Commun. Korean Math. Soc. **37** (2022), No. 2, pp. 585–594
https://doi.org/10.4134/CKMS.c200423
pISSN: 1225-1763 / eISSN: 2234-3024

GRADIENT RICCI SOLITONS WITH HALF HARMONIC WEYL CURVATURE AND TWO RICCI EIGENVALUES

Yutae Kang and Jongsu Kim

Abstract. In this article we classify four dimensional gradient Ricci solitons (M, g, f) with half harmonic Weyl curvature and at most two distinct Ricci-eigenvalues at each point. Indeed, we showed that, in a neighborhood V of each point in some open dense subset of M, (V, g) is isometric to one of the following:

(i) an Einstein manifold.

(ii) a domain in the Riemannian product $(\mathbb{R}^2, g_0) \times (N, \tilde{g})$, where g_0 is the flat metric on $\mathbb{R}^2$ and $(N, \tilde{g})$ is a two dimensional Riemannian manifold of constant curvature $\lambda \neq 0$.

(iii) a domain in $\mathbb{R} \times W$ with the warped product metric $ds^2 + h(s)^2\tilde{g}$, where $\tilde{g}$ is a constant curved metric on a three dimensional manifold W.

1. Introduction

In this paper, we study four dimensional gradient Ricci solitons with half harmonic Weyl curvature. A Riemannian manifold (M, g) is called a gradient Ricci soliton if there exist a smooth function f on M and a real constant λ such that

$$Ric + \nabla df = \lambda g, \tag{1}$$

where Ric denotes the Ricci tensor. The gradient Ricci solitons are important in Hamilton's Ricci flow theory as singularity models of the flow, so their classification is important in the study of the Ricci flow.

In four dimension, complete locally conformally flat shrinking gradient Ricci solitons are classified in [2] and Bach-flat shrinking ones in [3]. Gradient Ricci solitons with harmonic Weyl curvature are classified in [7].

A special feature of dimensionality four is that the Hodge $*$-operator splits the Weyl curvature tensor $W = W^+ + W^-$. Gradient Ricci solitons with $W^+ = 0$ or $W^- = 0$ are studied in [4]. We say that a Riemannian manifold has *half harmonic Weyl curvature* if $\delta W^+ = 0$ or $\delta W^- = 0$, where δ is the

Received November 8, 2020; Accepted November 23, 2020.

2010 *Mathematics Subject Classification.* 53C21, 53C25.

Key words and phrases. Gradient Ricci soliton, half harmonic Weyl curvature.

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. 2020R1A2B5B01001862).

divergence operator. In [8], Wu, Wu and Wylie proved that a four dimensional shrinking gradient Ricci soliton with half harmonic Weyl curvature is either Einstein, or a finite quotient of $\mathbb{S}^3 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}^2$ or $\mathbb{R}^4$.

Based on the works of [7] and [8], in this paper we characterize four dimensional gradient Ricci solitons with half harmonic Weyl curvature and at most two distinct Ricci-eigenvalues at each point.

Theorem 1. *Let (M, g, f) be a four dimensional (not necessarily complete) gradient Ricci soliton with half harmonic Weyl curvature and at most two distinct Ricci eigenvalues at each point.*

Then for each point in some open dense subset of M, there exists a neighborhood V such that (V, g) is isometric to one of the following:

(i) *an Einstein manifold.*

(ii) *a domain in the Riemannian product $(\mathbb{R}^2, g_0) \times (N, \tilde{g})$, where g_0 is the flat metric on $\mathbb{R}^2$ and $(N, \tilde{g})$ is a two dimensional Riemannian manifold of constant curvature $\lambda \neq 0$.*

(iii) *a domain in $\mathbb{R} \times W^3$ with the warped product metric $ds^2 + h(s)^2\tilde{g}$, where $\tilde{g}$ is a constant curved metric on a three dimensional manifold W^3.*

To prove Theorem 1 we depend on two crucial tools. First, ∇f is a Ricci-eigen vector field as proved in [8]. The second is the local dependence of all Ricci eigenvalues only on f. Even with these tools, there arises considerable technical difficulty in extending the arguments in [7] to this article. The main problem is that the tensor $Ric - \frac{R}{6}g$ is no longer a Codazzi tensor in our case of $\delta W^+ = 0$; note that Codazzi tensor property is well exploited in [7]. So we obtain some equations related to the tensor $Ric - \frac{R}{6}g$ in Lemma 4. Using these equations we analyze two essential cases depending on the distinctiveness of Ricci-eigenvalues.

This paper is organized as follows. In Section 2, we develop some formulas about gradient Ricci solitons and the equation $\delta W^+ = 0$. In Section 3, we divide the whole proof of Theorem 1 into two cases and resolve the first case. In Section 4, we do the second case and finish the proof of Theorem 1.

