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GEOMETRIC PROPERTIES ON (j,k)-SYMMETRIC FUNCTIONS RELATED TO STARLIKE AND CONVEX FUNCTION

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Abstract. For $j = 0, 1, 2, \ldots, k-1$; $k \geq 2$; and $-1 \leq B < A \leq 1$, we have introduced the functions classes denoted by $\mathcal{ST}_{[j,k]}(A,B)$ and $\mathcal{K}_{[j,k]}(A,B)$, respectively, called the generalized (j,k)-symmetric starlike and convex functions. We first proved the sharp bounds on $|f(z)|$ and $|f'(z)|$. Various radii related problems, such as radius of (j,k)-symmetric starlikeness, convexity, strongly starlikeness and parabolic starlikeness are determined. The quantity $|a_3^2 - a_5|$, which provide the initial bound on Zalcman functional is obtained for the functions in the family $\mathcal{ST}_{[j,k]}$. Furthermore, the sharp pre-Schwarzian norm is also established for the case when f is a member of $\mathcal{K}_{[j,k]}(\alpha)$ for all $0 \leq \alpha < 1$.

1. Introduction and preliminaries

We denote by $\mathcal{H}$, the family of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the condition of normalization $f(0) = f'(0) - 1 = 0$. Thus, all functions in the family $\mathcal{H}$ have the Taylor-Maclaurin series of the form:

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad (a_1 = 1, z \in \mathbb{D}). \tag{1.1}$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{H}$ that are univalent in $\mathbb{D}$. Suppose that $f, g \in \mathcal{H}$, then f is subordinate to g in $\mathbb{D}$ (cf. [12]) denoted by $f \prec g$ if there exists a function $w \in \mathcal{H}$ satisfies the Schwarz condition $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{D}).$$

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Let $\mathcal{P}[A, B]$, with $-1 \leq B < A \leq 1$, denote the class of functions p analytic in $\mathbb{D}$ that satisfy the subordination relation $p(z) \prec \frac{1+Az}{1+Bz}$ and has the series of the form:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \qquad (z \in \mathbb{D}),$$

with $p(0) = 1, \Re(p(z)) > 0$. This class was introduced and studied by Janowski [8]. In particular, $\mathcal{P}[1, -1] := \mathcal{P}$ is the usual Caratheodory class. Most important basic subclasses of the family $\mathcal{S}$ includes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively, the family of univalent starlike functions and convex functions of order α in $\mathbb{D}$ for all $0 \leq \alpha < 1$. For fixed integer $k \geq 1$, corresponding k-fold symmetric function (cf. [16] also see [3, 22]), denoted by $f_k(z)$, is of the form

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \frac{f(\epsilon^\nu z)}{\epsilon^\nu}, \quad (z \in \mathbb{D}), \tag{1.2}$$

where $\epsilon = e^{2\pi i/k}$. The set $\mathcal{S}^k$ denotes all such space of k-fold symmetric univalent functions. Similarly, (j, k)-symmetric functions is defined over a k-fold symmetric set Ω as $f : \Omega \to \mathbb{C}$ if $f(z) = \frac{1}{\epsilon^j} f(\epsilon z)$ for all $z \in \Omega$, $\epsilon = e^{\frac{2\pi i}{k}}$ and $j = 0, 1, 2, \ldots, k-1; k \geq 2$. Here we have considered the domain Ω as $\mathbb{D}$. $\mathcal{S}^{(j,k)}$ denoted as the space of (j, k)-symmetrical functions. In particular, $\mathcal{S}^{(0,2)}, \mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are known as the family of even, odd, and k-fold symmetric functions, respectively. The above family $\mathcal{S}^{(j,k)}$ is introduced and studied in [11]. For more recent work on the related class we refer [18] and the references therein. Every function $f_{j,k}$ is in the class $\mathcal{S}^{(j,k)}$ has the series representation:

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z), \quad (z \in \mathbb{D}),$$

where

$$f_{j,k}(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \epsilon^{-\nu j} f(\epsilon^\nu z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \epsilon^{-\nu j} \left(\sum_{n=1}^{\infty} a_n (\epsilon^\nu z)^n \right) \quad (z \in \mathbb{D}). \tag{1.3}$$

Further it is easy to see that

$$f_{j,k}(z) = \sum_{n=1}^{\infty} \psi_{n,j}\ a_n z^n, \quad (a_1 = 1, z \in \mathbb{D}) \tag{1.4}$$

where $\psi_{n,j} = \frac{1}{k} \sum_{\nu=0}^{k-1} \epsilon^{(n-j)\nu} = \begin{cases} 1; & n = lk + j, \\ 0; & n \neq lk + j \end{cases}$ $(l \in \mathbb{N}_0)$ and note that $f_{0,2}(z) = f(z)$. Motivated by the above concept, we mainly considered the functions class $\mathcal{ST}_{[j,k]}(A, B)$ and $\mathcal{K}_{[j,k]}(A, B)$, respectively called the family of generalized (j, k)-symmetric starlike and convex functions, and defined in term of subordination as follows.

Definition 1. A function $f \in \mathcal{H}$ is said to be in the class of $\mathcal{ST}_{[j,k]}(A,B)$, $(-1 \leq B < A \leq 1)$ if the following subordination condition satisfied:

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1+Az^{j+k-1}}{1+Bz^{j+k-1}}, \quad (z \in \mathbb{D},\ j=0,1,2,\ldots,k-1;\ k \geq 1). \tag{1.5}$$

Definition 2. A function $f \in \mathcal{H}$ is said to be in the class of $\mathcal{K}_{[j,k]}(A,B)$, $(-1 \leq B < A \leq 1)$ if the following subordination condition satisfied:

$$\frac{(zf'(z))'}{f'_{j,k}(z)} \prec \frac{1+Az^{j+k-1}}{1+Bz^{j+k-1}}, \quad (z \in \mathbb{D},\ j=0,1,2,\ldots,k-1;\ k \geq 1). \tag{1.6}$$

For the various choices of the parameters the above defined families unifies various classical as well as recently studied subclasses of $\mathcal{S}$. In Table 1, we provide some relevant connections for the family $\mathcal{ST}_{[j,k]}(A,B)$. Here, for an abbreviation, we set $p(z) = \frac{zf'(z)}{f_{j,k}(z)}$.

TABLE 1. For the certain values of j,k,A and B, the class $\mathcal{ST}_{[j,k]}(A,B)$ is represents in terms of subordination.

