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ON THE INDEX AND BIDERIVATIONS OF SIMPLE MALCEV ALGEBRAS

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Abstract. Let $(M, [\ ,\ ])$ be a finite dimensional Malcev algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. We first prove that, $(M, [\ ,\ ])$ (with $[M, M] \neq 0$) is simple if and only if $ind(M) = 1$ (i.e., M admits a unique (up to a scalar multiple) invariant scalar product). Further, we characterize the form of skew-symmetric biderivations on simple Malcev algebras. In particular, we prove that the simple seven dimensional non-Lie Malcev algebra has no nontrivial skew-symmetric biderivation.

1. Introduction

In this paper, we consider finite dimensional Malcev algebras over an algebraically closed field $\mathbb{F}$ of characteristic 0.

Malcev algebras was introduced by A. I. Malcev in [18] with the name of Moufang-Lie algebras as tangent algebras of analytic Moufang loops, its present name being given by A. A. Sagle in [21]. They are closely connected with the alternative algebras in the same way as Lie algebras are related to associative algebras. A Malcev algebra $(M, [\ ,\ ])$ is called quadratic (or pseudo-Euclidean) if it is endowed with a nondegenerate symmetric bilinear form ψ which is invariant, that is,

$$\psi([x, y], z) = \psi(x, [y, z]),\ \forall x, y, z \in M.$$

In this case, ψ is called an invariant scalar product on M. An inductive description of quadratic Malcev algebras and Malcev superalgebras was already established by H. Albuquerque and S. Benayadi [1]. Since Malcev algebras are naturally a generalization of Lie algebras, the study of Malcev algebras has often proceeded by determining which properties of Lie algebras apply, in an appropriate form, to Malcev algebras. Let us denote by $\mathcal{F}(M)$ the linear space of all symmetric invariant bilinear forms on Malcev algebra M and let $\mathcal{B}(M)$

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be the subspace of $\mathcal{F}(M)$ spanned by the set of invariant scalar products on M. The dimension of $\mathcal{B}(M)$ is called the index of M and noted by $ind(M)$.

I. Bajo and S. Benayadi [2] proved that over an algebraically closed field $\mathbb{F}$ of characteristic 0, a Lie algebra $(M, [\ ,\ ])$, with $[M, M] \neq 0$ is simple if and only if $\dim \mathcal{B}(M) = 1$. The analogous result was obtained in [3] for Jordan algebras, in [28] for Lie triple systems and in [5] for alternative algebras. Our first main result (Corollary 3.7) is to give the analogous result for Malcev algebras.

Next, we study skew-symmetric biderivations on simple Malcev algebras. Commuting maps and biderivations arose first in the associative ring theory [6, 7]. Then, many authors have made considerable efforts to make the study of these maps very successful, see for example [9, 10, 14, 16, 23–25]. To study biderivations and commuting linear maps of Schrödininger-Virasoro Lie algebra, in [24], the authors use the $\mathbb{Z}$-graduation of this algebra. Furthermore, in [8] the authors give a general method to characterise biderivations and commuting linear maps for a large class of Lie algebras, the results obtained show under certain conditions the crucial relationship between biderivations and commuting linear maps of a Lie algebra $\mathcal{L}$ and the elements of $cent(M)$, with $cent(M)$ denote the centroid of an $\mathcal{L}$-modules M. The way used in [23] requires the use of root systems of the simple Lie algebra. In this paper, using a classification theorem in [11] and a computational method, we extend the study of skew-symmetric biderivations to simple Malcev algebras. Our second main result (Theorem 4.4) is to prove the following: Let $(M, [\ ,\ ])$ be a simple finite-dimensional Malcev algebra over an algebraically closed field of characteristic zero $\mathbb{F}$. Then, every skew-symmetric biderivation δ of M is of the form $\delta(x, y) = \lambda[x, y]$, $x, y \in M$ for some $\lambda \in \mathbb{F}$.

This paper is organized as follows. In the first section we recall general definitions and examples related to Malcev algebras. Section 2 is dedicated to give examples of quadratic Malcev algebras and to prove an analogous result (Corollary 3.1 in [2]) for Malcev algebras. The last section aims to characterize the form of skew-symmetric biderivations on simple Malcev algebras. In particular, we prove that every skew-symmetric biderivation on the simple seven dimensional non-Lie Malcev algebra is trivial. Then, we deduce our second main result (Theorem 4.4).

