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CERTAIN REDUCTION AND TRANSFORMATION FORMULAS FOR THE KAMPÉ DE FÉRIET FUNCTION

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Abstract. In 2014, Liu and Wang established a large number of interesting reduction, transformation and summation formulas for the Kampé de Fériet function. Inspired by the work, we aim to find further several transformation and reduction formulas for the Kampé de Fériet function. Theses formulas are mainly based on the formulas given by Liu and Wang [33].

1. Introduction

The familiar and classical Pochhammer symbol $(a)_\nu$ is defined as follows [35]:

$$(1.1) \qquad (a)_\nu = \begin{cases} a(a+1)\cdots(a+n-1), & \nu = n \in \mathbb{N}, \\ 1, & \nu = 0, \end{cases}$$
$$= \frac{\Gamma(a+\nu)}{\Gamma(a)},$$

and $\Gamma(a)$ is the well-knowns Gamma function whose Euler's integral is defined by [44]:

$$(1.2) \qquad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \ (\mathrm{Re}(z) > 0).$$

In terms of Pochhammer's symbol, we define the generalized hypergeometric function ${}_pF_q$ for complex parameters and argument as follows [1, 2, 5, 43, 44, 46] by

$$(1.3) \qquad {}_pF_q \left[\begin{array}{ccc} a_1, & \ldots, & a_p \\ & & \\ b_1, & \ldots, & b_q \end{array} ; \ x \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}$$

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when $p = q$ then this series converges for $|x| < \infty$, but when $q = p - 1$ convergence occurs when $|x| < 1$. However, when only one of the numerator parameters a_j is a negative integer or zero, then the series always converges since it is simply a polynomial in x of degree $-a_j$. For detailed account of ${}_pF_q$ including its properties, we refer standard text [1, 2, 5, 43, 44, 46].

It is interesting to mention here that for the hypergeometric function ${}_2F_1$, we have, four classical summation theorems namely Gauss, Gauss second, Kummer and Bailey and for generalized hypergeometric function ${}_3F_2$, we have four classical summation theorems namely Watson, Dixon, Whipple and Saalschütz. These theorems have wide applications. During 1992-2011, the above mentioned classical summation theorems have been generalized in the most general form by several researchers. For this, we refer some of the interesting research papers by Rakha and Rathie [37], Kim, et al. [27] and Lavoie, et al. [30–32].

It is not out of place to mention here that the vast popularity and immense usefulness of the hypergeometric function ${}_2F_1$, generalized hypergeometric function ${}_pF_q$ and confluent hypergeometric function ${}_1F_1$ in one variable have inspired and stimulated a number of researchers to discover and study hypergeometric functions in two and more variables.

A serious and very significant study of the function in two variables was initiated by Appell [4] who discovered the so-called four functions namely F_1, F_2, F_3 and F_4 which, in literature, popularly known as the Appell's functions which are natural generalization of the Gaussian hypergemetric function. The confluence forms of the Appell's functions were studied by Humbert [22]. A complete literature of the above mentioned Appell functions and their confluent forms is available in the standard text of Erdélyi et al. [20] and also in [3, 4]. Later on, the four Appell's functions and their confluent forms were generalized by Kampé de Fériet [24], who introduced a more general function in two variables. The notation defined by Kampé de Fériet for his double hypergeometric functions was subsequently abbreviated by Burchanall and Chaundy [6, 7]. In this paper, however, we shall recall the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation suggested by Srivastava and Panda [47].

So, for this, let (h_H) denote the sequence of H parameters $(h_1, h_2, \dots, h_H)$ and for $n \in \mathbb{N}_0$, define the Pochhammer symbol

$$((h_H))_n := (h_1)_n \cdots (h_H)_n,$$

with the convention that when $n = 0$, the product is interpreted as unity. With this, we define the generalized Kampé de Fériet function in the following manner:

$$F^{H:A;B}_{G:C;D}\left[\begin{array}{lll} (h_H): & (a_A); & (b_B) \\ & & \\ (g_G): & (c_C); & (d_D) \end{array} ; x,\ y\right] \tag{1.4}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((h_H))_{m+n}((a_A))_m((b_B))_n x^m y^n}{((g_G))_{m+n}((c_C))_m((d_D))_n m!n!}.$$

For more details about this function (including its convergence conditions), we refer research paper [45].

For the Kampé de Fériet function, there are lots of transformation formulas available in the literature. Early works can be found in the following research papers [8, 9, 19, 21, 23, 25, 29, 34, 38–40, 42, 49, 50]

Moreover, researchers [10–18, 26, 33, 36, 41, 48] made further research and obtained many interesting and useful results in recent years.

In addition, the well known Beta function $B(a, b)$ is defined by the first integral and known to be evaluated as the second one in the following manner:

$$B(a,b) = \begin{cases} \int_0^1 t^{a-1}(1-t)^{b-1}dt, & ;(\mathrm{Re}(a)>0, \mathrm{Re}(b)>0) \\ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, & ;(a,b\in\mathbb{C}\setminus\mathbb{Z}_0^-). \end{cases} \tag{1.5}$$

Krattenthaler and Rao [28] made a systematic use of so-called Beta integral method, a method of establishing new hypergeometric identities in one and two variables from the old ones by mainly using the Beta integral in (1.5) based on the MATHEMATICA package HYP, to illustrate several interesting identities for the hypergeometric function and the Kampé de Fériet functions of two variables.

The paper is organized as follows. In Section 3, we establish twenty one reduction formulas for the Kampé de Fériet function. In Section 4, we establish twenty one transformation formulas and finally in Section 5, we further obtain twenty reduction formulas with the help of the results given in Sections 3 and 4.

The results established in this paper are simple, interesting, easily established and may be potentially useful. It hoped that these results will be a definite contribution in the theory of hypergeometric function in two variables.

2. Results required

In our present investigation, we shall require the following results due to Liu and Wang [33].

$$F^{1:1;1}_{1:0;0}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon \\ & & ;x,\ x \\ \beta: & -; & - \end{array}\right] = (1-x)^{-\alpha}. \tag{2.1}$$

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon,\gamma \\ & & ;x,\ x \\ \beta: & -; & \gamma+\beta \end{array}\right] \tag{2.2}$$

$$= (1-x)^{\beta-\alpha-\epsilon}\ {}_2F_1\left[\begin{array}{ll} \beta-\epsilon, & \gamma+\beta-\alpha \\ & ;\ x \\ \gamma+\beta & \end{array}\right].$$

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon, 1+\frac{1}{2}\alpha \\ & & \\ \beta: & -; & \frac{1}{2}\alpha \end{array} ;x,\ x\right] \tag{2.3}$$

$$= (1-x)^{\beta-\alpha-\epsilon}\ {}_2F_1\left[\begin{array}{ll} \beta-\epsilon, & 1+\frac{1}{2}\beta \\ & \\ \frac{1}{2}\beta & \end{array} ;\ x\right].$$

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon, \frac{1}{2}(\alpha-\beta) \\ & & \\ \beta: & -; & 1+\frac{1}{2}(\alpha+\beta) \end{array} ;x,\ x\right] \tag{2.4}$$

$$= (1-x)^{\beta-\alpha-\epsilon}\ {}_3F_2\left[\begin{array}{lll} \beta-\epsilon, & 1+\frac{1}{2}\beta, & \frac{1}{2}(\beta-\alpha) \\ & & \\ \frac{1}{2}\beta, & 1+\frac{1}{2}(\alpha+\beta) & \end{array} ;\ x\right].$$

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon, 1+\frac{1}{2}\alpha, \frac{1}{2}(\alpha-\beta) \\ & & \\ \beta: & -; & \frac{1}{2}\alpha, 1+\frac{1}{2}(\alpha+\beta) \end{array} ;x,\ x\right] \tag{2.5}$$

$$= (1-x)^{\beta-\alpha-\epsilon}\ {}_2F_1\left[\begin{array}{ll} \beta-\epsilon, & \frac{1}{2}(\beta-\alpha) \\ & \\ 1+\frac{1}{2}(\alpha+\beta) & \end{array} ;\ x\right].$$