$\mathcal{ST}_{[j,k]}(A,B)$	Subordination relation	Relevant connections
$\mathcal{ST}_{[1,k]}(A,B) =: \mathcal{S}^*_k(A,B)$	$p(z) \prec \frac{1+Az^k}{1+Bz^k}$	[2]
$\mathcal{ST}_{[1,1]}(A,B) =: \mathcal{S}^*(A,B)$	$p(z) \prec \frac{1+Az}{1+Bz}$	[8]
$\mathcal{ST}_{[1,1]}(1,-1) =: \mathcal{S}^*$	$p(z) \prec \frac{1+z}{1-z}$	[13]
$\mathcal{ST}_{[j,k]}(1-2\alpha,-1) =: \mathcal{ST}_{[j,k]}(\alpha); \alpha \in [0,1)$	$p(z) \prec \frac{1+(1-2\alpha)z^{j+k-1}}{1-z^{j+k-1}}$	[18]
$\mathcal{ST}_{[1,1]}(1-2\alpha,-1) =: \mathcal{S}^*(\alpha); \alpha \in [0,1)$	$p(z) \prec \frac{1+(1-2\alpha)z}{1-z}$	[15]
$\mathcal{ST}_{[1,1]}(1,0) =: \mathcal{S}^*(1,0)$	$p(z) \prec 1+z$	[19]
$\mathcal{ST}_{[1,1]}(e^{i\beta}(e^{i\beta}-2\alpha\cos\beta),-1) =: \mathcal{S}_\beta(\alpha)$, $\alpha \in [0,1)$ and $\beta \in (-\pi/2,\pi/2)$,	$p(z) \prec \frac{1+e^{i\beta}(e^{i\beta}-2\alpha\cos\beta)z}{1-z}$	[20]
$\mathcal{ST}_{[1,1]}(A,B) =: \mathcal{S}^{(j,k)}(A,B)$	$p(z) \prec \frac{1+Az}{1+Bz}$	[17]

Organization of the paper is as follows. In Section 2, we first provide the sharp bounds on the growth and distortion problems. Various radii related problems for the functions class $\mathcal{ST}_{[j,k]}(A,B)$ and $\mathcal{K}_{[j,k]}(A,B)$ are discussed in Section 3. Sharp Zalcman-type bounds for $\mathcal{ST}_{[j,k]}$ and pre-Schwarzian norm for functions in $\mathcal{K}_{[j,k]}(\alpha)$ are illustrated in Sections 4 and 5, respectively.

2. Main results

Lemma 2.1. *Suppose that $p(z)$ is analytic in $\mathbb{D}$. Then $p(z) \prec \frac{1+Az^{j+k-1}}{1+Bz^{j+k-1}}$ if and only if*

$$\left| p(z) - \frac{1-ABr^{2(j+k-1)}}{1-B^2r^{2(j+k-1)}} \right| \leq \begin{cases} \frac{(A-B)r^{j+k-1}}{1-B^2r^{2(j+k-1)}}; & B \neq 0; \\ Ar^{j+k-1}; & B = 0, \ (|z| \leq r < 1). \end{cases} \tag{2.1}$$

The estimate (2.1) *is sharp bound for* $|z| = r$.

Proof. Define $w(z)$ by

$$w(z) = \frac{1 + Az^{j+k-1}}{1 + Bz^{j+k-1}}, \ j = 0, 1, 2, \dots; k \geq 1, \ z \in \mathbb{D}.$$

This implies that

$$w(z) + Bz^{j+k-1} w(z) = 1 + Az^{j+k-1}.$$

Taking modulus on both sides of the above equation and squaring the resulting equation yields

$$|w(z)|^2 - 2\left[w(z)\left(\frac{1 - ABr^{2(j+k-1)}}{1 - B^2 r^{2(j+k-1)}}\right)\right] \leq \frac{A^2 r^{2(j+k-1)} - 1}{1 - B^2 r^{2(j+k-1)}}.$$

Further computation gives

$$|w(z)|^2 - 2\left[w(z)\left(\frac{1 - ABr^{2(j+k-1)}}{1 - B^2 r^{2(j+k-1)}}\right)\right] + \left|\frac{1 - ABr^{2(j+k-1)}}{1 - B^2 r^{2(j+k-1)}}\right|^2$$
$$\leq \frac{A^2 r^{2(j+k-1)} - 1}{1 - B^2 r^{2(j+k-1)}} + \left|\frac{1 - ABr^{2(j+k-1)}}{1 - B^2 r^{2(j+k-1)}}\right|^2.$$

This implies that

$$\left|w(z) - \frac{1 - ABr^{2(j+k-1)}}{1 - B^2 r^{2(j+k-1)}}\right| \leq \frac{(A - B) r^{j+k-1}}{1 - B^2 r^{2(j+k-1)}}, \ (|z| \leq r < 1).$$

Hence using the application of subordination relation we have

$$p(z) \prec \frac{1 + Az^{j+k-1}}{1 + Bz^{j+k-1}}$$

if and only if

$$\left|p(z) - \frac{1 - ABr^{2(j+k-1)}}{1 - B^2 r^{2(j+k-1)}}\right| \leq \frac{(A - B) r^{j+k-1}}{1 - B^2 r^{2(j+k-1)}}, \qquad (B \neq 0, |z| \leq r < 1).$$

For the case $B = 0$, then

$$p(z) \prec 1 + Az^{j+k-1}$$

if and only if

$$|p(z) - 1| \leq Ar^{j+k-1}, \qquad (|z| \leq r < 1).$$

This completes the proof of Lemma 2.1. □

Remark 2.2. The family $\mathcal{ST}_{[j,k]}(A, B)$ is defined under the condition $-1 \leq B < A \leq 1$ such that the set of values of the functional $\frac{zf'(z)}{f_{j,k}(z)}$ lies in the right side of the half plane inside the disk with center $\frac{1-ABr^{2(j+k-1)}}{1-B^2r^{2(j+k-1)}}$ and radius $\frac{(A-B)r^{j+k-1}}{1-B^2r^{2(j+k-1)}}$. So that $\mathcal{ST}_{[j,k]}(A, B) \subset \mathcal{S}^*$ whenever $-1 \leq B < A \leq 1$.

Remark 2.3. The family $\mathcal{K}_{[j,k]}(A,B)$ is defined under the condition $-1 \le B < A \le 1$ such that the set of values of the functional and $\frac{(zf'(z))'}{f'_{j,k}(z)}$ lies in the right side of the half plane inside the disk with center $\frac{(B-A)r^{2(j+k-1)}}{1-B^2r^{2(j+k-1)}}$ and radius $\frac{(A-B)r^{j+k-1}}{1-B^2r^{2(j+k-1)}}$. So that $\mathcal{K}_{[j,k]}(A,B) \subset \mathcal{K}$ whenever $-1 \le B < A \le 1$.

The next theorem based on growth and distortion theorems for the family $\mathcal{ST}_{[j,k]}(A,B)$ and $\mathcal{K}_{[j,k]}(A,B)$.