2. Preliminaries

In this section, We recall some definitions and facts related to Malcev algebras. The theory of Malcev algebras is well developed, see for example ([11–13, 20, 21, 26, 27]).

Definition. A Malcev algebra $(M, [\cdot,\cdot])$ is a vector space M with a bilinear map $[\cdot,\cdot] : M \times M \to M$ satisfying:

(1) $[x, y] = -[y, x]$, (skew-symmetry)
(2) $J_M(x, y, [x, z]) = [J_M(x, y, z), x]$, (Malcev identity)

for all x, y, z from M, where J_M is the Jacobian of M defined by

$$J_M(x,y,z) = [[x,y],z] + [[y,z],x] + [[z,x],y] \text{ for all } x,y,z \in M.$$

Definition. A Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is a vector space $\mathfrak{g}$ with a skew-symmetric bracket $[\cdot,\cdot]$ such that $J_{\mathfrak{g}}(x,y,z) = 0$ for all $x,y,z \in \mathfrak{g}$.

Remark 2.1. It is easy to check that every Lie algebra is a Malcev algebra, because Lie algebras in particular satisfy the Malcev identity.

However, there are Malcev algebras which are non-Lie algebras.

Example 2.2 ([21]). Let M has a basis $\{e_1, e_2, e_3, e_4\}$ with multiplication table:

$[\cdot,\cdot]$	e_1	e_2	e_3	e_4
e_1	0	$-e_2$	$-e_3$	0
e_2	e_2	0	$2e_4$	0
e_3	e_3	$-2e_4$	0	0
e_4	$-e_4$	0	0	0

With few calculations, one can easily show that M is a Malcev algebra. Moreover, $J_M(e_1, e_2, [e_1, e_3]) = 6e_4 = [J_M(e_1, e_2, e_3), e_1]$ and hence M is not a Lie algebra.

Now, we give a proposition which is very useful in the sequel.

Proposition 2.3 ([11]). *A simple Malcev algebra is either a simple Lie algebra or isomorphic to the 7-dim simple* (*non-Lie*) *Malcev algebras* $M(\alpha,\beta,\gamma)$, *where* α, β, γ *are scalars in* $\mathbb{F}$ *with* $\alpha\beta\gamma \neq \{0\}$.

Recall that a Malcev algebra M is called simple if it has no ideals except itself and zero, and $[M,M] \neq \{0\}$.

Remark 2.4. The simple non-Lie Malcev algebra $M(\alpha,\beta,\gamma)$, is isomorphic to the 7-dimensional Malcev algebra $\mathbf{B}$ constructed by Sagle ([22], Theorem 7.12) which is defined by the following multiplication table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	$-\alpha e_2$	$-\alpha e_3$	$-\alpha e_4$	αe_5	αe_6	αe_7
e_2	αe_2	0	$2e_7$	$-2e_6$	e_1	0	0
e_3	αe_3	$-2e_7$	0	$2e_5$	0	e_1	0
e_4	αe_4	$2e_6$	$-2e_5$	0	0	0	e_1
e_5	$-\alpha e_5$	$-e_1$	0	0	0	αe_4	$-\alpha e_3$
e_6	$-\alpha e_6$	0	$-e_1$	0	$-\alpha e_4$	0	αe_2
e_7	$-\alpha e_7$	0	0	$-e_1$	αe_3	$-\alpha e_2$	0

Now, we recall Loos's construction [17] of Lie triple systems from Malcev algebras.

Definition ([19]). A Lie triple system is a vector space $\mathbf{T}$ equipped with a trilinear product $\{a,b,c\}$ satisfying the following three properties:

(1) $\{x,y,z\} = -\{y,x,z\}$,

(2) $\{x,y,z\}+\{y,z,x\}+\{z,x,y\}=0$,
(3) $\{x,y,\{z,t,w\}\}-\{z,t,\{x,y,w\}\}=\{\{x,y,z\},t,w\}+\{z,\{x,y,t\},w\}$,

for all $x,y,z,t,w\in\mathbf{T}$.

Example 2.5 ([17])**.** Let $(M,[\ ,\ ])$ be a Malcev algebra, then the pair $(M,\{\cdot,\cdot\})$ where $\{\cdot,\cdot\}: M\times M\times M\longrightarrow M$ defined by

$$\{x,y,z\}=2[[x,y],z]-[[y,z],x]-[[z,x],y],\ \forall x,y,z\in M$$

is a Lie triple system.

A general study of Lie triple systems are considered in [15, 19].