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon, 1+\frac{1}{2}\alpha, \frac{1}{2}(\alpha-\beta-1) \\ & & \\ \beta: & -; & \frac{1}{2}\alpha, \frac{1}{2}(\alpha+\beta+3) \end{array} ;x,\ x\right] \tag{2.6}$$

$$= (1-x)^{\beta-\alpha-\epsilon}\ {}_3F_2\left[\begin{array}{lll} \beta-\epsilon, & 1+\frac{1}{2}\beta, & \frac{1}{2}(\beta-\alpha-1) \\ & & \\ \frac{1}{2}\beta, & \frac{1}{2}(\alpha+\beta+3) & \end{array} ;\ x\right].$$

$$F^{0:2;3}_{1:0;1}\left[\begin{array}{lll} -: & \alpha,\epsilon; & \beta-\alpha, \beta-\epsilon, \gamma \\ & & \\ \beta: & -; & \delta \end{array} ;x,\ -\tfrac{x}{1-x}\right] \tag{2.7}$$

$$= (1-x)^{\beta-\alpha-\epsilon}\ {}_3F_2\left[\begin{array}{lll} \beta-\alpha, & \beta-\epsilon, & \delta-\gamma \\ & & \\ \beta, & \delta & \end{array} ;\ x\right].$$

$$F^{0:2;4}_{1:0;2}\left[\begin{array}{lll} -: & \alpha,\epsilon; & \beta-\alpha, \beta-\epsilon, \gamma, 1+\frac{1}{2}\gamma \\ & & \\ \beta: & -; & \delta, \frac{1}{2}\gamma \end{array} ;x,\ -\tfrac{x}{1-x}\right] \tag{2.8}$$

$$= (1-x)^{\beta-\alpha-\epsilon}\ {}_4F_3\left[\begin{array}{llll} \beta-\alpha, & \beta-\epsilon, & \delta-\gamma-1, & \gamma-\delta+2 \\ & & & \\ \beta, & \delta, & \gamma-\delta+1 & \end{array} ;\ x\right].$$

$$F^{0:2;2}_{1:0;0}\left[\begin{array}{lll} -: & \alpha,\epsilon; & \beta-\epsilon,\gamma \\ & & ;x,\ -\frac{x}{1-x} \\ \beta: & -; & - \end{array}\right] = (1-x)^{-\alpha}\ {}_2F_1\left[\begin{array}{c} \beta-\epsilon,\quad \alpha+\gamma \\ ;\quad -\frac{x}{1-x} \\ \beta \end{array}\right]. \tag{2.9}$$

$$F^{0:2;3}_{1:0;1}\left[\begin{array}{lll} -: & \alpha,\epsilon; & \beta-\epsilon,\gamma,\delta \\ & & ;x,\ -\frac{x}{1-x} \\ \beta: & -; & \alpha+\gamma+\delta \end{array}\right] = (1-x)^{-\alpha}\ {}_3F_2\left[\begin{array}{c} \beta-\epsilon,\quad \alpha+\gamma,\quad \alpha+\delta \\ ;\quad -\frac{x}{1-x} \\ \beta,\quad \alpha+\gamma+\delta \end{array}\right]. \tag{2.10}$$

$$F^{0:2;3}_{1:0;1}\left[\begin{array}{lll} -: & \alpha,\epsilon; & \beta-\epsilon,\gamma,1+\frac{1}{2}\gamma \\ & & ;x,\ -\frac{x}{1-x} \\ \beta: & -; & \frac{1}{2}\gamma \end{array}\right] = (1-x)^{-\alpha}\ {}_3F_2\left[\begin{array}{c} \beta-\epsilon,\quad \alpha+\gamma,\quad 1+\frac{1}{2}(\alpha+\gamma) \\ ;\quad -\frac{x}{1-x} \\ \beta,\quad \frac{1}{2}(\alpha+\gamma) \end{array}\right]. \tag{2.11}$$

$$F^{0:2;3}_{1:0;1}\left[\begin{array}{lll} -: & \alpha,\epsilon; & \beta-\epsilon,\gamma,-\frac{1}{2}\alpha \\ & & ;x,\ -\frac{x}{1-x} \\ \beta: & -; & 1+\frac{1}{2}\alpha+\gamma \end{array}\right] = (1-x)^{-\alpha}\ {}_4F_3\left[\begin{array}{c} \alpha+\gamma,\quad 1+\frac{1}{2}(\alpha+\gamma),\quad \frac{1}{2}\gamma,\quad \beta-\epsilon \\ ;\quad -\frac{x}{1-x} \\ \frac{1}{2}(\alpha+\gamma),\quad 1+\frac{1}{2}\alpha+\gamma,\quad \beta \end{array}\right]. \tag{2.12}$$

$$F^{0:2;4}_{1:0;2}\left[\begin{array}{lll} -: & \alpha,\epsilon; & \beta-\epsilon,\gamma,1+\frac{1}{2}\gamma,-\frac{1}{2}\alpha \\ & & ;x,\ -\frac{x}{1-x} \\ \beta: & -; & \frac{1}{2}\gamma,1+\frac{1}{2}\alpha+\gamma \end{array}\right] = (1-x)^{-\alpha}\ {}_3F_2\left[\begin{array}{c} \beta-\epsilon,\quad \alpha+\gamma,\quad \frac{1}{2}\alpha \\ ;\quad -\frac{x}{1-x} \\ \beta,\quad 1+\frac{1}{2}\alpha+\gamma \end{array}\right]. \tag{2.13}$$

$$F^{0:2;4}_{1:0;2}\left[\begin{array}{lll} -: & \alpha,\epsilon; & \beta-\epsilon,\gamma,1+\frac{1}{2}\gamma,-\frac{\alpha+1}{2} \\ & & ;x,\ -\frac{x}{1-x} \\ \beta: & -; & \frac{1}{2}\gamma,\frac{3}{2}+\frac{1}{2}\alpha+\gamma \end{array}\right] = (1-x)^{-\alpha}\ {}_4F_3\left[\begin{array}{c} \alpha+\gamma,\quad 1+\frac{1}{2}(\alpha+\gamma),\quad \frac{1}{2}(\alpha-1),\quad \beta-\epsilon \\ ;\quad -\frac{x}{1-x} \\ \frac{1}{2}(\alpha+\gamma),\quad \frac{3}{2}+\frac{1}{2}\alpha+\gamma,\quad \beta \end{array}\right]. \tag{2.14}$$

$$F^{2:0;1}_{1:0;1}\left[\begin{array}{c} \alpha,\gamma: \ -; \ \epsilon \\ \beta: \ -; \ \beta+\epsilon \end{array} ;x,\ -x\right] = (1-x)^{-\alpha}\ {}_2F_1\left[\begin{array}{c} \alpha,\ \ \beta+\epsilon-\gamma \\ \beta+\epsilon \end{array} ;\ -\tfrac{x}{1-x}\right]. \tag{2.15}$$

$$F^{2:0;1}_{1:0;1}\left[\begin{array}{c} \alpha,\gamma: \ -; \ 1+\frac{1}{2}\gamma \\ -: \ -; \ \frac{1}{2}\gamma \end{array} ;x,\ -x\right] = (1-x)^{-\alpha}\ {}_2F_1\left[\begin{array}{c} \alpha,\ \ 1+\frac{1}{2}\beta \\ \frac{1}{2}\beta \end{array} ;\ -\tfrac{x}{1-x}\right]. \tag{2.16}$$

$$F^{2:0;1}_{1:0;1}\left[\begin{array}{c} \alpha,\gamma: \ -; \ \frac{1}{2}(\gamma-\beta) \\ \beta: \ -; \ 1+\frac{1}{2}(\gamma+\beta) \end{array} ;x,\ -x\right] = (1-x)^{-\alpha}\ {}_3F_2\left[\begin{array}{c} \alpha,\ \ 1+\frac{1}{2}\beta,\ \ \frac{1}{2}(\beta-\gamma) \\ \frac{1}{2}\beta,\ \ 1+\frac{1}{2}(\gamma+\beta) \end{array} ;\ -\tfrac{x}{1-x}\right]. \tag{2.17}$$

$$F^{2:0;2}_{1:0;2}\left[\begin{array}{c} \alpha,\gamma: \ -; \ 1+\frac{1}{2}\gamma, \frac{1}{2}(\gamma-\beta) \\ \beta: \ -; \ \frac{1}{2}\gamma, 1+\frac{1}{2}(\gamma+\beta) \end{array} ;x,\ -x\right] = (1-x)^{-\alpha}\ {}_2F_1\left[\begin{array}{c} \alpha,\ \ \frac{1}{2}(\beta-\gamma) \\ 1+\frac{1}{2}(\gamma+\beta) \end{array} ;\ -\tfrac{x}{1-x}\right]. \tag{2.18}$$

$$F^{2:0;2}_{1:0;2}\left[\begin{array}{c} \alpha,\gamma: \ -; \ 1+\frac{1}{2}\gamma, \frac{1}{2}(\gamma-\beta-1) \\ \beta: \ -; \ \frac{1}{2}\gamma, \frac{1}{2}(3+\gamma+\beta) \end{array} ;x,\ -x\right] = (1-x)^{-\alpha}\ {}_3F_2\left[\begin{array}{c} \alpha,\ \ 1+\frac{1}{2}\beta,\ \ \frac{1}{2}(\beta-\gamma-1) \\ \frac{1}{2}\beta,\ \ \frac{1}{2}(3+\gamma+\beta) \end{array} ;\ -\tfrac{x}{1-x}\right]. \tag{2.19}$$