Theorem 2.4. *If* $f(z) \in \mathcal{K}_{[j,k]}(A,B)$, *then*

$$L(r) \le |f'_{j,k}(r)| \le H(r),$$

where

$$L(r) := \left(1+Br^{j+k-1}\right)^{\frac{-(1+B)(A-B)}{2B^2(j+k-1)}}\left(1-Br^{j+k-1}\right)^{\frac{-(1-B)(A-B)}{2B^2(j+k-1)}},$$
$$H(r) := \left(1+Br^{j+k-1}\right)^{\frac{(B-1)(A-B)}{2B^2(j+k-1)}}\left(1-Br^{j+k-1}\right)^{\frac{(-1-B)(A-B)}{2B^2(j+k-1)}}.$$

Proof. Suppose $f(z) \in \mathcal{K}_{[j,k]}(A,B)$. Then

$$\Re\left\{\frac{(zf'(z))'}{f'_{j,k}(z)}\right\} \ge 0, \quad (z \in \mathbb{D}). \tag{2.2}$$

Therefore, upon replacing z by $\epsilon^\nu z$ in (2.2), we get

$$\Re\left\{\frac{f'(\epsilon^\nu z)+z\epsilon^\nu f''(\epsilon^\nu z)}{f'_{j,k}(\epsilon^\nu z)}\right\} \ge 0, \quad (z \in \mathbb{D},\ \nu = 0,1,\ldots,k-1). \tag{2.3}$$

From the relation $f'_{j,k}(\epsilon^\nu z) = \epsilon^{\nu j-\nu} f'_{j,k}(\epsilon^\nu z)$, (2.3) yields

$$\Re\left\{\frac{\epsilon^{\nu-\nu j}f'(\epsilon^\nu z)+z\epsilon^{2\nu-\nu j}f''(\epsilon^\nu z)}{f'_{j,k}(z)}\right\} \ge 0, \quad (z \in \mathbb{D}). \tag{2.4}$$

Which implies that,

$$\Re\left\{\frac{\sum_{\nu=0}^{k-1}\epsilon^{\nu-\nu j}f'(\epsilon^\nu z)+z\sum_{\nu=0}^{k-1}\epsilon^{2\nu-\nu j}f''(\epsilon^\nu z)}{f'_{j,k}(z)}\right\} \ge 0,$$

or,

$$\Re\left\{\frac{f'_{j,k}(z)+zf''_{j,k}(z)}{f'_{j,k}(z)}\right\} \ge 0, \quad (z \in \mathbb{D}),$$

this implies $f_{j,k}(z) \in \mathcal{K}_{[j,k]}(A,B)$. Hence, application of subordination and ([7], Theorem 2), conclude that

$$g(|z|<r) \le G(|z| \le r^{j+k-1}), \quad r < 1.$$

The functional $g(z) = \frac{zf''_{j,k}(z)}{f'_{j,k}(z)}$ is a disc with center at $\frac{(A-B)r^{2(j+k-1)}}{1-B^2r^{2(j+k-1)}}$ and its radius is $\frac{(A-B)r^{j+k-1}}{1-B^2r^{2(j+k-1)}}$. Thus, for $z \in \mathbb{D}$ we have

$$\left| g(z) - \frac{(A-B)|z|^{2(j+k-1)}}{1-B^2|z|^{2(j+k-1)}} \right| \leq \frac{(A-B)|z|^{j+k-1}}{1-B^2|z|^{2(j+k-1)}}. \tag{2.5}$$

After simplification we get the result for $0 < |z| = r < 1$,

$$\left| \frac{f''_{j,k}(r)}{f'_{j,k}(r)} - \frac{(A-B)r^{2(j+k-1)-1}}{1-B^2r^{2(j+k-1)}} \right| \leq \frac{(A-B)r^{j+k-2}}{1-B^2r^{2(j+k-1)}}. \tag{2.6}$$

Upon integrating with respect to r (2.6), we get

$$\begin{aligned} &\left| \log f'_{j,k}(r) + \frac{(A-B)}{2B^2(j+k-1)} \log(1-B^2r^{2(j+k-1)}) \right| \\ &\leq \frac{(A-B)}{2B(j+k-1)} \log\left(\frac{1+Br^{j+k-1}}{1-Br^{j+k-1}} \right), \end{aligned}$$

or

$$\begin{aligned} &\frac{-(A-B)}{2B(j+k-1)} \log\left(\frac{1+Br^{j+k-1}}{1-Br^{j+k-1}} \right) \\ &\leq \log|f'_{j,k}(r)| + \frac{(A-B)}{2B^2(j+k-1)} \log(1-B^2r^{2(j+k-1)}) \\ &\leq \frac{(A-B)}{2B(j+k-1)} \log\left(\frac{1+Br^{j+k-1}}{1-Br^{j+k-1}} \right), \end{aligned}$$

or

$$\begin{aligned} &\frac{-(A-B)}{2B^2(j+k-1)} \left[\log(1+Br^{j+k-1})^{1+B}(1-Br^{j+k-1})^{1-B} \right] \\ &\leq \log|f'_{j,k}(r)| \leq \frac{(A-B)}{2B^2(j+k-1)} \left[\log(1+Br^{j+k-1})^{B-1}(1-Br^{j+k-1})^{-1-B} \right], \end{aligned}$$

or

$$\begin{aligned} &(1+Br^{j+k-1})^{\frac{-(A-B)(1+B)}{2B^2(j+k-1)}} (1-Br^{j+k-1})^{\frac{-(A-B)(1-B)}{2B^2(j+k-1)}} \\ &\leq |f'_{j,k}(r)| \leq (1+Br^{j+k-1})^{\frac{(A-B)(B-1)}{2B^2(j+k-1)}} (1-Br^{j+k-1})^{\frac{(A-B)(-1-B)}{2B^2(j+k-1)}}. \end{aligned}$$

This completes the proof of Theorem 2.4. □

Theorem 2.5. *Suppose that* $f(z) \in \mathcal{K}_{[j,k]}(A,B)$. *Then*

$$\begin{aligned} &\frac{1}{r}\int_0^r \frac{1-Ax^{j+k-1}}{1-Bx^{j+k-1}} (1+Bx^{j+k-1})^{\frac{-(A-B)(1+B)}{2B^2(j+k-1)}} (1-Bx^{j+k-1})^{\frac{-(A-B)(1-B)}{2B^2(j+k-1)}}\,dx \\ &\leq |f'(z)| \\ &\leq \frac{1}{r}\int_0^r \frac{1+Ax^{j+k-1}}{1+Bx^{j+k-1}} (1+Bx^{j+k-1})^{\frac{(A-B)(B-1)}{2B^2(j+k-1)}} (1-Bx^{j+k-1})^{\frac{(A-B)(-1-B)}{2B^2(j+k-1)}}\,dx. \end{aligned}$$

Proof. Suppose $f(z) \in \mathcal{K}_{[j,k]}(A,B)$. The function $\frac{1+Az^{j+k-1}}{1+Bz^{j+k-1}}$ maps the disc $|z| \leq r$ on to the interior of the circle with the line segment

$$\left[\frac{1-Ar^{j+k-1}}{1-Br^{j+k-1}}, \frac{1+Ar^{j+k-1}}{1+Br^{j+k-1}}\right]$$

as diameter and $\frac{(zf'(z))'}{f'_{j,k}(z)} \prec \frac{1+Az^{j+k-1}}{1+Bz^{j+k-1}}$. From the above fact, we have

$$\frac{1-Ar^{j+k-1}}{1-Br^{j+k-1}} \leq \left|\frac{(zf'(z))'}{f'_{j,k}(z)}\right| \leq \frac{1+Ar^{j+k-1}}{1+Br^{j+k-1}}.$$