3. Malcev algebras admitting a unique quadratic structure

In this section, we will prove the first main result: a Malcev algebra is simple if and only if it admits a unique (up to a scalar multiple) invariant scalar product.

Definition. Let $(M,[\ ,\ ])$ be a Malcev algebra and $\psi: M\times M\longrightarrow\mathbb{F}$ be a bilinear form. ψ will be called:

(1) symmetric if $\psi(x,y)=\psi(y,x)$ for all $x,y\in M$;
(2) nondegenerate if $\psi(x,y)=0$ for all $y\in M\Rightarrow x=0$ and if $\psi(x,y)=0$ for all $x\in M\Rightarrow y=0$;
(3) invariant if $\psi([x,y],z)=\psi(x,[y,z])$ for all $x,y,z\in M$.

If ψ is symmetric, nondegenerate and invariant, (M,ψ) will be called a quadratic Malcev algebra and ψ will be called an invariant scalar product.

Example 3.1 ([17])**.** The Killing form of a semisimple Malcev algebra M over a field of characteristic zero is nondegenerate.

It follows that every simple or semisimple Malcev algebra is quadratic.

Another interesting quadratic Malcev algebra is giving as follows:

Let $(M,[\ ,\ ])$ be a Malcev algebra and M^* be the dual vector space of the underlying vector space of M. An easy computation proves that the following product $\star$ defines a Malcev algebra structure on the vector space $\widetilde{M}=M\oplus M^*$:

$$(x+f)\star(y+h):=[x,y]+f\circ L_y+h\circ R_x\ \text{ for all } (x,f),(y,h)\in\widetilde{M}.$$

Where L_x(resp. R_x) is the left multiplication (resp. the right multiplication) by x in the Malcev algebra $(M,[\ ,\ ])$.

Moreover, if we consider the bilinear form $\psi:\widetilde{M}\times\widetilde{M}\to\mathbb{F}$ defined by:

$$\psi(x+f,y+h)=f(y)+h(x)\ \text{ for all } (x,f),(y,h)\in\widetilde{M},$$

then $(\widetilde{M},\psi)$ is a quadratic Malcev algebra called the trivial T^*-extension of M and noted by T^*_0M ([4]). For more details about quadratic Malcev algebras (see [1]).

Definition. Let $(\mathbf{T}, \{\cdot,\cdot,\cdot\})$ be a Lie triple system. A non-degenerate symmetric bilinear form ψ on $\mathbf{T}$ is said to be invariant on the Lie triple system $(\mathbf{T}, \{\cdot,\cdot,\cdot\})$ if it satisfies:

$$\psi(R(x,y)z,t) = \psi(z, R(y,x)t) \text{ for all } x,y,z,t \in \mathbf{T}.$$

Where $R(x,y)$ is the endomorphism of $\mathbf{T}$ defined by $R(x,y)(z) = \{x,y,z\}$, for all $z \in \mathbf{T}$. Such a form is called an invariant scalar product on $(\mathbf{T}, \{\cdot,\cdot,\cdot\})$.

Proposition 3.2. *Let $(M, [\ ,\ ], \psi)$ be a quadratic Malcev algebra, then ψ is an invariant scalar product on the Lie triple system $(M, \{\cdot,\cdot,\cdot\})$ obtained by Example 2.5.*

Proof. Let $x,y,z,t \in M$.

$$\begin{aligned}
\psi(\{x,y,z\},t) &= 2\psi([[x,y],z],t) - \psi([[y,z],x],t) - \psi([[z,x],y],t)\\
&= 2\psi([x,y],[z,t]) - \psi([y,z],[x,t]) - \psi([z,x],[y,t])\\
&= 2\psi([[y,x],t],z) - \psi([[x,t],y],z) - \psi([[t,y],x],z)\\
&= \psi(z, 2[[y,x],t] - [[x,t],y] - [[t,y],x])\\
&= \psi(z, \{y,x,t\}).
\end{aligned}$$

This means that

$$\psi(R(x,y)z,t) = \psi(z, R(y,x)t). \qquad \square$$

Proposition 3.3. *Let M be a simple Malcev algebra. Then $\dim \mathcal{B}(M) = 1$.*