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{c} \alpha: \ \epsilon; \ \beta-\epsilon,\gamma \\ \beta: \ -; \ \delta \end{array} ;x,\ x\right] = (1-x)^{-\alpha}\ {}_3F_2\left[\begin{array}{c} \alpha,\ \ \beta-\epsilon,\ \ \delta-\gamma \\ \beta,\ \ \delta \end{array} ;\ -\tfrac{x}{1-x}\right]. \tag{2.20}$$

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon,\gamma,1+\frac{1}{2}\gamma \\ & & ;x,\ x \\ \beta: & -; & \delta,\frac{1}{2}\gamma \end{array}\right] \tag{2.21}$$

$$= (1-x)^{-\alpha}\ {}_4F_3\left[\begin{array}{llll} \alpha, & \beta-\epsilon, & \delta-\gamma-1, & \delta-\gamma+2 \\ & & & ;\ -\frac{x}{1-x} \\ \beta, & \delta, & \gamma-\delta+1 & \end{array}\right].$$

3. Reduction formulas for the Kampé de Fériet function

In this section, the following twenty one interesting reduction formulas have been established.

$$F^{2:1;1}_{2:0;0}\left[\begin{array}{lll} \alpha,\lambda: & \epsilon; & \beta-\epsilon \\ & & ;1,\ 1 \\ \beta,\mu: & -; & - \end{array}\right] = \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\alpha)\Gamma(\mu-\lambda)}. \tag{3.1}$$

$$F^{2:1;2}_{2:0;1}\left[\begin{array}{lll} \alpha,\lambda: & \epsilon; & \beta-\epsilon,\gamma \\ & & ;1,\ 1 \\ \beta,\mu: & -; & \gamma+\beta \end{array}\right] \tag{3.2}$$

$$= \frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}\ {}_3F_2\left[\begin{array}{lll} \beta-\epsilon, & \gamma+\beta-\alpha, & \lambda \\ & & ;\ 1 \\ \gamma+\beta, & \mu+\beta-\alpha-\epsilon & \end{array}\right].$$

$$F^{2:1;2}_{2:0;1}\left[\begin{array}{lll} \alpha,\lambda: & \epsilon; & \beta-\epsilon,1+\frac{1}{2}\alpha \\ & & ;1,\ 1 \\ \beta,\mu: & -; & \frac{1}{2}\alpha \end{array}\right] \tag{3.3}$$

$$= \frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}\ {}_3F_2\left[\begin{array}{lll} \beta-\epsilon, & 1+\frac{1}{2}\beta, & \lambda \\ & & ;\ 1 \\ \frac{1}{2}\beta, & \mu+\beta-\alpha-\epsilon & \end{array}\right].$$

$$F^{2:1;2}_{2:0;1}\left[\begin{array}{lll} \alpha,\lambda: & \epsilon; & \beta-\epsilon,\frac{1}{2}(\alpha-\beta) \\ & & ;1,\ 1 \\ \beta,\mu: & -; & 1+\frac{1}{2}(\alpha+\beta) \end{array}\right] \tag{3.4}$$

$$= \frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}$$

$$\times\ {}_4F_3\left[\begin{array}{llll} \beta-\epsilon, & 1+\frac{1}{2}\beta, & \frac{1}{2}(\beta-\alpha), & \lambda \\ & & & ;\ 1 \\ \frac{1}{2}\beta, & 1+\frac{1}{2}(\beta+\alpha), & \mu+\beta-\alpha-\epsilon & \end{array}\right].$$

$$F^{2:1;3}_{2:0;2}\left[\begin{array}{lll} \alpha,\lambda: & \epsilon; & \beta-\epsilon,1+\frac{1}{2}\alpha,\frac{1}{2}(\alpha-\beta) \\ & & ;1,\ 1 \\ \beta,\mu: & -; & \frac{1}{2}\alpha,1+\frac{1}{2}(\alpha+\beta) \end{array}\right] \tag{3.5}$$

$$= \frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}\ {}_3F_2\left[\begin{array}{lll} \beta-\epsilon, & \frac{1}{2}(\beta-\alpha), & \lambda \\ & & ;\ 1 \\ 1+\frac{1}{2}(\alpha+\beta), & \mu+\beta-\alpha-\epsilon & \end{array}\right].$$

$$F^{2:1;3}_{2:0;2}\left[\begin{matrix}\alpha,\lambda: & \epsilon; & \beta-\epsilon, 1+\frac{1}{2}\alpha, \frac{1}{2}(\alpha-\beta+1) \\ \beta,\mu: & -; & \frac{1}{2}\alpha, \frac{1}{2}(\alpha+\beta+3)\end{matrix} ;1,\ 1\right] \tag{3.6}$$
$$= \frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}$$
$$\times\ {}_4F_3\left[\begin{matrix}\beta-\epsilon, & 1+\frac{1}{2}\beta, & \frac{1}{2}(\beta-\alpha+1), & \lambda \\ \frac{1}{2}\beta, & \frac{1}{2}(\alpha+\beta+3), & \mu+\beta-\alpha-\epsilon & \end{matrix} ;\ 1\right].$$

$$F^{1:2;3}_{1:1;2}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\beta-\alpha,\gamma \\ \beta: & \mu; & \delta, 1-\mu+\lambda\end{matrix} ;1,\ 1\right] \tag{3.7}$$
$$= \frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}$$
$$\times\ {}_4F_3\left[\begin{matrix}\beta-\alpha, & \beta-\epsilon, & \delta-\gamma, & \lambda \\ \beta, & \delta, & \mu+\beta-\alpha-\epsilon & \end{matrix} ;\ 1\right].$$

$$F^{1:2;4}_{1:1;3}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\beta-\alpha,\gamma,1+\frac{1}{2}\gamma \\ \beta: & \mu; & \delta, \frac{1}{2}\gamma, 1-\mu+\lambda\end{matrix} ;1,\ 1\right] \tag{3.8}$$
$$= \frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}$$
$$\times\ {}_5F_4\left[\begin{matrix}\beta-\alpha, & \beta-\epsilon, & \delta-\gamma-1, & \gamma-\delta+2, & \lambda \\ \beta, & \delta, & \gamma-\delta+1, & \mu+\beta-\alpha-\epsilon & \end{matrix} ;\ 1\right].$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\gamma \\ \beta: & \mu; & 1-\mu+\lambda\end{matrix} ;1,\ 1\right] \tag{3.9}$$
$$= \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}\ {}_3F_2\left[\begin{matrix}\beta-\epsilon, & \alpha+\gamma, & \lambda \\ \beta, & 1-\mu+\alpha+\lambda & \end{matrix} ;\ 1\right].$$

$$F^{1:2;3}_{1:1;2}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\gamma,\delta \\ \beta: & \mu; & \alpha+\gamma+\delta, 1-\mu+\lambda\end{matrix} ;1,\ 1\right] \tag{3.10}$$
$$= {}_4F_3\left[\begin{matrix}\beta-\epsilon, & \alpha+\gamma, & \alpha+\delta, & \lambda \\ \beta, & \alpha+\gamma+\delta, & 1-\mu+\lambda & \end{matrix} ;\ 1\right].$$