2. Gradient Ricci solitons with half harmonic Weyl curvature

On an oriented Riemannian 4-manifold (M, g), the Hodge $*$-operator splits any two form into the self-dual part and the anti-self dual part [1, Chap. 13]. This induces the Weyl tensor W to split into $W = W^+ + W^-$ where $W^\pm$ are the self-dual and anti-self-dual part of W, respectively. Indeed, given an oriented orthonormal basis e_1, e_2, e_3, e_4 of T_pM, for any pair (ij), $1 \leq i \neq j \leq 4$, we denote $(i'j')$ to be the another pair defined by $(iji'j') = \sigma(1234)$ for some even permutation σ on $\{1, 2, 3, 4\}$. Then the Weyl tensor is well known to satisfy $W_{ijkl} = W_{i'j'k'l'}$. Then $W^\pm_{ijkl} = \frac{1}{4}(W_{ijkl} \pm W_{ijk'l'} \pm W_{i'j'kl} + W_{i'j'k'l'}) = \frac{1}{2}(W_{ijkl} \pm W_{ijk'l'})$. As the change of orientation on a manifold simply interchanges W^+ and W^-, we may treat $\delta W^+ = 0$ condition below, instead of the condition of half harmonic Weyl curvature.

Denoting by R and R_{ij} the scalar curvature and some Ricci tensor components, respectively, we recall the following formulas.

Lemma 1. *For a gradient Ricci soliton (M, g, f) the following holds.*

(i) $\nabla_k R_{jl} - \nabla_l R_{jk} = \nabla^i R_{ijkl} = R_{ijkl}\nabla^i f$.

(ii) $\nabla_i R = 2\nabla^j R_{ij} = 2R_{ij}\nabla^j f$.

(iii) $\nabla_k R_{jl} - \nabla_l R_{jk} + \nabla_{k'} R_{jl'} - \nabla_{l'} R_{jk'} = R_{ijkl}\nabla^i f + R_{ijk'l'}\nabla^i f$
$= 4\nabla^i W^+_{ijkl} + \frac{1}{6}(\nabla_k R g_{jl} - \nabla_l R g_{jk}) + \frac{1}{6}(\nabla_{k'} R g_{jl'} - \nabla_{l'} R g_{jk'})$.

In the above, (i) and (ii) are well known [5] and (iii) is from the equation (3) in [8]. Next lemma stems from [3].

Lemma 2. *Let (M, g, f) be a gradient Ricci soliton with $\delta W^+ = 0$. Let c be a regular value of f and $\Sigma_c = f^{-1}(c)$. Then the following holds.*

(i) $E_1 = \frac{\nabla f}{|\nabla f|}$ *is an eigenvector of Ric, whenever* $\nabla f \neq 0$.

(ii) $|\nabla f|$, R *and* $Ric(E_1, E_1)$ *is constant on a connected component of* Σ_c.

(iii) *There is a locally defined function s such that* $ds = \frac{df}{|\nabla f|}$ *and* $\nabla s = \frac{\nabla f}{|\nabla f|} = E_1$.

(iv) *Near a point in Σ_c, the metric g can be written as*

$$g = ds^2 + \sum_{i,j>1} g_{ij}(s, x_2, x_3, x_4)dx_i \otimes dx_j,$$

where x_2, x_3, x_4 is a local coordinates system on Σ_c.

(v) $\nabla_{E_1} E_1 = 0$.

Proof. (i) is from Lemma 2.4 in [8]. The proofs of (ii)∼(v) come from that of Lemma 2.3 of [7]. □

A gradient Ricci soliton (M, g, f) is well known to be real analytic. Recall from [6] that letting $E_r(x)$ be the number of distinct eigenvalues of Ric_x for $x \in M$, the set $M_r = \{x \in M \mid E_r \text{ is constant in a neighborhoodof } x\}$ is an open dense subset of M and in each connected component of M_r, the eigenvalues are well-defined and differentiable functions.

We will consider a point $p \in \Sigma_c$, for a regular value c of f, and local orthonormal Ricci-eigenvector fields E_i, $i = 1, 2, 3, 4$ in a neighborhood of p such that $E_1 = \frac{\nabla f}{|\nabla f|}$ and E_2, E_3, E_4 are tangent to Σ_c. We let λ_i be the Ricci-eigenvalues corresponding to E_i. The frame field $\{E_i\}$ will be called *adapted*.