Proof. By ([17], Corollary 2), the Lie triple system $(M, \{\cdot,\cdot,\cdot\})$ obtained by Example 2.5 is simple. If ψ_1, ψ_2 are two invariant scalar products on $(M, [\ ,\ ])$, then ψ_1, ψ_2 become two invariant scalar products on the Lie triple system $(M, \{\cdot,\cdot,\cdot\})$. By using ([28], Corollary 4.6) there is a nonzero scalar λ such that

$$\psi_1(x,y) = \lambda\psi_2(x,y),\ \forall x,y \in M.$$

Then, $\dim \mathcal{B}(M) = 1$. $\square$

Lemma 3.4. *If M is a Malcev algebra admitting an invariant scalar product, then $\mathcal{B}(M) = \mathcal{F}(M)$.*

Proof. Let ψ be an invariant scalar product on M and $\phi \in \mathcal{F}(M)$. Let $\mathcal{M}(\psi)$ and $\mathcal{M}(\phi)$ be associated matrices of ψ and ϕ in some fixed basis of M. Then, for $\lambda \in \mathbb{F}$ the determinant $\det(\mathcal{M}(\phi) - \lambda\mathcal{M}(\psi))$ is a polinomial in λ. Hence, we can find $\lambda_0 \in \mathbb{F}$ such that $\det(\mathcal{M}(\phi) - \lambda_0\mathcal{M}(\psi)) \neq 0$. This proves that $\phi - \lambda_0\psi$ is nondegenerate and thus $\phi = (\phi - \lambda_0\psi) + \lambda_0\psi$ is nondegenerate. $\square$

Definition. Let $(M, [\ ,\ ], \psi)$ be a quadratic Malcev algebra and I an arbitrary vector subspace of M.

(1) I is called an ideal (resp. a subalgebra) of M if and only if $[I, M] \subset I$ (resp. $[I, I] \subset I$).

(2) I is called nondegenerate if the restriction of ψ to $I\times I$ is nondegenerate, otherwise, it is called degenerate.
(3) We say that (M,ψ) is irreducible if every ideal of M is degenerate.

Lemma 3.5. *Any Malcev algebra M such that $\dim \mathcal{B}(M)=1$ is irreducible.*

Proof. Suppose that $\dim \mathcal{B}(M)=1$. Then by Lemma 3.4, every nonzero symmetric invariant bilinear form on M is nondegenerate. Moreover, since M is a quadratic Malcev algebra, then, (see [4])

$$M=\bigoplus_{i=1}^{n} I_i,$$

where for all $1\leq i\leq n$, I_i is a nondegenerate irreducible ideal, and for all $i\neq j$, I_i and I_j are orthogonal. If ψ_1 denotes the Killing form of I_1 then the bilinear form φ on M defined by $\varphi(x,y)=\psi_1(x,y)$ whenever $x,y\in I_1$ and $\varphi(x,y)=0$ otherwise, is a degenerate invariant symmetric bilinear form, which contradicts the result in Lemma 3.4. Then M is irreducible. □

Proposition 3.6. *Let M be a Malcev algebra with $[M,M]\neq 0$. If the vector space $\mathcal{B}(M)$ is one-dimensional, then M is a simple Malcev algebra.*

Proof. By Lemma 3.4, we deduce that $\dim \mathcal{F}(M)=1$ and hence every nonzero symmetric invariant bilinear form on M is nondegenerate. This implies that the Killing form of M is nondegenerate and M is semisimple Malcev algebra. Let $M=M_1\oplus\cdots\oplus M_n$ be the decomposition of M into the direct sum of simple ideals. If ψ_1 denotes the Killing form of M_1 then the bilinear form ϕ on M defined by $\phi(x,y)=\psi_1(x,y)$ whenever $x,y\in M_1$ and $\phi(x,y)=0$ otherwise, is a degenerate invariant symmetric bilinear form, which contradicts the result in Lemma 3.4. □

Corollary 3.7. *Let M be a Malcev algebra with $[M,M]\neq 0$. Then, M is simple if and only if $\dim \mathcal{B}(M)=1$.*

4. Biderivations on simple Malcev algebras

In this section, we characterize the form of skew-symmetric biderivations on finite-dimensional simple Malcev algebras. For finite-dimensional simple Lie algebras the form of skew-symmetric biderivations were characterized in ([8]). Therefore, thanks to Proposition (2.3), we only have to characterize the form of skew-symmetric biderivations on seven-dimensional simple non-Lie Malcev algebra $\mathbf{B}$.

Definition. Let M be a Malcev algebra, a linear map $D: M\longrightarrow M$ is called derivation on M, if it satisfies the following identity:

$$D([x,\,y])=[D(x),\,y]+[x,\,D(y)]$$

for all $x,\,y\in M$.