$$F^{1:2;3}_{1:1;2}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\gamma,1+\frac{1}{2}\gamma \\ \beta: & \mu; & \frac{1}{2}\gamma,1-\mu+\lambda\end{matrix};1,\ 1\right] \tag{3.11}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_4F_3\left[\begin{matrix}\beta-\epsilon, & \alpha+\gamma, & 1+\frac{1}{2}(\alpha+\gamma), & \lambda \\ \beta, & \frac{1}{2}(\alpha+\gamma), & 1-\mu+\alpha+\lambda & \end{matrix};\ 1\right].$$

$$F^{1:2;3}_{1:1;2}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\gamma,-\frac{1}{2}\gamma \\ \beta: & \mu; & 1+\frac{1}{2}\alpha+\gamma,1-\mu+\lambda\end{matrix};1,\ 1\right] \tag{3.12}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_5F_4\left[\begin{matrix}\alpha+\gamma, & 1+\frac{1}{2}(\alpha+\gamma), & \frac{1}{2}\alpha, & \beta-\epsilon, & \lambda \\ \frac{1}{2}(\alpha+\gamma), & 1+\frac{1}{2}\alpha+\gamma, & \beta, & 1-\mu+\alpha+\lambda & \end{matrix};\ 1\right].$$

$$F^{1:2;4}_{1:1;3}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\gamma,1+\frac{1}{2}\gamma,-\frac{1}{2}\gamma \\ \beta: & \beta; & \frac{1}{2}\gamma,1+\frac{1}{2}\alpha+\gamma,1-\mu+\lambda\end{matrix};1,\ 1\right] \tag{3.13}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_4F_3\left[\begin{matrix}\beta-\epsilon, & \alpha+\gamma, & \frac{1}{2}\alpha, & \lambda \\ \beta, & ,1+\frac{1}{2}\alpha+\lambda, & 1-\mu+\alpha+\lambda & \end{matrix};\ 1\right].$$

$$F^{1:2;4}_{1:1;3}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\gamma,1+\frac{1}{2}\gamma,-\frac{\alpha+1}{2} \\ \beta: & \mu; & \frac{1}{2}\gamma,\frac{3}{2}+\frac{1}{2}\alpha+\gamma,1-\mu+\lambda\end{matrix};1,\ 1\right] \tag{3.14}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_5F_4\left[\begin{matrix}\alpha+\gamma, & 1+\frac{1}{2}(\alpha+\gamma), & \frac{1}{2}(\alpha-1), & \beta-\epsilon, & \lambda \\ \frac{1}{2}(\alpha+\gamma), & \frac{3}{2}+\frac{1}{2}\alpha+\gamma, & \beta, & 1-\mu+\alpha+\lambda & \end{matrix};\ 1\right].$$

$$F^{3:0;1}_{2:0;1}\left[\begin{matrix}\alpha,\gamma,\lambda: & -; & \epsilon \\ \beta,\mu: & -; & \beta+\epsilon\end{matrix};1,\ -1\right] \tag{3.15}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}\ {}_3F_2\left[\begin{matrix}\alpha, & \beta+\epsilon-\gamma, & \lambda \\ \beta+\epsilon, & 1-\mu+\alpha+\lambda & \end{matrix};\ 1\right].$$

$$F^{3:0;1}_{2:0;1}\left[\begin{matrix}\alpha,\gamma,\lambda: & -; & 1+\frac{1}{2}\gamma \\ & & & ;1,\ -1 \\ \beta,\mu: & -; & \frac{1}{2}\gamma\end{matrix}\right] = \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}\,{}_3F_2\left[\begin{matrix}\alpha, & 1+\frac{1}{2}\beta, & \lambda \\ & & & ;\quad 1 \\ \frac{1}{2}\beta, & 1-\mu+\alpha+\lambda\end{matrix}\right]. \tag{3.16}$$

$$F^{3:0;1}_{2:0;1}\left[\begin{matrix}\alpha,\gamma,\lambda: & -; & \frac{1}{2}(\gamma-\beta) \\ & & & ;1,\ -1 \\ \beta,\mu: & -; & 1+\frac{1}{2}(\gamma+\beta)\end{matrix}\right] = \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)} \times\ {}_4F_3\left[\begin{matrix}\alpha, & 1+\frac{1}{2}\beta, & \frac{1}{2}(\beta-\gamma), & \lambda \\ & & & & ;\quad 1 \\ \frac{1}{2}\beta, & 1+\frac{1}{2}(\gamma+\beta) & 1-\mu+\alpha+\lambda\end{matrix}\right]. \tag{3.17}$$

$$F^{3:0;2}_{2:0;2}\left[\begin{matrix}\alpha,\gamma,\lambda: & -; & 1+\frac{1}{2}\gamma, \frac{1}{2}(\gamma-\beta-1) \\ & & & ;1,\ -1 \\ \beta,\mu: & -; & \frac{1}{2}\gamma, \frac{1}{2}(3+\gamma+\beta)\end{matrix}\right] = \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}\,{}_3F_2\left[\begin{matrix}\alpha, & \frac{1}{2}(\beta-\gamma), & \lambda \\ & & & ;\quad 1 \\ 1+\frac{1}{2}(\beta+\gamma), & 1-\mu+\alpha+\lambda\end{matrix}\right]. \tag{3.18}$$

$$F^{3:0;2}_{2:0;2}\left[\begin{matrix}\alpha,\gamma,\lambda: & -; & 1+\frac{1}{2}\gamma, \frac{1}{2}(\gamma-\beta-1) \\ & & & ;1,\ -1 \\ \beta,\mu: & -; & \frac{1}{2}\gamma, \frac{1}{2}(3+\gamma+\beta)\end{matrix}\right] = \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)} \times\ {}_4F_3\left[\begin{matrix}\alpha, & 1+\frac{1}{2}\beta, & \frac{1}{2}(\beta-\gamma-1), & \lambda \\ & & & & ;\quad 1 \\ \frac{1}{2}\beta, & \frac{1}{2}(3+\gamma+\beta) & 1-\mu+\alpha+\lambda\end{matrix}\right]. \tag{3.19}$$

$$F^{2:1;2}_{2:0;1}\left[\begin{matrix}\alpha,\lambda: & \epsilon; & \beta-\epsilon,\gamma \\ & & & ;1,\ 1 \\ \beta,\mu: & -; & \delta\end{matrix}\right] = \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)} \times\ {}_4F_3\left[\begin{matrix}\alpha, & \beta-\epsilon, & \delta-\gamma, & \lambda \\ & & & & ;\quad 1 \\ \beta, & \delta & 1-\mu+\alpha+\lambda\end{matrix}\right]. \tag{3.20}$$

$$(3.21)\qquad F_{2:0;2}^{2:1;3}\left[\begin{array}{lll} \alpha,\lambda: & \epsilon; & \beta-\epsilon,\gamma,1+\frac{1}{2}\gamma \\ & & ;1,\ 1 \\ \beta,\mu: & -; & \delta,\frac{1}{2}\gamma \end{array}\right]$$
$$=\frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_5F_4\left[\begin{array}{ccccc} \alpha, & \beta-\epsilon, & \delta-\gamma-1, & \delta-\gamma+2 & \lambda \\ & & & & ;\ 1 \\ \beta, & \delta & \gamma-\delta+1 & 1-\mu+\alpha+\lambda & \end{array}\right].$$

Proof. The derivation of the reduction formulas (3.1) to (3.21) are quite straight forward. We shall establish only one of them, say, for example, (3.2) and rest can be proven on a similar fashion.