Lemma 3. *For a gradient Ricci soliton (M, g, f) with $\delta W^+ = 0$ and for a local adapted frame field $\{E_i\}$, setting $\zeta_i = -\langle \nabla_{E_i} E_i, E_1\rangle$, for $i > 1$, we have;*

$$\nabla_{E_i} E_1 = \zeta_i E_i = \frac{1}{|\nabla f|}(\lambda - \lambda_i)E_i. \tag{2}$$

Proof. From the gradient Ricci soliton equation, for $i > 1$,

$$\nabla_{E_i} E_1 = \nabla_{E_i}\left(\frac{\nabla f}{|\nabla f|}\right) = \frac{\nabla_{E_i}\nabla f}{|\nabla f|} = \frac{\nabla^2 f(E_i)}{|\nabla f|}$$
$$= \frac{(-Ric + \lambda g)(E_i)}{|\nabla f|} = \frac{1}{|\nabla f|}(\lambda - \lambda_i)E_i.$$

Then

$$\zeta_i = -\langle \nabla_{E_i} E_i, E_1\rangle = \langle E_i, \nabla_{E_i} E_1\rangle = \frac{1}{|\nabla f|}(\lambda - \lambda_i). \qquad \square$$

Lemma 4. *For a Riemannian metric g of dimension four with $\delta W^+ = 0$, consider orthonormal vector fields E_i, $i = 1, \ldots, 4$ such that $Ric(E_i, \cdot) = \lambda_i g(E_i, \cdot)$. Let $A = Ric - \frac{R}{2(n-1)}$. Setting $\Gamma_{ij}^k := \langle \nabla_{E_i} E_j, E_k\rangle$, the following holds;*

$$\begin{aligned}&(\lambda_j - \lambda_k)\Gamma_{ij}^k + \nabla_{E_i}\langle E_k, AE_j\rangle - (\lambda_i - \lambda_k)\Gamma_{ji}^k - \nabla_{E_j}\langle E_k, AE_i\rangle\\ &+ (\lambda_{j'} - \lambda_k)\Gamma_{i'j'}^k + \nabla_{E_{i'}}\langle E_k, AE_{j'}\rangle - (\lambda_{i'} - \lambda_k)\Gamma_{j'i'}^k\\ &- \nabla_{E_{j'}}\langle E_k, AE_{i'}\rangle = 0\end{aligned} \tag{3}$$

for any $i, j, k = 1, \ldots, n$ with $i \neq j$.

Proof. We have $0 = -2(\delta W^+)_{kij} = 2\nabla^l W_{lkij}^+ = \nabla^l(W_{lkij} + W_{lki'j'})$. Since $-\delta W = \frac{1}{2}d^\nabla A$ [1, 16.3], we obtain

$$0 = (d^\nabla A)_{ijk} + (d^\nabla A)_{i'j'k} = \nabla_i A_{jk} - \nabla_j A_{ik} + \nabla_{i'} A_{j'k} - \nabla_{j'} A_{i'k}.$$

We have got

$$\begin{aligned}&\langle(\nabla_{E_i} A)E_j, E_k\rangle - \langle(\nabla_{E_j} A)E_i, E_k\rangle + \langle(\nabla_{E_{i'}} A)E_{j'}, E_k\rangle\\ &- \langle(\nabla_{E_{j'}} A)E_{i'}, E_k\rangle = 0.\end{aligned} \tag{4}$$

The tensor $A = Ric - \frac{R}{6}$ with eigenfunctions $\lambda_i - \frac{R}{6}$ satisfies

$$\begin{aligned}\langle(\nabla_{E_i} A)E_j, E_k\rangle =& -\langle \nabla_{E_i} E_j, AE_k\rangle - \langle \nabla_{E_i} E_k, AE_j\rangle + \nabla_{E_i}\langle E_k, AE_j\rangle\\ =& -(\lambda_k - \frac{R}{6})\langle \nabla_{E_i} E_j, E_k\rangle - (\lambda_j - \frac{R}{6})\langle \nabla_{E_i} E_k, E_j\rangle\\ &+ \nabla_{E_i}\langle E_k, AE_j\rangle\\ =& (\lambda_j - \lambda_k)\langle \nabla_{E_i} E_j, E_k\rangle + \nabla_{E_i}\langle E_k, AE_j\rangle.\end{aligned}$$

Putting this into (4), we can get (3). $\square$

Putting $i = 2$, $j = 3$, $i' = 1$, $j' = 4$ and $k = 4$ into (3), we get (5) below. Similarly, we can obtain (6)∼(13).

