1-\frac{1}{\sigma}\right)H(z)+\frac{1}{\sigma}p(z),\ (\sigma\geq 1).$$

As $H(z)\prec h(z)$ by Theorem 3.5 and $p(z)\prec h(z)$ by given hypothesis. Therefore

$$\left(1-\frac{1}{\sigma}\right)H(z)+\frac{1}{\sigma}p(z)\prec h(z),$$

which completes the proof. □

Theorem 3.7. *The class* $S^*(\mu,\nu;h)$ *is preserved under the Bernardi operator given as*:

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt,\ c=1,2,3,\dots. \tag{12}$$

Proof. We can write (12) as:

$$\begin{aligned}F_{\mu,\upsilon}(z)=L_{\mu,\upsilon}F(z)&=\frac{c+1}{z^c}\int_0^z t^{c-1}L_{\mu,\upsilon}f(t)dt\\&=\sum_{n=1}^{\infty}\frac{c+1}{n+c}z^n*L_{\mu,\upsilon}f(z).\end{aligned}$$

It has been shown in [2] that $\left(\sum_{n=1}^{\infty} \frac{c+1}{n+c} z^n\right)$ is convex in E and since $L_{\mu,\upsilon} f \in S^*(h)$. It follows from the result in [6] that $F_{\mu,\upsilon} \in S^*(h)$ in E. This completes the proof. □

Theorem 3.8. *For* $m \geq 2$,

$$K_m(\mu, \upsilon; h) \subset K_m(\mu + 1, \upsilon + 1; h).$$

Proof. Let $f \in K_m(\mu, \upsilon; h)$. Then there exists $g \in S^*(\mu, \upsilon; h)$ such that

$$\frac{z\left(L_{\mu,\nu} f(z)\right)'}{L_{\mu,\nu} g(z)} \in P_m(h). \tag{13}$$

Using identity (5) in (13) we get

$$\begin{aligned} \frac{z\left(L_{\mu,\nu} f(z)\right)'}{L_{\mu,\nu} g(z)} &= \frac{z L_{\mu,\nu}(z f'(z))}{L_{\mu,\nu} g(z)} \\ &= \frac{z L_{\mu+1,\nu+1}(z f'(z))' + \left(\frac{\mu+\nu+1}{2}\right) L_{\mu+1,\nu+1}(z f'(z))}{z(L_{\mu+1,\nu+1} g(z))' + \left(\frac{\mu+\nu+1}{2}\right) L_{\mu+1,\nu+1} g(z)} \\ &= \frac{\frac{z L_{\mu+1,\nu+1}(z f'(z))'}{L_{\mu+1,\nu+1} g(z)} + \left(\frac{\mu+\nu+1}{2}\right) \frac{L_{\mu+1,\nu+1}(z f'(z))}{L_{\mu+1,\nu+1} g(z)}}{\frac{z(L_{\mu+1,\nu+1} g(z))'}{L_{\mu+1,\nu+1} g(z)} + \left(\frac{\mu+\nu+1}{2}\right)}. \end{aligned} \tag{14}$$

Set

$$\frac{z L_{\mu+1,\nu+1}(z f'(z))'}{L_{\mu+1,\nu+1} g(z)} = p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z) \tag{15}$$

and

$$\frac{z(L_{\mu+1,\nu+1} g(z))'}{L_{\mu+1,\nu+1} g(z)} = p_0(z). \tag{16}$$

If we consider a special case of Theorem 3.1, it follows $p_0 \in P$. Since

$$Re(p_0(z)) = \Re\left\{\frac{z(L_{\mu+1,\nu+1} g(z))'}{L_{\mu+1,\nu+1} g(z)}\right\} > 0,$$

we have

$$Re(h_0(z)) = \Re\left\{\frac{1}{p_0(z) + \frac{\mu+\nu+1}{2}}\right\} > 0. \tag{17}$$

Thus with some computations using (14), (15), (16) and (17) we get:

$$[p_i(z) + h_0(z).z p_i'(z)] \prec h(z), \; i = 1, 2. \tag{18}$$

Now we apply Lemma 2.2, it follows from (18) that $p_i \prec h$, $i = 1, 2$ for $z \in E$ and consequently $p \in P_m(h)$. This completes the proof that $f \in K_m(\mu + 1, \upsilon + 1; h)$. □

4. Concluding remarks

Let μ and υ satisfy the conditions of Lemma 2.3. Then $f_{\mu,\nu}(z)$ given by (3) is convex in E and from this the following observations can easily be deduced:

(i) $S^*(\mu, \upsilon; h) \subset S^*(h)$, $C(\mu, \upsilon; h) \subset C(h)$.

(ii) $M_\sigma(\mu, \nu; h) \subset S^*(h)$, $\sigma \geq 0$. $M_\sigma(\mu, \nu; h) \subset C(h)$, $\sigma \geq 1$. That is for μ, υ write conditions of Lemma 2.3 f belonging to the classes $S^*(\mu, \upsilon; h), C(\mu, \upsilon; h)$ and $M_\sigma(\mu, \nu; h)$ is starlike in E.

(iii) $f \in K_2\,(\mu, \upsilon; h)$ is close-to-convex in E.

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