By Theorem 2.4, we get

$$\begin{aligned}
&\frac{1-Ar^{j+k-1}}{1-Br^{j+k-1}}(1+Br^{j+k-1})^{\frac{-(A-B)(1+B)}{2B^2(j+k-1)}}(1-Br^{j+k-1})^{\frac{-(A-B)(1-B)}{2B^2(j+k-1)}}\\
&\leq |(zf'(z))'|\\
&\leq \frac{1+Ar^{j+k-1}}{1+Br^{j+k-1}}(1+Br^{j+k-1})^{\frac{(A-B)(B-1)}{2B^2(j+k-1)}}(1-Br^{j+k-1})^{\frac{(A-B)(-1-B)}{2B^2(j+k-1)}}.
\end{aligned}$$

On integrating both sides of the above inequalities, we get

$$\begin{aligned}
&\frac{1}{r}\int_0^r \frac{1-Ax^{j+k-1}}{1-Bx^{j+k-1}}(1+Bx^{j+k-1})^{\frac{-(A-B)(1+B)}{2B^2(j+k-1)}}(1-Bx^{j+k-1})^{\frac{-(A-B)(1-B)}{2B^2(j+k-1)}}dx\\
&\leq |f'(z)|\\
&\leq \frac{1}{r}\int_0^r \frac{1+Ax^{j+k-1}}{1+Bx^{j+k-1}}(1+Bx^{j+k-1})^{\frac{(A-B)(B-1)}{2B^2(j+k-1)}}(1-Bx^{j+k-1})^{\frac{(A-B)(-1-B)}{2B^2(j+k-1)}}dx.
\end{aligned}$$

This completes the proof of Theorem 2.5. □

Theorem 2.6. *Let $f(z) \in \mathcal{ST}_{[j,k]}(A,B)$. Then*

$$\begin{aligned}
&\int_0^r \frac{1-Ax^{j+k-1}}{1-Bx^{j+k-1}}(1+Bx^{j+k-1})^{\frac{-(A-B)(1+B)}{2B^2(j+k-1)}}(1-Bx^{j+k-1})^{\frac{-(A-B)(1-B)}{2B^2(j+k-1)}}dx\\
&\leq |f(z)|\\
&\leq \int_0^r \frac{1+Ax^{j+k-1}}{1+Bx^{j+k-1}}(1+Bx^{j+k-1})^{\frac{(A-B)(B-1)}{2B^2(j+k-1)}}(1-Bx^{j+k-1})^{\frac{(A-B)(-1-B)}{2B^2(j+k-1)}}dx,
\end{aligned}$$

Proof. Using the well-known result

$$f(z) \in \mathcal{K}_{[j,k]}(A,B) \Leftrightarrow zf'(z) \in \mathcal{ST}_{[j,k]}(A,B),$$

get the desired proof of Theorem 2.6. □

3. Radius problems

In this section, we shall deal with various radii related problems associated with the above defined families. In general, our aim is to determine the largest number $R \in (0,1)$ such that for any given two subfamilies $\mathcal{E}$ and $\mathcal{F}$ of $\mathcal{H}$, with each $f \in \mathcal{E}$, we have $r^{-1}f(rz) \in \mathcal{F}$, where $r \leq R$. We call R as the $\mathcal{F}$-radius of

$\mathcal{E}$. The number R is best possible if there exists a function $f_0 \in \mathcal{F}$ such that $r^{-1} f_0(rz) \notin \mathcal{E}$, where $r > R$.

3.1. $\mathcal{ST}_{[j,k]}(\alpha)$-radius for the family $\mathcal{ST}_{[j,k]}(A, B)$

Theorem 3.1. *For $0 \leq \alpha < 1$, $\mathcal{ST}_{[j,k]}(\alpha)$-radius of functions in the family $\mathcal{ST}_{[j,k]}(A, B)$ is given by*

$$R_1(\alpha) := \min\left\{\frac{2(1-\alpha)}{(A-B)+\sqrt{(A-B)^2+4(AB-\alpha B^2)(1-\alpha)}}, 1\right\}. \tag{3.1}$$

Following function shows that the estimate is best possible.

$$f(z) = \begin{cases} z(1+Bz^{j+k-1})^{\frac{A-B}{B(j+k-1)}}; & B \neq 0, \\ z\exp(\frac{Az^{j+k-1}}{j+k-1}); & B = 0. \end{cases} \tag{3.2}$$

Proof. Let $f \in \mathcal{ST}_{[j,k]}(\alpha)$. Then

$$\Re\left\{\frac{zf'(z)}{f_{j,k}(z)}\right\} > \alpha \quad (0 \leq \alpha < 1, z \in \mathbb{D}). \tag{3.3}$$

Definition 1 together with Lemma 2.1, gives

$$\Re\left(\frac{zf'(z)}{f_{j,k}(z)}\right) \geq \frac{1-ABr^{2(j+k-1)}}{1-B^2r^{2(j+k-1)}} - \frac{(B-A)r^{j+k-1}}{1-B^2r^{2(j+k-1)}}. \tag{3.4}$$

From (3.3) and the inequality (3.4), we see that

$$\Re\left(\frac{zf'(z)}{f_{j,k}(z)}\right) \geq \frac{1-ABr^{2(j+k-1)}}{1-B^2r^{2(j+k-1)}} - \frac{(B-A)r^{j+k-1}}{1-B^2r^{2(j+k-1)}} \geq \alpha$$

is true if

$$(AB-\alpha B^2)r^{2(j+k-1)} + (B-A)r^{j+k-1} + (\alpha-1) \leq 0.$$

Further simplification yields

$$R_1(\alpha) = \frac{2(1-\alpha)}{(A-B)+\sqrt{(A-B)^2+4(AB-\alpha B^2)(1-\alpha)}}.$$

Indeed, for the sharpness, we consider

$$\frac{zf'(z)}{f_{j,k}(z)} = \frac{1+Az^{j+k-1}}{1+Bz^{j+k-1}} = 1 + \frac{(A-B)z^{j+k-1}}{1+Bz^{j+k-1}}.$$

The case when $B \neq 0$, we have

$$\frac{f'(z)}{f_{j,k}(z)} - \frac{1}{z} = \frac{(A-B)z^{j+k-2}}{1+Bz^{j+k-1}},$$

or

$$\log\left(\frac{f}{z}\right) = \frac{(A-B)}{B(j+k-1)}\log(1+Bz^{j+k-1}).$$

This implies that

$$f(z) = z(1+Bz^{j+k-1})^{\frac{(A-B)}{B(j+k-1)}}.$$

The case when $B=0$, we have

$$\frac{f'(z)}{f_{j,k}(z)}-\frac{1}{z}=Az^{j+k-2}.$$

On integration, we get

$$f(z)=z\exp\left(\frac{Az^{j+k-1}}{(j+k-1)}\right).$$

□

If we put $\alpha=0$ in Theorem 3.1, then the following result obtain:

Theorem 3.2. *The $\mathcal{ST}_{[j,k]}$-radius of functions in the family of $\mathcal{ST}_{[j,k]}(A,B)$ is given by*

$$R_2=\frac{2}{(A-B)+\sqrt{(A-B)^2+4AB}}. \tag{3.5}$$

The radius is sharp for the function defined in (3.2).