In order to establish the result (3.2), we proceed as follows. Multiply both sides of the equation (2.2) by $x^{\lambda-1}(1-x)^{\mu-\lambda-1}$ and integrating with respect to x between the interval $[0,1]$, we have

$$(3.22)\qquad \int_0^1 x^{\lambda-1}(1-x)^{\mu-\lambda-1}F_{1:0;1}^{1:1;2}\left[\begin{array}{lll} \alpha: & \epsilon; & \beta-\epsilon,\gamma \\ & & ;x,\ x \\ \beta: & -; & \gamma+\beta \end{array}\right]dx$$
$$=\int_0^1 x^{\lambda-1}(1-x)^{\mu-\lambda-\alpha-\epsilon+\beta-1}\ {}_2F_1\left[\begin{array}{ll} \beta-\epsilon, & \gamma+\beta-\alpha \\ & ;\ x \\ \gamma+\beta & \end{array}\right]dx.$$

Now, let us donate the left-hand side of (3.22) by I_1 and expressing the Kampé de Fériet function by its definition, changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series involved in the process, we have

$$I_1=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\epsilon)_m(\beta-\epsilon)_n(\gamma)_n}{(\beta)_{m+n}(\gamma+\beta)_n m!n!}\int_0^1 x^{\lambda+m+n-1}(1-x)^{\mu-\lambda-1}dx.$$

Evaluating the beta integral, we have after some algebra

$$I_1=\frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\lambda)_{m+n}(\epsilon)_m(\beta-\epsilon)_n(\gamma)_n}{(\beta)_{m+n}(\mu)_{m+n}(\gamma+\beta)_n m!n!}.$$

Summing up the series with the help of the definition of Kampé de Fériet function, we have

$$(3.23)\qquad I_1=\frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)}F_{2:0;1}^{2:1;2}\left[\begin{array}{lll} \alpha,\lambda & \epsilon; & \beta-\epsilon,\gamma \\ & & ;1,\ 1 \\ \beta,\mu: & -; & \gamma+\beta \end{array}\right].$$

Now, denoting the right-hand side of (3.22) by I_2, expressing ${}_2F_1$ as a series and proceeding the same as above and after a little simplification, we have

$$I_2=\frac{\Gamma(\lambda)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu+\beta-\alpha-\epsilon)}\sum_{n=0}^{\infty}\frac{(\beta-\epsilon)_n(\gamma+\beta-\alpha)_n(\lambda)_n}{(\gamma+\beta)_n(\mu+\beta-\alpha-\epsilon)_n n!}.$$

Summing up the series, we have

$$(3.24)\quad I_2 = \frac{\Gamma(\lambda)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu+\beta-\alpha-\epsilon)}\ {}_3F_2\left[\begin{matrix}\beta-\epsilon, & \gamma+\beta-\alpha, & \lambda \\ \gamma+\beta, & \mu+\beta-\alpha-\epsilon & \end{matrix};\ 1\right].$$

Finally, substituting the values of (3.23) and (3.24) in (3.22), we arrive at the desired reduction formula (3.2). This completes the proof of (3.2). □

4. Transformation formulas for the Kampé de Fériet function

In this section, we shall establish the following twenty interesting transformation formulas for the Kampé de Fériet function.

$$(4.1)\quad F^{2:1;2}_{2:0;1}\left[\begin{matrix}\alpha,\lambda: & \epsilon; & \beta-\epsilon,\gamma \\ \beta,\mu: & -; & \gamma+\beta\end{matrix};1,\ 1\right] = F^{1:1;2}_{1:0;1}\left[\begin{matrix}\lambda: & \alpha+\epsilon-\beta; & \beta-\epsilon,\gamma+\beta-\alpha \\ \mu: & -; & \gamma+\beta\end{matrix};1,\ 1\right].$$

$$(4.2)\quad F^{2:1;2}_{2:0;1}\left[\begin{matrix}\alpha,\lambda: & \epsilon; & \beta-\epsilon,1+\frac{1}{2}\alpha \\ \beta,\mu: & -; & \frac{1}{2}\alpha\end{matrix};1,\ 1\right] = F^{1:1;2}_{1:0;1}\left[\begin{matrix}\lambda: & \alpha+\epsilon-\beta; & \beta-\epsilon,1+\frac{1}{2}\beta \\ \mu: & -; & \frac{1}{2}\beta\end{matrix};1,\ 1\right].$$

$$(4.3)\quad F^{2:1;2}_{2:0;1}\left[\begin{matrix}\alpha,\lambda: & \epsilon; & \beta-\epsilon,\frac{1}{2}(\alpha-\beta) \\ \beta,\mu: & -; & 1+\frac{1}{2}(\alpha+\beta)\end{matrix};1,\ 1\right] = F^{1:1;3}_{1:0;2}\left[\begin{matrix}\lambda: & \alpha+\epsilon-\beta; & \beta-\epsilon,1+\frac{1}{2}\beta,\frac{1}{2}(\beta-\alpha) \\ \mu: & -; & \frac{1}{2}\beta,1+\frac{1}{2}(\alpha+\beta)\end{matrix};1,\ 1\right].$$

$$(4.4)\quad F^{2:1;3}_{2:0;2}\left[\begin{matrix}\alpha,\lambda: & \epsilon; & \beta-\epsilon,1+\frac{1}{2}\alpha,\frac{1}{2}(\alpha-\beta) \\ \beta,\mu: & -; & \frac{1}{2}\alpha,1+\frac{1}{2}(\alpha+\beta)\end{matrix};1,\ 1\right] = F^{1:1;2}_{1:0;1}\left[\begin{matrix}\lambda: & \alpha+\epsilon-\beta; & \beta-\epsilon,\frac{1}{2}(\beta-\alpha) \\ \mu: & -; & 1+\frac{1}{2}(\alpha+\beta)\end{matrix};1,\ 1\right].$$

$$(4.5)\quad F^{2:1;3}_{2:0;2}\left[\begin{matrix}\alpha,\lambda: & \epsilon; & \beta-\epsilon,1+\frac{1}{2}\alpha,\frac{1}{2}(\alpha-\beta+1) \\ \beta,\mu: & -; & \frac{1}{2}\alpha,\frac{1}{2}(\alpha+\beta+3)\end{matrix};1,\ 1\right]$$

$$= F^{1:1;3}_{1:0;2}\left[\begin{matrix}\lambda: & \alpha+\epsilon-\beta; & \beta-\epsilon, 1+\frac{1}{2}\beta, \frac{1}{2}(\beta-\alpha-1) \\ & & \\ \mu: & -; & \frac{1}{2}\beta, \frac{1}{2}(\alpha+\beta+3)\end{matrix} ;1,\ 1\right].$$

$$F^{1:2;3}_{1:1;2}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\beta-\alpha,\gamma \\ & & \\ \beta: & \mu; & \delta, 1-\mu+\lambda\end{matrix} ;1,\ 1\right] \tag{4.6}$$

$$= F^{1:1;3}_{1:0;2}\left[\begin{matrix}\lambda: & \alpha+\epsilon-\beta; & \beta-\alpha,\beta-\epsilon,\delta-\gamma \\ & & \\ \mu: & -; & \beta,\delta\end{matrix} ;1,\ 1\right].$$

$$F^{1:2;4}_{1:1;3}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\beta-\alpha,\gamma,1+\frac{1}{2}\gamma \\ & & \\ \beta: & \mu; & \delta,\frac{1}{2}\gamma,1-\mu+\lambda\end{matrix} ;1,\ 1\right] \tag{4.7}$$

$$= F^{1:1;4}_{1:0;3}\left[\begin{matrix}\lambda: & \alpha+\epsilon-\beta; & \beta-\alpha,\beta-\epsilon,\delta-\gamma-1,\gamma-\delta+2 \\ & & \\ \mu: & -; & \beta,\delta,\gamma-\delta+1\end{matrix} ;1,\ 1\right].$$