Definition. A bilinear map $\delta : M \times M \longrightarrow M$ is called skew-symmetric biderivation on M, if it satisfies the following identities:

$$\delta(x, y) = -\delta(y, x),$$
$$\delta(x, [y, z]) = [\delta(x, y), z] + [y, \delta(x, z)]$$

for all $x, y, z \in M$.

Now, let δ be a skew-symmetric biderivation of the simple seven dimensional non-Lie Malcev algebra $\mathbf{B}$ and $x, y \in \mathbf{B}$ such that $x = \sum_{i=1}^{7} x_i e_i$ and $y = \sum_{i=1}^{7} y_i e_i$. Then by the bilinearity of δ, we obtain,

$$\delta(x, y) = \sum_{i=1}^{7}\sum_{j=1}^{7} x_i y_j \delta(e_i, e_j).$$

The map $\delta_x : \mathbf{B} \longrightarrow \mathbf{B}$, which is defined by $\delta_x(y) = \delta(x, y)$ is a derivation of $\mathbf{B}$, since δ is a skew-symmetric biderivation of $\mathbf{B}$. Therefore, we reduce our study to characterize the form of derivations of the algebra $\mathbf{B}$.

The following lemma gives the form of every derivation on $\mathbf{B}$, we obtain this by computations with a Maple mathematical software.

Lemma 4.1. *Let D be a derivation of the simple seven dimensional non-Lie Malcev algebra $\mathbf{B}$. Then, the matrix of D is of the form*:

$$M_D = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ \alpha a_{15} & a_{22} & a_{23} & a_{24} & 0 & \frac{-\alpha a_{14}}{2} & \frac{\alpha a_{13}}{2} \\ \alpha a_{16} & a_{32} & a_{33} & a_{34} & \frac{\alpha a_{14}}{2} & 0 & \frac{-\alpha a_{12}}{2} \\ \alpha a_{17} & a_{42} & a_{43} & -a_{33}-a_{22} & \frac{-\alpha a_{13}}{2} & \frac{\alpha a_{12}}{2} & 0 \\ \alpha a_{12} & 0 & a_{17} & -a_{16} & -a_{22} & -a_{32} & -a_{42} \\ \alpha a_{13} & -a_{17} & 0 & a_{15} & -a_{23} & -a_{33} & -a_{43} \\ \alpha a_{14} & a_{16} & -a_{15} & 0 & -a_{24} & -a_{34} & a_{33}+a_{22} \end{pmatrix},$$

where α, $a_{ij} \in \mathbb{F}$.

By Lemma 4.1, the matrix M_{e_i} of δ_{e_i}, for $i = 1, \ldots, 7$ is of the form

$$M_{e_i} = \begin{pmatrix} 0 & a^i_{12} & a^i_{13} & a^i_{14} & a^i_{15} & a^i_{16} & a^i_{17} \\ \alpha a^i_{15} & a^i_{22} & a^i_{23} & a^i_{24} & 0 & \frac{-\alpha a^i_{14}}{2} & \frac{\alpha a^i_{13}}{2} \\ \alpha a^i_{16} & a^i_{32} & a^i_{33} & a^i_{34} & \frac{\alpha a^i_{14}}{2} & 0 & \frac{-\alpha a^i_{12}}{2} \\ \alpha a^i_{17} & a^i_{42} & a^i_{43} & -a^i_{33}-a^i_{22} & \frac{-\alpha a^i_{13}}{2} & \frac{\alpha a^i_{12}}{2} & 0 \\ \alpha a^i_{12} & 0 & a^i_{17} & -a^i_{16} & -a^i_{22} & -a^i_{32} & -a^i_{42} \\ \alpha a^i_{13} & -a^i_{17} & 0 & a^i_{15} & -a^i_{23} & -a^i_{33} & -a^i_{43} \\ \alpha a^i_{14} & a^i_{16} & -a^i_{15} & 0 & -a^i_{24} & -a^i_{34} & a^i_{33}+a^i_{22} \end{pmatrix}.$$