3.2. $\mathcal{K}_{[j,k]}(\alpha)$-radius for the family $\mathcal{K}_{[j,k]}(A,B)$

Theorem 3.3. *For $0\le\alpha<1$, the $\mathcal{K}_{[j,k]}(\alpha)$-radius of functions in the family of $\mathcal{K}_{[j,k]}(A,B)$ is given by*

$$R_3(\alpha):=\min\left\{\frac{2(\alpha-1)}{(A-B)+\sqrt{(A-B)^2+4(-A+B+B^2-\alpha B^2)(\alpha-1)}},1\right\}. \tag{3.6}$$

The bound is sharp for the function defined in (3.2).

Proof. From (2.5) we have

$$\Re\left(\frac{zf''_{j,k}(z)}{f'_{j,k}(z)}\right)\ge\frac{(A-B+B^2)r^{2(j+k-1)}-(A-B)r^{j+k-1}-1}{1-B^2r^{2(j+k-1)}}. \tag{3.7}$$

If $f\in\mathcal{K}_{[j,k]}(\alpha)$, then

$$\Re\left(1+\frac{zf''_{j,k}(z)}{f'_{j,k}(z)}\right)>\alpha. \tag{3.8}$$

From (3.7) and (3.8), we see that

$$\Re\left(1+\frac{zf''_{j,k}(z)}{f'_{j,k}(z)}\right)\ge\frac{(A-B-B^2)r^{2(j+k-1)}-(A-B)r^{j+k-1}-1}{1-B^2r^{2(j+k-1)}}\ge\alpha$$

is true if

$$(-A+B+B^2-\alpha B^2)r^{2(j+k-1)}+(A-B)r^{j+k-1}+(\alpha-1)\le 0.$$

Upon simplification we get the required result (3.6). □

If we put $\alpha=0$ in Theorem 3.3, then the following result obtain:

Theorem 3.4. *The $\mathcal{K}_{[j,k]}$-radius of functions in the family of $\mathcal{K}_{[j,k]}(A,B)$ is*

$$R_4 = \frac{2}{(A-B)+\sqrt{(A-B)^2+4(A-B+B^2)}}. \tag{3.9}$$

The radius is sharp for the function defined in (3.2).

3.3. Radius of (j,k)-symmetric strongly starlikeness of order γ

A function $f \in \mathcal{H}$ is said to be (j,k)-symmetric strongly starlike of order $\gamma;\ 0<\gamma\le 1$ if it satisfies the subordination relation

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \left(\frac{1+z^{j+k-1}}{1-z^{j+k-1}}\right)^{\gamma} \quad (j=0,1,2,\dots,k-1; k\ge 1, z\in\mathbb{D}).$$

Or, equivalently, we have

$$\left|\arg\frac{zf'(z)}{f_{j,k}(z)}\right| \le \frac{\pi}{2}\gamma.$$

For our requirement we recall the following important result due to Gangadharan et al.

Lemma 3.5 ([6])**.** *If c is any point in $|\arg w|\le \frac{\pi}{2}\gamma$ and if $R_c \le \Re[c]\sin\frac{\pi}{2}\gamma - \Im[c]\cos\frac{\pi}{2}\gamma$, $\Im[c]\ge 0$. The disk $|w-c|\le R_c$ is contained in the sector $|\arg w|\le \frac{\pi}{2}\gamma$, $0<\gamma\le 1$. In particular when $\Im[c]=0$, the condition becomes $R_c \le c\sin\frac{\pi}{2}\gamma$.*

In this subsection we compute radius of (j,k)-symmetric strongly starlikeness for the class $\mathcal{ST}_{[j,k]}(A,B)$.

Theorem 3.6. *Let $f\in\mathcal{ST}_{[j,k]}(A,B)$ and $0<\gamma\le 1$. Then the radius of (j,k)-symmetric strongly starlike of order γ in $|z|<R(\gamma)$ given by*

$$R_5(\gamma) = \frac{2\sin\frac{\pi}{2}\gamma}{(B-A)+\sqrt{(B-A)^2+4AB\sin^2\frac{\pi}{2}\gamma}}.$$

Proof. From Lemma 2.1, we have

$$\left|\frac{zf'(z)}{f_{j,k}(z)} - \frac{1-ABr^{2(j+k-1)}}{1-B^2r^{2(j+k-1)}}\right| \le \frac{(B-A)r^{j+k-1}}{1-B^2r^{2(j+k-1)}}.$$

Set

$$c = \frac{1-ABr^{2(j+k-1)}}{1-B^2r^{2(j+k-1)}} \text{ and } R_c = \frac{(B-A)r^{j+k-1}}{1-B^2r^{2(j+k-1)}}.$$

Since $\Im(AB)=0$, Lemma 3.5 yields

$$\frac{(B-A)r^{j+k-1}}{1-B^2r^{2(j+k-1)}} \le \frac{1-ABr^{2(j+k-1)}}{1-B^2r^{2(j+k-1)}}\sin\left(\frac{\pi}{2}\gamma\right)$$

or

$$-ABr^{2(j+k-1)}\sin\frac{\pi}{2}\gamma - (B-A)r^{j+k-1} + \sin\frac{\pi}{2}\gamma \ge 0.$$

Since $\sin\frac{\pi}{2}\gamma \ge 0$, therefore, the above inequality gives the required radius $R_5(\gamma)$. This completes the proof of Theorem 3.6. □

3.4. Radius of (j,k)-symmetric parabolic starlikeness of order β

A function $f \in \mathcal{H}$ is said to be (j,k)-symmetric parabolic starlike of order β, denoted by $SP_{[j,k]}(\beta)$, if it satisfy the following inequality

$$\Re\left(\frac{zf'(z)}{f_{j,k}(z)}\right) > \left|\frac{zf'(z)}{f_{j,k}(z)} - 1\right| + \beta, \quad (\beta \in [-1,1), z \in \mathbb{D}). \tag{3.10}$$

Geometrically, the values of the functional $\frac{zf'(z)}{f_{j,k}(z)}$ lies in the parabolic region

$$\Theta = \left\{ w = u + iv : v^2 < 2(1-\beta)\left(u - \frac{1+\beta}{2}\right)\right\}. \tag{3.11}$$

The $SP_{[j,k]}(\beta)$-radius of functions in the family of $\mathcal{ST}_{[j,k]}(A,B)$ are determined.