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\gamma \\ & & \\ \beta: & \mu; & 1-\mu+\lambda\end{matrix} ;1,\ 1\right] \tag{4.8}$$

$$= F^{1:1;2}_{0:1;2}\left[\begin{matrix}\lambda: & \alpha; & \beta-\epsilon,\alpha+\gamma \\ & & \\ -: & \mu; & \beta,1-\mu-\lambda\end{matrix} ;1,\ 1\right].$$

$$F^{1:2;3}_{1:1;2}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\gamma,\delta \\ & & \\ \beta: & \mu; & \alpha+\gamma+\delta,1-\mu+\lambda\end{matrix} ;1,\ 1\right] \tag{4.9}$$

$$= F^{1:1;3}_{0:1;2}\left[\begin{matrix}\lambda: & \alpha; & \beta-\epsilon,\alpha+\gamma,\alpha+\delta \\ & & \\ -: & \mu; & \beta,\alpha+\gamma+\delta,1-\mu+\lambda\end{matrix} ;1,\ 1\right].$$

$$F^{1:2;3}_{1:1;2}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\gamma,1+\frac{1}{2}\gamma \\ & & \\ \beta: & \mu; & \frac{1}{2}\gamma,1-\mu+\lambda\end{matrix} ;1,\ 1\right] \tag{4.10}$$

$$= F^{1:1;3}_{0:1;3}\left[\begin{matrix}\lambda: & \alpha; & \beta-\epsilon,\alpha+\gamma,1+\frac{1}{2}(\alpha+\gamma) \\ & & \\ -: & \mu; & \beta,\frac{1}{2}(\alpha+\gamma),1-\mu+\lambda\end{matrix} ;1,\ 1\right].$$

$$F^{1:2;3}_{1:1;2}\left[\begin{matrix}\lambda: & \alpha,\epsilon; & \beta-\epsilon,\gamma,-\frac{1}{2}\gamma \\ & & \\ \beta: & \mu; & 1+\frac{1}{2}\alpha+\gamma,1-\mu+\lambda\end{matrix} ;1,\ 1\right] \tag{4.11}$$

$$= F^{1:1;4}_{0:1;4}\left[\begin{array}{lll} \lambda: & \alpha; & \beta-\epsilon, \alpha+\gamma, 1+\frac{1}{2}(\alpha+\gamma), \frac{1}{2}\alpha \\ & & \\ -: & \mu; & \beta, \frac{1}{2}(\alpha+\gamma), 1+\frac{1}{2}\alpha+\gamma, 1-\mu+\lambda \end{array} ;1,\ 1\right].$$

(4.12)
$$F^{1:2;4}_{1:1;3}\left[\begin{array}{lll} \lambda: & \alpha, \epsilon; & \beta-\epsilon, \gamma, 1+\frac{1}{2}\gamma, -\frac{1}{2}\gamma \\ & & \\ \beta: & \beta; & \frac{1}{2}\gamma, 1+\frac{1}{2}\alpha+\gamma, 1-\mu+\lambda \end{array} ;1,\ 1\right]$$
$$= F^{1:1;3}_{0:1;3}\left[\begin{array}{lll} \lambda: & \alpha; & \beta-\epsilon, \alpha+\gamma, \frac{1}{2}\alpha \\ & & \\ -: & \mu; & \beta, 1+\frac{1}{2}\alpha+\gamma, 1-\mu+\lambda \end{array} ;1,\ 1\right].$$

(4.13)
$$F^{1:2;4}_{1:1;3}\left[\begin{array}{lll} \lambda: & \alpha, \epsilon; & \beta-\epsilon, \gamma, 1+\frac{1}{2}\gamma, -\frac{\alpha+1}{2} \\ & & \\ \beta: & \mu; & \frac{1}{2}\gamma, \frac{3}{2}+\frac{1}{2}\alpha+\gamma, 1-\mu+\lambda \end{array} ;1,\ 1\right]$$
$$= F^{1:1;4}_{0:1;4}\left[\begin{array}{lll} \lambda: & \alpha; & \beta-\epsilon, \alpha+\gamma, 1+\frac{1}{2}(\alpha+\gamma), \frac{1}{2}(\alpha-1) \\ & & \\ -: & \mu; & \beta, \frac{1}{2}(\alpha+\gamma), \frac{3}{2}+\frac{1}{2}\alpha+\gamma, 1-\mu+\lambda \end{array} ;1,\ 1\right].$$

(4.14)
$$F^{3:0;1}_{2:0;1}\left[\begin{array}{lll} \alpha, \gamma, \lambda: & -; & \epsilon \\ & & \\ \beta, \mu: & -; & \beta+\epsilon \end{array} ;1,\ -1\right]$$
$$= F^{1:1;2}_{0:1;2}\left[\begin{array}{lll} \lambda: & \alpha; & \alpha, \beta+\epsilon-\gamma \\ & & \\ -: & \mu; & \beta+\epsilon, \gamma-\mu+\lambda \end{array} ;1,\ 1\right].$$

(4.15)
$$F^{3:0;1}_{2:0;1}\left[\begin{array}{lll} \alpha, \gamma, \lambda: & -; & 1+\frac{1}{2}\gamma \\ & & \\ \beta, \mu: & -; & \frac{1}{2}\gamma \end{array} ;1,\ -1\right]$$
$$= F^{1:1;2}_{0:1;2}\left[\begin{array}{lll} \lambda: & \alpha; & \alpha, 1+\frac{1}{2}\beta \\ & & \\ -: & \mu; & \frac{1}{2}\beta, 1-\mu+\lambda \end{array} ;1,\ 1\right].$$

(4.16)
$$F^{3:0;1}_{2:0;1}\left[\begin{array}{lll} \alpha, \gamma, \lambda: & -; & \frac{1}{2}(\gamma-\beta) \\ & & \\ \beta, \mu: & -; & 1+\frac{1}{2}(\gamma+\beta) \end{array} ;1,\ -1\right]$$
$$= F^{1:1;3}_{0:1;3}\left[\begin{array}{lll} \lambda: & \alpha; & \alpha, 1+\frac{1}{2}\beta, \frac{1}{2}(\beta-\gamma) \\ & & \\ -: & \mu; & \frac{1}{2}\beta, 1+\frac{1}{2}(\gamma+\beta), 1-\mu+\lambda \end{array} ;1,\ 1\right].$$

(4.17)
$$F^{3:0;2}_{2:0;2}\left[\begin{array}{lll} \alpha, \gamma, \lambda: & -; & 1+\frac{1}{2}\gamma, \frac{1}{2}(\gamma-\beta-1) \\ & & \\ \beta, \mu: & -; & \frac{1}{2}\gamma, \frac{1}{2}(3+\gamma+\beta) \end{array} ;1,\ -1\right]$$

$$= F^{1:1;2}_{0:1;2}\left[\begin{matrix} \lambda: & \alpha; & \alpha, \frac{1}{2}(\beta-\gamma) \\ -: & \mu; & 1+\frac{1}{2}(\gamma+\beta), 1-\mu+\lambda \end{matrix} ;1,\ 1\right].$$

$$F^{3:0;2}_{2:0;2}\left[\begin{matrix} \alpha,\gamma,\lambda: & -; & 1+\frac{1}{2}\gamma, \frac{1}{2}(\gamma-\beta-1) \\ \beta,\mu: & -; & \frac{1}{2}\gamma, \frac{1}{2}(3+\gamma+\beta) \end{matrix} ;1,\ -1\right] \tag{4.18}$$

$$= F^{1:1;3}_{0:1;3}\left[\begin{matrix} \lambda: & \alpha; & \alpha, 1+\frac{1}{2}\beta, \frac{1}{2}(\beta-\mu-1) \\ -: & \mu; & \frac{1}{2}\beta, 1+\frac{1}{2}(2+\gamma+\beta), 1-\mu+\lambda \end{matrix} ;1,\ 1\right].$$

$$F^{2:1;2}_{2:0;1}\left[\begin{matrix} \alpha,\lambda: & \epsilon; & \beta-\epsilon,\gamma \\ \beta,\mu: & -; & \delta \end{matrix} ;1,\ 1\right] \tag{4.19}$$

$$= F^{1:1;3}_{0:1;3}\left[\begin{matrix} \lambda: & \alpha; & \alpha,\beta-\epsilon,\delta-\gamma \\ -: & \mu; & \beta,\delta,1-\mu+\lambda \end{matrix} ;1,\ 1\right].$$

$$F^{2:1;3}_{2:0;2}\left[\begin{matrix} \alpha,\lambda: & \epsilon; & \beta-\epsilon,\gamma,1+\frac{1}{2}\gamma \\ \beta,\mu: & -; & \delta,\frac{1}{2}\gamma \end{matrix} ;1,\ 1\right] \tag{4.20}$$

$$= F^{1:1;4}_{0:1;4}\left[\begin{matrix} \lambda: & \alpha; & \alpha,\beta-\epsilon,\delta-\gamma-1,\delta-\gamma+2 \\ -: & \mu; & \beta,\delta,\gamma-\delta+1,1-\mu+\lambda \end{matrix} ;1,\ 1\right].$$

Proof. The derivation of the reduction formulas (4.1) to (4.20) are quite straight forward. We shall establish only one of them, say, for example, (4.1) and rest can be proven on a similar fashion.