$$(\lambda_3 - \lambda_4)\Gamma_{23}^4 - (\lambda_2 - \lambda_4)\Gamma_{32}^4 + (\lambda_4 - \frac{R}{6})' - (\lambda_1 - \lambda_4)\zeta_4 = 0. \tag{5}$$

$$(\lambda_4 - \lambda_2)\Gamma_{34}^2 - (\lambda_3 - \lambda_2)\Gamma_{43}^2 + (\lambda_2 - \frac{R}{6})' - (\lambda_1 - \lambda_2)\zeta_2 = 0. \tag{6}$$

$$(\lambda_2 - \lambda_3)\Gamma_{42}^3 - (\lambda_4 - \lambda_3)\Gamma_{24}^3 + (\lambda_3 - \frac{R}{6})' - (\lambda_1 - \lambda_3)\zeta_3 = 0. \tag{7}$$

$$(\lambda_2-\lambda_3)\Gamma_{12}^3+(\lambda_3-\lambda_4)\Gamma_{33}^4=E_4(\lambda_3). \tag{8}$$

$$(\lambda_2-\lambda_3)\Gamma_{13}^2+(\lambda_2-\lambda_4)\Gamma_{22}^4=E_4(\lambda_2). \tag{9}$$

$$(\lambda_4-\lambda_2)\Gamma_{12}^4+(\lambda_4-\lambda_3)\Gamma_{44}^3=E_3(\lambda_4). \tag{10}$$

$$(\lambda_4-\lambda_2)\Gamma_{14}^2+(\lambda_2-\lambda_3)\Gamma_{22}^3=E_3(\lambda_2). \tag{11}$$

$$(\lambda_3-\lambda_4)\Gamma_{13}^4+(\lambda_4-\lambda_2)\Gamma_{44}^2=E_2(\lambda_4). \tag{12}$$

$$(\lambda_3-\lambda_4)\Gamma_{14}^3+(\lambda_3-\lambda_2)\Gamma_{33}^2=E_2(\lambda_3). \tag{13}$$

3. Analysis of the space when $\lambda_1=\lambda_2\neq\lambda_3=\lambda_4$

Let (M,g,f) be a gradient Ricci soliton with $\delta W^+=0$ and exactly two distinct Ricci eigenvalues at each point. There exists an adapted frame field E_j, $j=1,2,3,4$, with the corresponding eigenvalues λ_j in a neighborhood of a point p of M with $\nabla f(p)\neq 0$. After re-ordering of E_2,E_3,E_4 if necessary, we may only have to consider two cases;

(i) $\lambda_1=\lambda_2\neq\lambda_3=\lambda_4$,

(ii) $\lambda_1\neq\lambda_2=\lambda_3=\lambda_4$.

The case (i) shall be considered in this section and (ii) in Section 4.

We assume in this section that $\lambda_1=\lambda_2\neq\lambda_3=\lambda_4$. From Lemma 2(ii), the Ricci eigenvalues $\lambda_1=\lambda_2$ and $\lambda_3=\lambda_4$ are constant on a connected component of a regular level hypersurface Σ_c of f and so depend only on the local variable s in Lemma 2(iii). Also, ζ_2 and $\zeta_3=\zeta_4$, in Lemma 3 depend on s only. Hence $E_i(\lambda_j)=E_i(\zeta_k)=0$ for $i>1$.

Lemma 5. *Let (M,g,f) be a four dimensional gradient Ricci soliton with $\delta W^+=0$. Suppose that $\lambda_1=\lambda_2\neq\lambda_3=\lambda_4$ for an adapted frame fields $\{E_i\}$ on an open subset of $\{\nabla f\neq 0\}$. Then the following holds on the open subset*;

$\nabla_{E_1}E_1=0$.

$\nabla_{E_i}E_1=\zeta_i(s)E_i$ *for* $i=2,3,4$, *with* $\zeta_i(s)=\frac{1}{|\nabla f|}(\lambda-\lambda_i)$.

$\nabla_{E_1}E_2=0$.

$\nabla_{E_2}E_2=-\zeta_2(s)E_1$.

$\nabla_{E_3}E_3=-\zeta_3E_1+\beta_3E_4$.

$\nabla_{E_4}E_4=-\zeta_4E_1+\beta_4E_3$.

$\nabla_{E_3}E_2=xE_4$, $\quad\nabla_{E_4}E_2=-xE_3$ *and* $\nabla_{E_2}E_3=\Gamma_{23}^4E_4$.

$\nabla_{E_3}E_4=-xE_2-\beta_3E_3$, $\quad\nabla_{E_4}E_3=xE_2-\beta_4E_4$ *and* $\nabla_{E_2}E_4=\Gamma_{24}^3E_4$.

$[E_1,E_2]=-\zeta_2E_2$ *and* $[E_3,E_4]=-2xE_2-\beta_3E_3+\beta_4E_4$.

Proof. Lemma 2(v) already gives $\nabla_{E_1}E_1=0$. The second formula is from (2).