Theorem 3.7. *For $\beta < 1$, $A, B \in \mathbb{R}$, $A < B$ and $|B| \leq 1$. Let R_6 be given by*

$$R_6 := \min\left\{1, \frac{2(1-\beta)}{(B-A) + \sqrt{(B-A)^2 + 4B^2(1-\alpha)^2}}\right\},$$

R_7 is defined in $(0,1]$ such that $1 \geq (B(1+\beta) - 2A)r^{j+k-1} + \beta$ for all $r \in [0, R_7]$ and R_8 is defined in $(0,1]$ such that $A + B(1-2\beta) \geq 2B^2(1-\beta)r^{2(j+k-1)}$ for all $r \in [0, R_8]$. If $f \in \mathcal{ST}_{[j,k]}(A,B)$, then the $SP_{[j,k]}(\beta)$-radius is given by

$$R_9 = \begin{cases} R_7; & R_7 \leq R_6, \\ R_8; & R_7 > R_6. \end{cases}$$

Proof. Since

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1 + Az^{j+k-1}}{1 + Bz^{j+k-1}},$$

it follows from Lemma 2.1 that

$$\left|\frac{zf'(z)}{f_{j,k}(z)} - \frac{1 - ABr^{2(j+k-1)}}{1 - B^2 r^{2(j+k-1)}}\right| \leq \frac{(B-A)r^{j+k-1}}{1 - B^2 r^{2(j+k-1)}}, \ (|z| \leq r < 1). \tag{3.12}$$

By letting $w(z) = \frac{zf'(z)}{f_{j,k}(z)} = u + \iota v$, the points lies the boundary of the disk in (3.12) are given by,

$$w(z) = \frac{1 - ABr^{2(j+k-1)}}{1 - B^2 r^{2(j+k-1)}} + \frac{(B-A)r^{j+k-1}}{1 - B^2 r^{2(j+k-1)}} e^{i\theta}$$

and hence,

$$\begin{aligned} \Re(w(z)) &= \frac{1 - ABr^{2(j+k-1)} + (B-A)r^{j+k-1}\cos\theta}{1 - B^2 r^{2(j+k-1)}}, \\ \Im(w(z)) &= \frac{(B-A)r^{j+k-1}}{1 - B^2 r^{2(j+k-1)}} \sin\theta. \end{aligned} \tag{3.13}$$

For $f \in SP_{[j,k]}(\beta)$, we have

$$\Re\left(\frac{zf'(z)}{f_{j,k}(z)}\right) > \left|\frac{zf'(z)}{f_{j,k}(z)} - 1\right| + \beta, \quad \beta \in [-1,1), z \in \mathbb{D}.$$

Or, equivalently

$$u > |(u + \iota v) - 1| + \beta. \tag{3.14}$$

Squaring both side of (3.14) yields

$$v^2 < 2u(1-\beta) + \beta^2 - 1.$$

Or

$$\Im(w(z))^2 < 2(1-\beta)\left(\Re w(z) - \frac{1+\beta}{2}\right). \tag{3.15}$$

Putting (3.13) in (3.15) we get,

$$\begin{aligned} &\left(\frac{(B-A)r^{j+k-1}}{1-B^2r^{2(j+k-1)}}\sin\theta\right)^2 \\ &< 2(1-\beta)\left(\frac{1-ABr^{2(j+k-1)} + (B-A)r^{j+k-1}\cos\theta}{1-B^2r^{2(j+k-1)}}\right) + \beta^2 - 1. \end{aligned} \tag{3.16}$$

Which leads to

$$\begin{aligned} L(x) := {} & (B-A)^2r^{2(j+k-1)}x^2 + 2(1-\beta)(B-A)(1-B^2r^{2(j+k-1)})r^{j+k-1}x \\ & + 2(1-\beta)(1-ABr^{2(j+k-1)})(1-B^2r^{2(j+k-1)}) \\ & + (\beta^2-1)(1-B^2r^{2(j+k-1)})^2 - (B-A)^2r^{2(j+k-1)} \geq 0, \end{aligned}$$

where we set $x = \cos\theta$. It is suffices to find $r(=R)$ such that $L(x) \geq 0$ for all $x \in [-1, 1]$. Note that

$$L'(x) = 2(B-A)^2r^{2(j+k-1)}x + 2(1-\beta)(B-A)(1-B^2r^{2(j+k-1)})r^{j+k-1},$$

which vanishes for

$$x = x_0 = \frac{-(1-\beta)(1-B^2r^{2(j+k-1)})}{(B-A)r^{j+k-1}}.$$

With $\beta < 1, A < B$ and $|B| \leq 1$, we have $x_0 \leq 0$. If $x_0 \leq -1$, we need $L(-1) \geq 0$ and if $-1 < x_0 < 0$, we need $L(x_0) \geq 0$.

Indeed, if $x_0 \leq -1$, then we have

$$(\beta B^2 - B^2)r^{2(j+k-1)} - (B-A)r^{j+k-1} + (1-\beta) \geq 0$$

which is equivalent to

$$R_6 = r \leq \frac{2(1-\beta)}{(B-A) + \sqrt{(B-A)^2 + 4B^2(1-\alpha)^2}}.$$

Therefore, the condition $L(-1) \geq 0$ is equivalent to

$$2(1 - ABr^{2(j+k-1)} - Br^{j+k-1} + Ar^{j+k-1}) \geq (1+\beta)(1-B^2r^{2(j+k-1)}).$$

That is

$$1 \geq (B(1+\beta) - 2A)r^{j+k-1} + \beta.$$

Also, we have $L(x_0)\geq 0$ implies that

$$(B-A)^2r^{2(j+k-1)}\left(\frac{(1-\beta)(1-B^2r^{2(j+k-1)})}{(B-A)r^{j+k-1}}\right)^2-2(1-\beta)(B-A)$$
$$(1-B^2r^{2(j+k-1)})r^{j+k-1}\times\left(\frac{(1-\beta)(1-B^2r^{2(j+k-1)})}{(B-A)r^{j+k-1}}\right)+2(1-\beta)$$
$$(1-ABr^{2(j+k-1)})(1-B^2r^{2(j+k-1)})+(\beta^2-1)(1-B^2r^{2(j+k-1)})^2$$
$$-(B-A)^2r^{2(j+k-1)}\geq 0.$$

This leads to

$$A+B(1-2\beta)\geq 2B^2(1-\beta)r^{2(j+k-1)}.$$

If $R_7\leq R_6$, then (3.12) will be lies in parabolic region (3.11) if and only if $r\leq R_7$. If $R_7>R_6$, then (3.12) will be lies in parabolic region (3.11) if and only if $r\leq R_8$. This completes the proof. □

4. Zalcman conjecture for the family of $\mathcal{ST}_{[j,k]}$

In 1960, Lawrence Zalcman conjectured that the coefficients of the family $\mathcal{S}$ satisfy the quantity $|a_n^2-a_{2n-1}|\leq(n-1)^2$, for each $n\geq 2$ with sharp bound for Koebe function and its rotation. Note that this conjecture implies the classical Fekete-Szegö inequality, namely, $|a_2^2-a_3|\leq 1$. Though the conjecture is settled for certain subfamilies of $\mathcal{S}$, the original problem is still open for $n>6$. For recent development of the problem we refer [10,14,21] and the reference therein. In this section, we considered the family of (j,k)-symmetric starlike function $\mathcal{ST}_{[j,k]}$ to evaluate the Zalcman functional for $n=3$. The problem is open for larger n.