Since δ is skew-symmetric, then, $\delta(e_i,\, e_i) = 0$ for $i = 1, \dots, 7$. Therefore

$$M_{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22}^1 & a_{23}^1 & a_{24}^1 & 0 & 0 & 0 \\ 0 & a_{32}^1 & a_{33}^1 & a_{34}^1 & 0 & 0 & 0 \\ 0 & a_{42}^1 & a_{43}^1 & -a_{33}^1 - a_{22}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{22}^1 & -a_{32}^1 & -a_{42}^1 \\ 0 & 0 & 0 & 0 & -a_{23}^1 & -a_{33}^1 & -a_{43}^1 \\ 0 & 0 & 0 & 0 & -a_{24}^1 & -a_{34}^1 & a_{33}^1 + a_{22}^1 \end{pmatrix},$$

$$M_{e_2} = \begin{pmatrix} 0 & 0 & a_{13}^2 & a_{14}^2 & a_{15}^2 & 0 & 0 \\ \alpha a_{15}^2 & 0 & a_{23}^2 & a_{24}^2 & 0 & \frac{-\alpha a_{14}^2}{2} & \frac{\alpha a_{13}^2}{2} \\ 0 & 0 & a_{33}^2 & a_{34}^2 & \frac{\alpha a_{14}^2}{2} & 0 & 0 \\ 0 & 0 & a_{43}^2 & -a_{33}^2 & \frac{-\alpha a_{13}^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha a_{13}^2 & 0 & 0 & a_{15}^2 & -a_{23}^2 & -a_{33}^2 & -a_{43}^2 \\ \alpha a_{14}^2 & 0 & -a_{15}^2 & 0 & -a_{24}^2 & -a_{34}^2 & a_{33}^2 \end{pmatrix},$$

$$M_{e_3} = \begin{pmatrix} 0 & a_{12}^3 & 0 & a_{14}^3 & 0 & a_{16}^3 & 0 \\ 0 & a_{22}^3 & 0 & a_{24}^3 & 0 & \frac{-\alpha a_{14}^3}{2} & 0 \\ \alpha a_{16}^3 & a_{32}^3 & 0 & a_{34}^3 & \frac{\alpha a_{14}^3}{2} & 0 & \frac{-\alpha a_{12}^3}{2} \\ 0 & a_{42}^3 & 0 & -a_{22}^3 & 0 & \frac{\alpha a_{12}^3}{2} & 0 \\ \alpha a_{12}^3 & 0 & 0 & -a_{16}^3 & -a_{22}^3 & -a_{32}^3 & -a_{42}^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha a_{14}^3 & a_{16}^3 & 0 & 0 & -a_{24}^3 & -a_{34}^3 & a_{22}^3 \end{pmatrix},$$

$$M_{e_4} = \begin{pmatrix} 0 & a_{12}^4 & a_{13}^4 & 0 & 0 & 0 & a_{17}^4 \\ 0 & a_{22}^4 & a_{23}^4 & 0 & 0 & 0 & \frac{\alpha a_{13}^4}{2} \\ 0 & a_{32}^4 & a_{33}^4 & 0 & 0 & 0 & \frac{-\alpha a_{12}^4}{2} \\ \alpha a_{17}^4 & a_{42}^4 & a_{43}^4 & 0 & \frac{-\alpha a_{13}^4}{2} & \frac{\alpha a_{12}^4}{2} & 0 \\ \alpha a_{12}^4 & 0 & a_{17}^4 & 0 & -a_{22}^4 & -a_{32}^4 & -a_{42}^4 \\ \alpha a_{13}^4 & -a_{17}^4 & 0 & 0 & -a_{23}^4 & -a_{33}^4 & -a_{43}^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{e_5} = \begin{pmatrix} 0 & a_{12}^5 & 0 & 0 & 0 & a_{16}^5 & a_{17}^5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha a_{16}^5 & a_{32}^5 & a_{33}^5 & a_{34}^5 & 0 & 0 & \frac{-\alpha a_{12}^5}{2} \\ \alpha a_{17}^5 & a_{42}^5 & a_{43}^5 & -a_{33}^5 & 0 & \frac{\alpha a_{12}^5}{2} & 0 \\ \alpha a_{12}^5 & 0 & a_{17}^5 & -a_{16}^5 & 0 & -a_{32}^5 & -a_{42}^5 \\ 0 & -a_{17}^5 & 0 & 0 & 0 & -a_{33}^5 & -a_{43}^5 \\ 0 & a_{16}^5 & 0 & 0 & 0 & -a_{34}^5 & a_{33}^5 \end{pmatrix},$$