From $E_i(\lambda_j)=0$ for $i>1$ and from (8)$\sim$(11), we can get $\Gamma_{12}^3=\Gamma_{12}^4=\Gamma_{22}^4=\Gamma_{22}^3=0$, so that $\nabla_{E_1}E_2=0$ and $\nabla_{E_2}E_2=-\zeta_2E_1$. From (12) and (13), $\Gamma_{33}^2=\Gamma_{44}^2=0$ so that $\nabla_{E_4}E_4=-\zeta_4E_1+\beta_4E_3$, where $\beta_4=\Gamma_{44}^3$ and $\nabla_{E_3}E_3=-\zeta_3E_1+\beta_3E_4$, where $\beta_3=\Gamma_{33}^4$. Moreover, $\nabla_{E_3}E_2=xE_4$, where $x=\Gamma_{32}^4$ and $\nabla_{E_2}E_3=\Gamma_{23}^4E_4$.

By comparing (5) and (7), $\Gamma^3_{42} = -\Gamma^4_{32} = -x$, so $\nabla_{E_4} E_2 = -xE_3$. Since $\Gamma^2_{34} = -\Gamma^4_{32} = -x$ and $\Gamma^3_{34} = -\Gamma^4_{33} = -\beta_3$, $\nabla_{E_3} E_4 = -xE_2 - \beta_3 E_3$. Similarly,

$$\nabla_{E_4} E_3 = xE_2 - \beta_4 E_4,$$
$$[E_1, E_2] = \nabla_{E_1} E_2 - \nabla_{E_2} E_1 = -\zeta_2 E_2 \text{ and}$$
$$[E_3, E_4] = \nabla_{E_3} E_4 - \nabla_{E_4} E_3 = -2xE_2 - \beta_3 E_3 + \beta_4 E_4.$$

We have proved all the formulas and note that this lemma actually do not need $\lambda_1 = \lambda_2$ in the hypothesis. □

One can directly compute the curvature components from Lemma 5 and obtain the following Ricci tensor components $R_{ii} = \lambda_i$.
$R_{11} = -\zeta_2^{'} - \zeta_2^2 - 2\zeta_3^{'} - 2\zeta_3^2$.
$R_{22} = -\zeta_2^{'} - \zeta_2^2 + 2x^2 - 2\zeta_2\zeta_3$.
$R_{33} = R_{44} = -\zeta_3^{'} - 2\zeta_3^2 - \zeta_2\zeta_3 - 2x\Gamma^4_{23} + E_3(\beta_4) + E_4(\beta_3) - \beta_3^2 - \beta_4^2$.
By evaluating the equation (1) on (E_i, E_i), we can get

$$f^{''} = \lambda - R_{11} = \lambda + \zeta_2^{'} + \zeta_2^2 + 2\zeta_3^{'} + 2\zeta_3^2, \tag{14}$$

$$\zeta_2 f^{'} = \lambda - R_{22} = \lambda + \zeta_2^{'} + \zeta_2^2 - 2x^2 + 2\zeta_2\zeta_3, \tag{15}$$

$$\zeta_3 f^{'} = \lambda - R_{33} = \lambda + \zeta_3^{'} + 2\zeta_3^2 + \zeta_2\zeta_3 - A, \tag{16}$$

where $A = -2x\Gamma^4_{23} + E_3(\beta_4) + E_4(\beta_3) - \beta_3^2 - \beta_4^2$. Using (14)∼(16), the equation (5) becomes

$$\begin{aligned} 0 &= -(\lambda_2 - \lambda_4)\Gamma^4_{32} + (\lambda_4 - \frac{R}{6})^{'} - (\lambda_1 - \lambda_4)\zeta_4 \\ &= (\zeta_2 - \zeta_3) f^{'} x + (\lambda - \zeta_3 f^{'})^{'} - \frac{R^{'}}{6} + (f^{''} - \zeta_3 f^{'})\zeta_3. \end{aligned}$$

We put $R^{'} = 2R_{11} f^{'} = 2(-\zeta_2^{'} - \zeta_2^2 - 2\zeta_3^{'} - 2\zeta_3^2) f^{'}$ into the above and, after removing $f^{'}$, we have

$$3x + \frac{\zeta_2^{'} - \zeta_3^{'}}{\zeta_2 - \zeta_3} + \zeta_2 + \zeta_3 = 0. \tag{17}$$

As $\lambda_1 = \lambda_2$, from Ricci tensor formulas we get

$$-x^2 + \zeta_2\zeta_3 = \zeta_3^{'} + \zeta_3^2. \tag{18}$$

As $\lambda_1 = \lambda - f^{''} = \lambda_2 = \lambda - \zeta_2 f^{'}$, we also get

$$\zeta_2 = \frac{f^{''}}{f^{'}}. \tag{19}$$

We put $R = 4\lambda - f^{''} - f^{'}(\zeta_2 + 2\zeta_3)$ into $R^{'} = 2R_{11} f^{'}$ and get

$$-\{f^{''} + f^{'}(\zeta_2 + 2\zeta_3)\}^{'} = -2(\zeta_2^{'} + \zeta_2^2 + 2\zeta_3^{'} + 2\zeta_3^2) f^{'},$$

which gives $\frac{f^{'''}}{f^{'}} + \frac{f^{''}}{f^{'}}(\zeta_2 + 2\zeta_3) = \zeta_2^{'} + 2\zeta_2^2 + 2\zeta_3^{'} + 4\zeta_3^2$.