Theorem 4.1. *If $f\in\mathcal{ST}_{[j,k]}$, then*

$$\begin{aligned}&(2-\psi_{2,j})^2(3-\psi_{3,j})^2(4-\psi_{4,j})(5-\psi_{5,j})|a_3^2-a_5|\\
\leq\ &(4-\psi_{4,j})(5-\psi_{5,j})(16\psi_{2,j}^2+4(2-\psi_{2,j})^2+16\psi_{2,j}(2-\psi_{2,j}))\\
&-(3-\psi_{3,j})(2-\psi_{2,j})(16\psi_{2,j}\psi_{3,j}\psi_{4,j}+8[\psi_{4,j}\psi_{3,j}(2-\psi_{2,j})\\
&+\psi_{4,j}\psi_{2,j}(3-\psi_{3,j})+\psi_{3,j}\psi_{2,j}(4-\psi_{4,j})]\\
&+4(3-\psi_{3,j})[\psi_{4,j}(2-\psi_{2,j})+\psi_{2,j}(4-\psi_{4,j})]+4\psi_{3,j}(4-\psi_{4,j})\\
&+2(4-\psi_{4,j})(3-\psi_{3,j})(2-\psi_{2,j})).\end{aligned}$$

Proof. It is known that $f\in\mathcal{ST}_{[j,k]}$ if and only if $\frac{zf'(z)}{f_{j,k}(z)}\in\mathcal{P}$. By (1.4) we have

$$\frac{zf'(z)}{f_{j,k}(z)}=p(z),$$

which implies that

$$zf'(z)=\left(1+\sum_{n=1}^{\infty}p_nz^n\right)\times f_{j,k}(z).$$

Using (1.1) and (1.4), we have

$$a_n = \frac{1}{n-\psi_{n,j}} \sum_{m=1}^{n-1} p_m \psi_{n-m,j}\ a_{n-m};\ \psi_{1,j}=1, n\geq 2. \tag{4.1}$$

From (4.1) we have

$$\begin{aligned} a_3 &= \frac{1}{(2-\psi_{2,j})(3-\psi_{3,j})}(p_1^2\psi_{2,j}+p_2(2-\psi_{2,j})),\\ a_5 &= \frac{1}{(2-\psi_{2,j})(3-\psi_{3,j})(4-\psi_{4,j})(5-\psi_{5,j})}(p_1^4\psi_{2,j}\psi_{3,j}\psi_{4,j}\\ &\quad + p_1^2p_2[\psi_{4,j}\psi_{3,j}(2-\psi_{2,j})+\psi_{4,j}\psi_{2,j}(3-\psi_{3,j})+\psi_{3,j}\psi_{2,j}(4-\psi_{4,j})]\\ &\quad + p_1p_3(3-\psi_{3,j})[\psi_{4,j}(2-\psi_{2,j})+\psi_{2,j}(4-\psi_{4,j})]\\ &\quad + p_2^2\psi_{3,j}(4-\psi_{4,j})+p_4(4-\psi_{4,j})(3-\psi_{3,j})(2-\psi_{2,j})), \end{aligned}$$

and hence, we have

$$\begin{aligned} &(2-\psi_{2,j})^2(3-\psi_{3,j})^2(4-\psi_{4,j})(5-\psi_{5,j})(a_3^2-a_5)\\ &= (4-\psi_{4,j})(5-\psi_{5,j})(p_1^4\psi_{2,j}^2+p_2^2(2-\psi_{2,j})^2+2p_1^2p_2\psi_{2,j}(2-\psi_{2,j}))\\ &\quad - (3-\psi_{3,j})(2-\psi_{2,j})(p_1^4\psi_{2,j}\psi_{3,j}\psi_{4,j}+p_1^2p_2[\psi_{4,j}\psi_{3,j}(2-\psi_{2,j})\\ &\quad + \psi_{4,j}\psi_{2,j}(3-\psi_{3,j})+\psi_{3,j}\psi_{2,j}(4-\psi_{4,j})]\\ &\quad + p_1p_3(3-\psi_{3,j})[\psi_{4,j}(2-\psi_{2,j})+\psi_{2,j}(4-\psi_{4,j})]\\ &\quad + p_2^2\psi_{3,j}(4-\psi_{4,j})+p_4(4-\psi_{4,j})(3-\psi_{3,j})(2-\psi_{2,j})). \end{aligned}$$

The inequality $|p_n|\leq 2$, which follows the required result. □

Remark 4.2. If we put $j=k=1$, then we have the result of [5] .

5. Pre-Schwarzian norm estimate for the family $\mathcal{K}_{[j,k]}(\alpha)$

Pre-Schwarzian derivative and its norm have wide applications in the theory of Teichmüller space (cf. [9, 25]). We recall here that, the norm of Pre-Schwarzian derivative is given by $T_f = f''/f'$, where $\|T_f\| = \sup_{|z|<1}(1-|z|^2)|T_f(z)|$. For $f\in\mathcal{S}$, we have $\|T_f\|\leq 6$ and for $f\in\mathcal{K}$, we have $\|T_f\|\leq 4$. Converse part of the above results which follows from Beckers theorem [4] shows that if $f\in\mathcal{H}$ and $\|T_f\|\leq 1$, then $f\in\mathcal{S}$. Yamashita [24] proved that if $f\in\mathcal{K}(\alpha)$, then $\|f\| = 4(1-\alpha)$ and for $f\in\mathcal{S}^*(\alpha)$ then $\|f\| = 6-4\alpha$. In a recent paper, Aghalary and Orouji [1] have estimated the sharp bound $\|T_f\|$ for functions in the class of α-spiral like functions of order ρ. Wherein the authors have also pointed various connections with other subfamilies of $\mathcal{S}$.