$$M_{e_6}=\begin{pmatrix} 0 & 0 & a^6_{13} & 0 & a^6_{15} & 0 & a^6_{17}\\ \alpha a^6_{15} & a^6_{22} & a^6_{23} & a^6_{24} & 0 & 0 & \frac{\alpha a^6_{13}}{2}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \alpha a^6_{17} & a^6_{42} & a^6_{43} & -a^6_{22} & \frac{-\alpha a^6_{13}}{2} & 0 & 0\\ \alpha a^6_{12} & 0 & a^6_{17} & 0 & -a^6_{22} & 0 & -a^6_{42}\\ \alpha a^6_{13} & -a^6_{17} & 0 & a^6_{15} & -a^6_{23} & 0 & -a^6_{43}\\ 0 & 0 & -a^6_{15} & 0 & -a^6_{24} & 0 & a^6_{22} \end{pmatrix},$$

$$M_{e_7}=\begin{pmatrix} 0 & a^7_{12} & 0 & a^7_{14} & a^7_{15} & a^7_{16} & 0\\ \alpha a^7_{15} & a^7_{22} & a^7_{23} & a^7_{24} & 0 & \frac{-\alpha a^7_{14}}{2} & 0\\ \alpha a^7_{16} & a^7_{32} & a^7_{33} & a^7_{34} & \frac{\alpha a^7_{14}}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -a^7_{16} & -a^7_{22} & -a^7_{32} & 0\\ 0 & 0 & 0 & a^7_{15} & -a^7_{23} & -a^7_{33} & 0\\ \alpha a^7_{14} & a^7_{16} & -a^7_{15} & 0 & -a^7_{24} & -a^7_{34} & 0 \end{pmatrix}.$$

Also, we have $\delta(e_1, e_i) = -\delta(e_i, e_1)$ for $i = 2, \ldots, 7$. Then, we deduce the following Equations:

$$\begin{gathered} a^1_{22} = -\alpha a^2_{15},\\ a^1_{32} = a^1_{42} = a^2_{13} = a^2_{14} = 0,\\ a^1_{33} = -\alpha a^3_{16},\\ a^1_{23} = a^1_{43} = a^3_{12} = a^3_{14} = 0, \end{gathered}\tag{4.1}$$

$$\begin{gathered} a^1_{33} + a^1_{22} = \alpha a^4_{17},\\ a^1_{24} = a^1_{34} = a^4_{12} = a^4_{13} = 0,\ a^4_{33} = -a^4_{22},\\ a^1_{22} = \alpha a^5_{12},\\ a^1_{24} = a^1_{23} = a^5_{16} = a^5_{17} = 0, \end{gathered}\tag{4.2}$$

$$\begin{gathered} a^1_{33} = \alpha a^6_{13},\\ a^1_{32} = a^1_{34} = a^6_{12} = a^6_{15} = a^6_{17} = 0,\\ a^1_{33} + a^1_{22} = -\alpha a^7_{14},\\ a^1_{42} = a^1_{43} = a^7_{15} = a^7_{16} = 0 \text{ and } a^7_{33} = -a^7_{22}. \end{gathered}\tag{4.3}$$

Since $\delta(e_2, e_i) = -\delta(e_i, e_2)$ for $i = 3, \ldots, 7$ and by using the above equations, we have

$$\begin{gathered} a^2_{15} = a^3_{16},\\ a^2_{23} = -a^3_{22},\ a^2_{33} = -a^3_{32},\ a^2_{43} = -a^3_{42},\ a^2_{24} = -a^4_{22},\ a^2_{34} = -a^4_{32},\ a^2_{33} = a^4_{42},\\ a^2_{15} = a^4_{17},\\ a^2_{15} = -a^5_{12},\\ a^2_{23} = -a^5_{17},\ a^2_{24} = a^5_{16} = a^2_{14} = a^5_{32} = a^2_{13} = a^5_{42} = 0,\ a^2_{33} = -a^6_{17},\\ a^2_{14} = a^6_{22} = a^2_{34} = a^6_{17} = a^6_{42} = 0 \text{ and } a^2_{33} = a^7_{16} = a^2_{13} = a^7_{22} = a^2_{43} = a^7_{12} = a^7_{32} = 0. \end{gathered}\tag{4.4}$$

Furthermore, since $\delta(e_3,\, e_i) = -\delta(e_i,\, e_3)$ for $i = 4, \dots, 7$, we get

$$a^3_{24} = -a^4_{23},\ a^3_{34} = -a^4_{33},\ a^3_{22} = a^4_{43},$$

$$a^3_{16} = a^4_{17},$$

$$a^3_{14} = a^5_{33} = a^5_{43} = a^3_{22} = a^3_{24} = a^5_{17} = 0,$$

$$a^3_{16} = -a^6_{13},$$

$$a^3_{14} = a^6_{23} = a^3_{12} = a^6_{43} = a^3_{32} = a^6_{43} = a^3_{34} = a^6_{15} = 0$$

and

$$a^3_{22} = a^7_{15} = a^2_{13} = a^7_{22} = a^3_{12} = a^7_{33} = a^3_{42} = a^7_{23} = 0.$$