Put (19) into the last equation to get $\zeta_2^{'} + \zeta_2^2 + \zeta_2(\zeta_2 + 2\zeta_3) = \zeta_2^{'} + 2\zeta_2^2 + 2\zeta_3^{'} + 4\zeta_3^2$, so we obtain

$$\zeta_2\zeta_3 = \zeta_3^{'} + 2\zeta_3^2. \tag{20}$$

We put (20) into (18) and get $x^2 = \zeta_3^2$.

Now we suppose that $\zeta_3 \neq 0$ and shall derive a contradiction.

We have $\zeta_2 = \frac{\zeta_3^{'}}{\zeta_3} + 2\zeta_3$ from (20). Put this into (15) and obtain

$$\begin{aligned}\zeta_2 f^{'} &= \lambda + \frac{\zeta_3^{''}}{\zeta_3} - \frac{(\zeta_3^{'})^2}{(\zeta_3)^2} + 2\zeta_3^{'} + \frac{(\zeta_3^{'})^2}{(\zeta_3)^2} + 4(\zeta_3)^2 + 4\zeta_3^{'} + 2\zeta_3^{'} + 2\zeta_3^2 \\ &= \lambda + \frac{\zeta_3^{''}}{\zeta_3} + 8\zeta_3^{'} + 6\zeta_3^2.\end{aligned} \tag{21}$$

As $x^2 = \zeta_3^2$, we have two possibilities; $x = \zeta_3$ and $x = -\zeta_3$.

If $x = \zeta_3$, (17) and (20) give $\frac{\zeta_2^{'} - \zeta_3^{'}}{\zeta_2 - \zeta_3} + \frac{\zeta_3^{'}}{\zeta_3} + 6\zeta_3 = 0$, so that $(\zeta_3(\zeta_2 - \zeta_3))^{'} + 6\zeta_3(\zeta_2 - \zeta_3)\zeta_3 = 0$. By (20) again, the last equation becomes

$$\zeta_3^{''} + 8\zeta_3\zeta_3^{'} + 6\zeta_3^3 = 0. \tag{22}$$

Using (22), the above (21) yields $\zeta_2 f^{'} = \lambda$. With (19), we have $f^{''} = \lambda$. From (19), we get $\zeta_2^{'} + \zeta_2^2 = 0$. (14) gives $\zeta_3^{'} + \zeta_3^2 = 0$. Then (20) gives $\zeta_2\zeta_3 = \zeta_3^2$, a contradiction since $\zeta_2 \neq \zeta_3$ and $\zeta_3 \neq 0$.

If $x = -\zeta_3$, (17) gives $\frac{\zeta_2^{'} - \zeta_3^{'}}{\zeta_2 - \zeta_3} + \frac{\zeta_3^{'}}{\zeta_3} = 0$. Integration yields $(\zeta_2 - \zeta_3)\zeta_3 = c$ for a constant c. Put this to (20) and get $\zeta_3^{'} + \zeta_3^2 = c$. Using this, (21) yields $\zeta_2 f^{'} = \lambda + 6c$. By (19), $f^{''} = \lambda + 6c$. We use (19) again to have $\zeta_2^{'} + \zeta_2^2 = (\frac{\lambda + 6c}{f^{'}})^{'} + (\frac{\lambda+6c}{f^{'}})^2 = 0$. Then (14) gives $c = 0$ so that $\zeta_3^{'} + \zeta_3^2 = 0$. Then (20) gives $\zeta_2\zeta_3 = \zeta_3^2$, a contradiction. We have shown that any of the two possibilities $x = \zeta_3$ and $x = -\zeta_3$ leads to a contradiction. So, $\zeta_3 \neq 0$ leads to a contradiction.

Now, $\zeta_3 = 0$. Then $x = 0$ and $\zeta_2^{'} + \zeta_2^2 = 0$ from (17), so $\zeta_2 = \frac{1}{s + c_1}$ for a constant c_1. (15) gives $f^{'} = \lambda(s + c_1)$ and $R_{11} = R_{22} = 0$. (16) gives $R_{33} = \lambda$.