In order to establish the result (4.1), we proceed as follows. In equation (2.2), if we replace $(1-x)^{\beta-\alpha-\epsilon}$ to ${}_1F_0\left[\begin{matrix} \alpha+\epsilon-\beta \\ - \end{matrix} ;\ x\right]$, then it takes the following form:

$$F^{1:1;2}_{1:0;1}\left[\begin{matrix} \alpha: & \epsilon; & \beta-\epsilon,\gamma \\ \beta: & -; & \gamma+\beta \end{matrix} ;x,\ x\right] \tag{4.21}$$

$$= {}_1F_0\left[\begin{matrix} \alpha+\epsilon-\beta \\ - \end{matrix} ;\ x\right] {}_2F_1\left[\begin{matrix} \beta-\epsilon, & \gamma+\beta-\alpha \\ \gamma+\beta & \end{matrix} ;\ x\right].$$

Now, multiply both sides of (4.21) by $x^{\lambda-1}(1-x)^{\mu-\lambda-1}$ and integrating with respect to x between the interval $[0,1]$, we have

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-\lambda-1} F^{1:1;2}_{1:0;1}\left[\begin{matrix} \alpha: & \epsilon; & \beta-\epsilon,\gamma \\ \beta: & -; & \gamma+\beta \end{matrix} ;x,\ x\right] dx \tag{4.22}$$

$$= \int_0^1 x^{\lambda-1}(1-x)^{\mu-\lambda-1} {}_1F_0 \left[\begin{matrix} \alpha+\epsilon-\beta \\ - \end{matrix} ; x \right]$$

$$\times {}_2F_1 \left[\begin{matrix} \beta-\epsilon, \quad \gamma+\beta-\alpha \\ \gamma+\beta \end{matrix} ; x \right] dx.$$

Now, if I_1 be the left-hand side of (4.22), then for evaluating this integral, we get the equation (3.23). Further, if I_2 be the right-hand side of (4.22), then

$$I_2 = \int_0^1 x^{\lambda-1}(1-x)^{\mu-\lambda-1} {}_1F_0 \left[\begin{matrix} \alpha+\epsilon-\beta \\ - \end{matrix} ; x \right]$$

$$\times {}_2F_1 \left[\begin{matrix} \beta-\epsilon, \quad \gamma+\beta-\alpha \\ \gamma+\beta \end{matrix} ; x \right] dx.$$

Expressing both ${}_1F_0$ and ${}_2F_1$ functions as a series, change the order of integration and summation, we have after some calculation

$$I_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha+\epsilon-\beta)_m(\beta-\epsilon)_n(\gamma+\beta-\alpha)_n}{(\gamma+\beta)_n n! m!} \int_0^1 x^{\lambda+m+n-1}(1-x)^{\mu-\lambda-1}dx.$$

Evaluating the beta-integral and using the identity

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

we have, after some algebra

$$I_2 = \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\lambda)_{m+n}(\alpha+\epsilon-\beta)_m(\beta-\epsilon)_n(\gamma+\beta-\alpha)_n}{(\mu)_{m+n}(\gamma+\beta)_n n! m!}.$$

Summing up the series with the help of the definition of Kampé de Fériet function, we have

$$\text{(4.23)} \quad I_2 = \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)} F^{1:1;2}_{1:0;1} \left[\begin{matrix} \lambda: & \alpha+\epsilon-\beta; & \beta-\epsilon, \gamma+\beta-\alpha \\ \mu: & -; & \gamma+\beta \end{matrix} ; 1, \ 1 \right].$$

Finally, equating the equations (3.23) and (4.23), we arrive at the desired transformation formula (4.1). This completes the proof of (4.1). □

In exactly the same manner, other results can be established.

5. Further reduction formulas for the Kampé de Fériet function

In this section, we shall establish the following twenty further reduction formulas for the Kampé de Fériet function.

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{lll}\lambda: & \alpha+\epsilon-\beta; & \beta-\epsilon, \gamma+\beta-\epsilon \\ & & ;1,\ 1 \\ \mu: & -; & \gamma+\beta\end{array}\right] \tag{5.1}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}\,{}_3F_2\left[\begin{array}{lll}\beta-\epsilon, & \gamma+\beta-\alpha, & \lambda \\ & & ;\quad 1 \\ \gamma+\beta, & \mu+\beta-\alpha-\epsilon & \end{array}\right].$$

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{lll}\lambda: & \alpha+\epsilon-\beta; & \beta-\epsilon, 1+\frac{1}{2}\beta \\ & & ;1,\ 1 \\ \mu: & -; & \frac{1}{2}\beta\end{array}\right] \tag{5.2}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}\,{}_3F_2\left[\begin{array}{lll}\beta-\epsilon, & 1+\frac{1}{2}\beta, & \lambda \\ & & ;\quad 1 \\ \frac{1}{2}\beta, & \mu+\beta-\alpha-\epsilon & \end{array}\right].$$

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{lll}\lambda: & \alpha+\epsilon-\beta; & \beta-\epsilon, 1+\frac{1}{2}\beta, \frac{1}{2}(\beta-\alpha) \\ & & ;1,\ 1 \\ \mu: & -; & \frac{1}{2}\beta, 1+\frac{1}{2}(\alpha+\beta)\end{array}\right] \tag{5.3}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}$$
$$\times\ {}_4F_3\left[\begin{array}{llll}\beta-\epsilon, & 1+\frac{1}{2}\beta, & \frac{1}{2}(\beta-\alpha), & \lambda \\ & & & ;\quad 1 \\ \frac{1}{2}\beta, & 1+\frac{1}{2}(\beta+\alpha), & \mu+\beta-\alpha-\epsilon & \end{array}\right].$$

$$F^{1:1;2}_{1:0;1}\left[\begin{array}{lll}\lambda: & \alpha+\epsilon-\beta; & \beta-\epsilon, \frac{1}{2}(\beta-\alpha) \\ & & ;1,\ 1 \\ \mu: & -; & 1+\frac{1}{2}(\alpha+\beta)\end{array}\right] \tag{5.4}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}\,{}_3F_2\left[\begin{array}{lll}\beta-\epsilon, & \frac{1}{2}(\beta-\alpha), & \lambda \\ & & ;\quad 1 \\ 1+\frac{1}{2}(\alpha+\beta), & \mu+\beta-\alpha-\epsilon & \end{array}\right].$$

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{lll}\lambda: & \alpha+\epsilon-\beta; & \beta-\epsilon, 1+\frac{1}{2}\beta, \frac{1}{2}(\beta-\alpha-1) \\ & & ;1,\ 1 \\ \mu: & -; & \frac{1}{2}\beta, \frac{1}{2}(\alpha+\beta+3)\end{array}\right] \tag{5.5}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}$$
$$\times\ {}_4F_3\left[\begin{array}{llll}\beta-\epsilon, & 1+\frac{1}{2}\beta, & \frac{1}{2}(\beta-\alpha+1), & \lambda \\ & & & ;\quad 1 \\ \frac{1}{2}\beta, & \frac{1}{2}(\alpha+\beta+3), & \mu+\beta-\alpha-\epsilon & \end{array}\right].$$

$$F^{1:1;3}_{1:0;2}\left[\begin{array}{l} \lambda: \quad \alpha+\epsilon-\beta; \quad \beta-\alpha, \beta-\epsilon, \delta-\gamma \\ \\ \mu: \qquad -; \qquad\qquad \beta, \delta \end{array} ;1,\ 1\right] \tag{5.6}$$

$$= \frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}$$

$$\times\ {}_4F_3\left[\begin{array}{llll} \beta-\alpha, & \beta-\epsilon, & \delta-\gamma, & \lambda \\ & & & ;\ 1 \\ \beta, & \delta, & \mu+\beta-\alpha-\epsilon & \end{array}\right].$$

$$F^{1:1;4}_{1:0;3}\left[\begin{array}{l} \lambda: \quad \alpha+\epsilon-\beta; \quad \beta-\alpha, \beta-\epsilon, \delta-\gamma-1, \gamma-\delta+2 \\ \\ \mu: \qquad -; \qquad\qquad \beta, \delta, \gamma-\delta+1 \end{array} ;1,\ 1\right] \tag{5.7}$$