Moreover, we have $\delta(e_4,\, e_i) = -\delta(e_i,\, e_4)$ for $i = 5, 6, 7$. It follows that:

$$a^4_{22} = a^5_{16} = a^4_{23} = a^5_{34} = 0,\ a^4_{33} = a^6_{15} = a^4_{32} = a^6_{24} = 0,$$

$$a^4_{17} = -a^7_{14}$$

and

$$a^4_{13} = a^7_{24} = a^4_{12} = a^7_{34} = a^4_{42} = a^7_{16} = a^4_{43} = a^7_{15} = 0.$$

Since $\delta(e_5,\, e_i) = -\delta(e_i,\, e_5)$ for $i = 6, 7$ and $\delta(e_6,\, e_7) = -\delta(e_7,\, e_6)$, we deduce that:

$$a^5_{12} = a^6_{13} \tag{4.5}$$

$$a^5_{12} = a^7_{14}.$$

$$a^6_{13} = a^7_{14}$$

and $a^6_{42} = a^7_{32} = 0$.

Therefore, from the above equations, the matrix M_{e_i} for $i = 1, \dots, 7$, becomes

$$M_{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a^1_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^1_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^1_{33} - a^1_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a^1_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a^1_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^1_{33} + a^1_{22} \end{pmatrix},$$

$$M_{e_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & a^2_{15} & 0 & 0 \\ \alpha a^2_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^2_{15} & 0 & 0 & 0 \\ 0 & 0 & -a^2_{15} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{e_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a^3_{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha a^3_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^3_{16} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a^3_{16} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{e_4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a^4_{17} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha a^4_{17} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^4_{17} & 0 & 0 & 0 & 0 \\ 0 & -a^4_{17} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{e_5} = \begin{pmatrix} 0 & a^5_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\alpha a^5_{12}}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{\alpha a^5_{12}}{2} & 0 \\ \alpha a^5_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{e_6} = \begin{pmatrix} 0 & 0 & a^6_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha a^6_{13}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-\alpha a^6_{13}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha a^6_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{e_7} = \begin{pmatrix} 0 & 0 & 0 & a^7_{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-\alpha a^7_{14}}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha a^7_{14}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha a^7_{14} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From (4.2), (4.1) and (4.4), we get

$$a^1_{33} = -2a^1_{22}. \tag{4.6}$$

By using (4.6), (4.3) and (4.5), we obtain

$$\frac{-3}{\alpha} a^1_{22} = 0. \tag{4.7}$$

Using (4.7) we deduce

$$\delta(e_i,\, e_j) = 0$$

for $i, j = 1, \ldots, 7$. Then $\delta(x,\, y) = 0$ for all $x,\, y \in \mathbf{B}$.

Furthermore, the following proposition holds:

Proposition 4.2. *The simple seven dimensional non-Lie Malcev algebra* $\mathbf{B}$ *has no nontrivial skew-symmetric biderivation.*

Lemma 4.3 ([8])**.** *Let* $(M, [\ ,\ ])$ *be a simple finite-dimensional Lie algebra over an algebraically closed field of characteristic zero* $\mathbb{F}$*. Then, every skew-symmetric biderivation* δ *of* M *is of the form* $\delta(x,\, y) = \lambda[x,\, y]$*,* $x,\, y \in M$*, for some* $\lambda \in \mathbb{F}$*.*

To conclude this article, we summarize our second main result which is a generalisation of Lemma 4.3.

Theorem 4.4. *Let* $(M, [\ ,\ ])$ *be a simple finite-dimensional Malcev algebra over an algebraically closed field of characteristic zero* $\mathbb{F}$*. Then, every skew-symmetric biderivation* δ *of* M *is of the form* $\delta(x,\, y) = \lambda[x,\, y]$*,* $x,\, y \in M$*, for some* $\lambda \in \mathbb{F}$*.*

Proof. From Proposition in [11], we have two cases.

In the case where M is a finite-dimensional simple Lie algebra, the result comes from Lemma 4.3.

In the case where M is the simple seven dimensional non-Lie Malcev algebra, the result comes from Proposition 4.2. □

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