From the connection formulas of Lemma 5, we see $\langle \nabla_{E_i} E_j, E_k \rangle = 0$ when either $i, j \in \{1, 2\}$ and $k \in \{3, 4\}$, or $k \in \{1, 2\}$ and $i, j \in \{3, 4\}$. This means that the distribution D_1 spanned by E_1, E_2 and D_2 spanned by E_3, E_4 both are not only integrable, but also totally geodesic. By the local de Rham theorem, a neighborhood of $p \in M$ with $\nabla f(p) \neq 0$ is isometric to a domain in the Riemannian product $\mathbb{R}^2 \times (N, \tilde{g})$, where $(N, \tilde{g})$ is a Riemannian manifold of constant curvature $\lambda \neq 0$. We have proved:

Proposition 1. *Let (M, g, f) be a four dimensional gradient Ricci soliton with $\delta W^+ = 0$. Suppose that $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$ for an adapted frame field $\{E_i\}$ on an open subset U of $\{\nabla f \neq 0\}$.*

Then for each point p in U, there exists a neighborhood V of p in U such that (V, g) is isometric to a domain in the Riemannian product $(\mathbb{R}^2, g_0) \times (N, \tilde{g})$, *where g_0 is the flat metric on $\mathbb{R}^2$ and $(N, \tilde{g})$ is a two dimensional Riemannian manifold of constant curvature $\lambda \neq 0$.*

4. Analysis of $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_4$ and the proof of Theorem 1

In this section we consider the case (ii) mentioned at the beginning of Section 3, i.e., $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_4$.

Proposition 2. *Let (M, g, f) be a four dimensional gradient Ricci soliton with $\delta W^+ = 0$. Suppose that $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_4$ for an adapted frame field $\{E_i\}$ on an open subset U of $\{\nabla f \neq 0\}$.*

Then for each point p in U, there exists a neighborhood V of p in U where g is a warped product;

$$g = ds^2 + h(s)^2 \tilde{g}$$

for a positive function h, where the Riemannian metric $\tilde{g}$ has constant curvature.

Proof. Near $p \in U$, choose a local coordinate system $(x_1 := s, x_2, x_4, x_4)$ from Lemma 2(iv) in which $g = ds^2 + \sum_{i,j\geq 2}^{4} g_{ij} dx_i \otimes dx_j$ with $g_{ij} := g_{ij}(x_1, x_2, x_3, x_4)$. For any u tangent to hypersurfaces $\Sigma_c = f^{-1}(c)$, if we put $u = u_2E_2 + u_3E_3 + u_4E_4$, then from (2) we have $\langle \nabla_u u, E_1 \rangle = -u_2^2\zeta_2 - u_3^2\zeta_3 - u_4^2\zeta_4$. Since $\lambda_2 = \lambda_3 = \lambda_4$, we have $\zeta_2 = \zeta_3 = \zeta_4$ and the second fundamental form $H^{E_1}(u, u) = -\langle \nabla_u u, E_1 \rangle = \zeta_2 g(u, u)$. Note that λ_i and ζ_i all depend only on s in Lemma 2(iii) because R and λ_1 depend only on s by Lemma 2(ii).

For $i, j \in \{2, 3, 4\}$, setting $\partial_i = \frac{\partial}{\partial_i}$ and $g_{ij} = g(\partial_i, \partial_j)$,

$$\begin{aligned}
\zeta_2 g_{ij} = H^{E_1}(\partial_i, \partial_j) &= -\langle \nabla_{\partial_i}\partial_j, \frac{\partial}{\partial s} \rangle = -\langle \sum_k \Gamma_{ij}^k \partial_k, \frac{\partial}{\partial s} \rangle \\
&= -\sum_k \langle \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})\partial_k, \frac{\partial}{\partial s} \rangle = \frac{1}{2}\frac{\partial}{\partial s} g_{ij}.
\end{aligned}$$

So, $\frac{1}{2}\frac{\partial}{\partial s} g_{ij} = \zeta_2 g_{ij}$. Integrating it, we get $g_{ij} = e^{C_{ij}} h(s)^2$ for a positive function $h := h(s)$, where $\zeta_2 = \frac{h'(s)}{h(s)}$ and each C_{ij} depends only on x_2, x_3, x_4.

Therefore g can be written as $g = ds^2 + h(s)^2 \tilde{g}$, where $\tilde{g}$ can be viewed as a Riemannian metric on a domain of (x_2, x_3, x_4)-space. The rest of argument in this proof originates from [6].

For $i, j \in \{2, 3, 4\}$, we have connection formulas and Ricci tensor components as follows;

$$\begin{aligned}
&\Gamma_{ij}^1 = -hh'\tilde{g}_{ij}, \quad \Gamma_{1j}^i = \frac{h'}{h}\delta_{ij}, \\
&R_{1i} = 0, \quad R_{11} = -3\frac{h''}{h}, \quad R_{ij} = -\tilde{g}_{ij}(hh'' + 2h'^2) + R_{ij}^{\tilde{g}},
\end{aligned}$$

where $R_{ij}^{\tilde{g}}$ is the Ricci tensor components of $\tilde{g}$.