In this section we considered the functions class $\mathcal{K}_{[j,k]}(A,B)$, which is defined by subordination as follows. For $-1\leq B<A\leq 1$ with $A\neq B$, every $f\in\mathcal{H}$ is said to be in $\mathcal{K}_{[j,k]}(A,B)$ if it satisfying the subordination relation:

$$1+\frac{zf''(z)}{f'_{j,k}(z)} \prec \frac{1+Az}{1+Bz}, \quad z\in\mathbb{D}.$$

If we take $A = 1-2\alpha$ and $B = -1$, then the class is defined by:

$$\mathcal{K}_{[j,k]}(\alpha) := \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'_{j,k}(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \right\}.$$

Consider the function $\Phi(z) : \mathbb{D} \to \mathbb{C}$ given by

$$\Phi(z) = \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1}; & \alpha \neq 1/2, \\ \log \frac{1}{1-z}; & \alpha = 1/2, \end{cases}$$

for which $1 + \frac{z\Phi''(z)}{\Phi'(z)} = \frac{1+(1-2\alpha)z}{1-z}$, then $\Phi(z) \in \mathcal{K}_{[j,k]}(\alpha)$. Note that Φ treat as an extremal function for the family $\mathcal{K}_{[j,k]}(\alpha)$. For $f \in \mathcal{K}_{[j,k]}(\alpha)$, we have $|a_2| \leq 1-\alpha$ and equality holds if and only if

$$f(z) = \bar{\mu}\Phi(\mu z), \tag{5.1}$$

μ is unimodular constant that is μ is complex number with $|\mu|^2 = \mu\bar{\mu} = 1$.

Thus, we have the following theorem.

Theorem 5.1. *For $0 \leq \alpha < 1$, the following two statements holds true.*

i) *Suppose that $f \in \mathcal{K}_{[j,k]}(\alpha)$, then $\|T_f\| = 2(1+\eta)$ if and only if f is of the form* (5.1).

ii) *If $f \in \mathcal{K}_{[j,k]}(\alpha)$ is not of the form* (5.1), *then*

$$\|T_f\| \leq 2(1+\eta)\frac{1+C+D}{D-C+3} \quad \textit{for } \eta = 1-2\alpha, \tag{5.2}$$

where

$$0 \leq C = \frac{2|a_2|}{1+\eta} \leq 1, \tag{5.3}$$

$$0 \leq D = \frac{|2a_3(3+3\eta) - 4a_2^2(2+\eta)|}{(1+\eta)(1+\eta-2|a_2|)} \leq 1+C < 2, \tag{5.4}$$

so that

$$\frac{1}{3} \leq \frac{1+C+D}{D-C+3} \leq \frac{1+C}{2} < 1.$$

Proof. Let us consider the function

$$F(z) \equiv F_\alpha(z) = \frac{1+\eta z}{1-z}, \quad \text{where} \quad \eta = 1-2\alpha,\ z \in \mathbb{D}. \tag{5.5}$$

Clearly F is univalent in $\mathbb{D}$ with $F'(0) = \eta+1$ and $F''(z) = 2(\eta+1)$. Geometrically, F maps conformally $\mathbb{D}$ onto $\Re(F(z)) > \alpha$. For $f \in \mathcal{K}_{[j,k]}(\alpha)$ then we set

$$g(z) = 1 + \frac{zf''(z)}{f'_{j,k}(z)}, \quad z \in \mathbb{D}.$$

Therefore, the composed function, $\phi \equiv F^{-1} \circ g : \mathbb{D} \longrightarrow \mathbb{D}$, is analytic in $\mathbb{D}$ with $\phi(0) = 0$ and $g = F \circ \phi$ which means g is subordinate to F. Since

$g'(0) = 2a_2, g''(0) = 12a_3 - 8a_2^2$. It is clear that $\phi(z) = \frac{g(z)-1}{g(z)+\eta}$ which follows that

$$\phi'(0) = \frac{2a_2}{1+\eta} \text{ and } \phi''(0) = \frac{4a_3(3+3\eta) - 8a_2^2(2+\eta)}{(1+\eta)^2}. \tag{5.6}$$

Hence, the function $\phi(z)$ satisfy the Schwarz lemma shows that

$$C =: |\phi'(0)| = \frac{|2a_2|}{1+\eta} \leq 1,$$

and further $C = 1$, i.e., equality holds if and only if

$$\phi(z) \equiv \mu z \tag{5.7}$$

with $|\mu| = 1$ or f is of the form (5.1). Further, it follows from $g = F \circ \phi$ that

$$\frac{f''(z)}{f'_{j,k}(z)} = \frac{\phi(z)(1+\eta)}{z(1-\phi(z))} \tag{5.8}$$

is analytic in $\mathbb{D}$. We need to prove that $\|f\| = 2(1+\eta)$ for which f of the form (5.1). In fact, from (5.8) we have

$$\begin{aligned}(1-|z|^2)\left|\frac{f''(z)}{f'_{j,k}(z)}\right| &= (1+\eta)\frac{(1-|z|^2)|\phi(z)|}{|z|(|1-\phi(z)|)} \\ &= \frac{(1+\eta)(1-|z|^2)}{|1-\mu z|} = (1+\eta)(1+x) \leq 2(1+\eta).\end{aligned}$$

Hence, the required result of (i) holds true for $z = \bar{\mu}x$, as $x \to 1$.

Next to proof of (ii), where $\phi(z)$ is not of the form of (5.1). It follows from [23] that

$$|\phi(z)| \leq |z|Q(|z|), \quad z \in \mathbb{D}, \tag{5.9}$$

where $Q(x) = \frac{x^2+Dx+C}{Cx^2+Dx+1}$, $0 \leq x \leq 1$. Here $D = \frac{|\phi''(0)|}{2(1-|\phi'(0)|)}$, which together with (5.6) provides the expression of D in terms of a_2 and a_3. By the application of Schwarz-Pick inequality at origin to $\chi(z) = \frac{\phi(z)}{z}$, where $|\chi| < 1$, we observe that

$$\frac{D}{1+|\phi'(0)|} = \frac{|\chi'(0)|}{1-|\chi(0)|^2} \leq 1.$$

Thus, $D \leq 1 + C = 1 + |\phi'(0)| = 1 + \frac{|2a_2|}{1+\eta} < 2$, by $|\phi'(0)| = C < 1$. Combining (5.8) and (5.9), we have

$$(1-|z|^2)\left|\frac{f''(z)}{f'_{j,k}(z)}\right| \leq (1+\eta)\frac{|\phi(z)|(1-|z|^2)}{|z|(1-|\phi(z)|)} = (1+\eta)G(|z|), \tag{5.10}$$

where $G(x) = \frac{(1+x)(x^2+Dx+C)}{x^2+x(D-C+1)+1}$, $0 \leq x \leq 1$. To prove that

$$G(x) \leq G(1) = \frac{2(C+D+1)}{D-C+3}. \tag{5.11}$$

Let $H(x)$ be the enumerator of the $G'(x)$. Then $H(0), H'(0), H''(0)$ are positive. Hence $H(x) \geq 0$ or $G(x)$ is increasing in $0 \leq x \leq 1$, which yields the condition (5.11). Combining (5.10) with (5.11), finally we get the result,

$$\|T_f\| \leq 2(1+\eta)\frac{1+C+D}{D-C+3}.$$

This completes the proof of Theorem 5.1. □

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