From Lemma 1(iii), we have $\nabla_1 R_{ij} - \nabla_j R_{i1} = \frac{\partial_1 R}{6} g_{ij}$ for $i, j \in \{2,3,4\}$. Hence

$$\begin{aligned}
\frac{\partial_1 R}{6} h^2 \tilde{g}_{ij} &= \frac{\partial_1 R}{6} g_{ij} = \nabla_1 R_{ij} - \nabla_i R_{1j} \\
&= \partial_1 R_{ij} - R(\nabla_{\partial_1} \partial_j, \partial_i) + R(\nabla_{\partial_i} \partial_j, \partial_1) \\
&= \partial_1 R_{ij} - \frac{h'}{h} R(\partial_j, \partial_i) - hh' R(\partial_1, \partial_1) \tilde{g}(\partial_i, \partial_j) \\
&= -\tilde{g}_{ij} \partial_1 (hh'' + 2h'^2) - \frac{h'}{h} \{-\tilde{g}_{ij}(hh'' + 2h'^2) + R^{\tilde{g}}_{ij}\} - hh' R_{11} \tilde{g}_{ij}.
\end{aligned}$$

Since R and $\partial_1 R$ depends only on s, we get $R^{\tilde{g}}_{ij} = H(s)\tilde{g}_{ij}$ for a function $H(s)$ of s only. So $\tilde{g}$ is a 3-dimensional Einstein metric. □

Proof of Theorem 1. The space (M, g, f) has half harmonic Weyl curvature, so by a change of orientation if necessary, we may assume $\delta W^+ = 0$. By treating each connected component of M, we may assume that M is connected. The space (M, g, f) is real analytic. So, if $\nabla f = 0$ on an open set, then $\nabla f = 0$ on M and g is Einstein. If $\nabla f \neq 0$ at a point, then $M_r \cap \{\nabla f \neq 0\}$ is an open dense subset of M.

If there is exactly one distinct Ricci eigenvalue in a neighborhood V of a point p in $M_r \cap \{\nabla f \neq 0\}$, then g is Einstein on V.

If there are exactly two distinct Ricci eigenvalues in a neighborhood V of a point p in $M_r \cap \{\nabla f \neq 0\}$, we have already reduced the proof to the two cases (i) and (ii) in the first paragraph of Section 3. Proposition 1 and Proposition 2 resolve the two cases. This finishes the proof of Theorem 1. □

References

[1] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10, Springer-Verlag, Berlin, 1987. `https://doi.org/10.1007/978-3-540-74311-8`

[2] H.-D. Cao and Q. Chen, *On locally conformally flat gradient steady Ricci solitons*, Trans. Amer. Math. Soc. **364** (2012), no. 5, 2377–2391. `https://doi.org/10.1090/S0002-9947-2011-05446-2`

[3] H.-D. Cao and Q. Chen, *On Bach-flat gradient shrinking Ricci solitons*, Duke Math. J. **162** (2013), no. 6, 1149–1169. `https://doi.org/10.1215/00127094-2147649`

[4] X. Chen and Y. Wang, *On four-dimensional anti-self-dual gradient Ricci solitons*, J. Geom. Anal. **25** (2015), no. 2, 1335–1343. `https://doi.org/10.1007/s12220-014-9471-8`

[5] B. Chow, S. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications. Part I*, Mathematical Surveys and Monographs, 135, American Mathematical Society, Providence, RI, 2007. `https://doi.org/10.1090/surv/135`

[6] A. Derdziński, *Classification of certain compact Riemannian manifolds with harmonic curvature and nonparallel Ricci tensor*, Math. Z. **172** (1980), no. 3, 273–280. `https://doi.org/10.1007/BF01215090`

[7] J. Kim, *On a classification of 4-d gradient Ricci solitons with harmonic Weyl curvature*, J. Geom. Anal. **27** (2017), no. 2, 986–1012. `https://doi.org/10.1007/s12220-016-9707-x`

[8] J.-Y. Wu, P. Wu, and W. Wylie, *Gradient shrinking Ricci solitons of half harmonic Weyl curvature*, Calc. Var. Partial Differential Equations **57** (2018), no. 5, Paper No. 141, 15 pp. `https://doi.org/10.1007/s00526-018-1415-x`

Yutae Kang
Department of Mathematics
Sogang University
Seoul 04107, Korea
Email address: `lubo@sogang.ac.kr`

Jongsu Kim
Department of Mathematics
Sogang University
Seoul 04107, Korea
Email address: `jskim@sogang.ac.kr`