$$= \frac{\Gamma(\mu)\Gamma(\mu+\beta-\alpha-\epsilon-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu+\beta-\alpha-\epsilon)}$$

$$\times\ {}_5F_4\left[\begin{array}{lllll} \beta-\alpha, & \beta-\epsilon, & \delta-\gamma-1, & \gamma-\delta+2, & \lambda \\ & & & & ;\ 1 \\ \beta, & \delta, & \mu+\beta-\alpha-\epsilon, & \gamma-\delta+1 & \end{array}\right].$$

$$F^{1:1;2}_{0:1;2}\left[\begin{array}{l} \lambda: \quad \alpha; \quad \beta-\epsilon, \alpha+\gamma \\ \\ -: \quad \mu; \quad \beta, 1-\mu-\lambda \end{array} ;1,\ 1\right] \tag{5.8}$$

$$= \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}\ {}_3F_2\left[\begin{array}{lll} \beta-\epsilon, & \alpha+\gamma, & \lambda \\ & & ;\ 1 \\ \beta, & 1-\mu+\alpha+\lambda & \end{array}\right].$$

$$F^{1:1;3}_{0:1;3}\left[\begin{array}{l} \lambda: \quad \alpha; \quad \beta-\epsilon, \alpha+\gamma, \alpha+\delta \\ \\ -: \quad \mu; \quad \beta, \alpha+\gamma+\delta, 1-\mu+\lambda \end{array} ;1,\ 1\right] \tag{5.9}$$

$$= {}_4F_3\left[\begin{array}{llll} \beta-\epsilon, & \alpha+\gamma, & \alpha+\delta, & \lambda \\ & & & ;\ 1 \\ \beta, & \alpha+\gamma+\delta, & 1-\mu+\lambda & \end{array}\right].$$

$$F^{1:1;3}_{0:1;3}\left[\begin{array}{l} \lambda: \quad \alpha; \quad \beta-\epsilon, \alpha+\gamma, 1+\frac{1}{2}(\alpha+\gamma) \\ \\ -: \quad \mu; \quad \beta, \frac{1}{2}(\alpha+\gamma), 1-\mu+\lambda \end{array} ;1,\ 1\right] \tag{5.10}$$

$$= \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$

$$\times\ {}_4F_3\left[\begin{array}{llll} \beta-\epsilon, & \alpha+\gamma, & 1+\frac{1}{2}(\alpha+\gamma), & \lambda \\ & & & ;\ 1 \\ \beta, & \frac{1}{2}(\alpha+\gamma), & 1-\mu+\alpha+\lambda & \end{array}\right].$$

$$F^{1:1;4}_{0:1;4}\left[\begin{matrix}\lambda: & \alpha; & \beta-\epsilon,\alpha+\gamma,1+\frac{1}{2}(\alpha+\gamma),\frac{1}{2}\alpha \\ -: & \mu; & \beta,\frac{1}{2}(\alpha+\gamma),1+\frac{1}{2}\alpha+\gamma,1-\mu-\lambda\end{matrix};1,\ 1\right] \tag{5.11}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_5F_4\left[\begin{matrix}\alpha+\gamma, & 1+\frac{1}{2}(\alpha+\gamma), & \frac{1}{2}\alpha, & \beta-\epsilon, & \lambda \\ \frac{1}{2}(\alpha+\gamma), & 1+\frac{1}{2}\alpha+\gamma, & \beta, & 1-\mu+\alpha+\lambda & \end{matrix};\ 1\right].$$

$$F^{1:1;3}_{0:1;3}\left[\begin{matrix}\lambda: & \alpha; & \beta-\epsilon,\alpha+\gamma,\frac{1}{2}\alpha \\ -: & \mu; & \beta,1+\frac{1}{2}\alpha+\gamma,1-\mu+\lambda\end{matrix};1,\ 1\right] \tag{5.12}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_4F_3\left[\begin{matrix}\beta-\epsilon, & \alpha+\gamma, & \frac{1}{2}\alpha, & \lambda \\ \beta, & ,1+\frac{1}{2}\alpha+\lambda, & 1-\mu+\alpha+\lambda & \end{matrix};\ 1\right].$$

$$F^{1:1;3}_{0:1;3}\left[\begin{matrix}\lambda: & \alpha; & \beta-\epsilon,\alpha+\gamma,1+\frac{1}{2}(\alpha+\gamma),\frac{1}{2}(\alpha-1) \\ -: & \mu; & \beta,\frac{1}{2}(\alpha+\gamma),\frac{3}{2}+\frac{1}{2}\alpha+\gamma,1-\mu+\lambda\end{matrix};1,\ 1\right] \tag{5.13}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_5F_4\left[\begin{matrix}\alpha+\gamma, & 1+\frac{1}{2}(\alpha+\gamma), & \frac{1}{2}(\alpha-1), & \beta-\epsilon, & \lambda \\ \frac{1}{2}(\alpha+\gamma), & \frac{3}{2}+\frac{1}{2}\alpha+\gamma, & \beta, & 1-\mu+\alpha+\lambda & \end{matrix};\ 1\right].$$

$$F^{1:1;2}_{0:1;2}\left[\begin{matrix}\lambda: & \alpha; & \alpha,\beta+\epsilon-\gamma \\ -: & \mu; & \beta+\epsilon,\gamma-\mu+\lambda\end{matrix};1,\ -1\right] \tag{5.14}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}\ {}_3F_2\left[\begin{matrix}\alpha, & \beta+\epsilon-\gamma, & \lambda \\ \beta+\epsilon, & 1-\mu+\alpha+\lambda & \end{matrix};\ 1\right].$$

$$F^{1:1;2}_{0:1;2}\left[\begin{matrix}\lambda: & \alpha; & \alpha,1+\frac{1}{2}\beta \\ -: & \mu; & \frac{1}{2}\beta,1-\mu+\lambda\end{matrix};1,\ -1\right] \tag{5.15}$$
$$=\frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}\ {}_3F_2\left[\begin{matrix}\alpha, & 1+\frac{1}{2}\beta, & \lambda \\ \frac{1}{2}\beta, & 1-\mu+\alpha+\lambda & \end{matrix};\ 1\right].$$

$$F^{1:1;3}_{0:1;3}\left[\begin{array}{lll}\lambda: & \alpha; & \alpha, 1+\frac{1}{2}\beta, \frac{1}{2}(\beta-\gamma)\\ -: & \mu; & \frac{1}{2}\beta, 1+\frac{1}{2}(\beta+\gamma), 1-\mu+\lambda\end{array};1,\ -1\right] \tag{5.16}$$
$$= \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_4F_3\left[\begin{array}{llll}\alpha, & 1+\frac{1}{2}\beta, & \frac{1}{2}(\beta-\gamma), & \lambda\\ \frac{1}{2}\beta, & 1+\frac{1}{2}(\gamma+\beta), & 1-\mu+\alpha+\lambda & \end{array};\ 1\right].$$

$$F^{1:1;2}_{0:1;2}\left[\begin{array}{lll}\lambda: & \alpha; & \alpha, \frac{1}{2}(\beta-\gamma)\\ -: & \mu; & 1+\frac{1}{2}(\beta+\gamma), 1-\mu+\lambda\end{array};1,\ -1\right] \tag{5.17}$$
$$= \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}\ {}_3F_2\left[\begin{array}{lll}\alpha, & \frac{1}{2}(\beta-\gamma), & \lambda\\ 1+\frac{1}{2}(\beta+\gamma), & 1-\mu+\alpha+\lambda & \end{array};\ 1\right].$$

$$F^{1:1;3}_{0:1;3}\left[\begin{array}{lll}\lambda: & \alpha; & \alpha, 1+\frac{1}{2}\beta, \frac{1}{2}(\beta-\mu-1)\\ -: & \mu; & \frac{1}{2}\beta, \frac{1}{2}(2+\beta+\gamma), 1-\mu+\lambda\end{array};1,\ 1\right] \tag{5.18}$$
$$= \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_4F_3\left[\begin{array}{llll}\alpha, & 1+\frac{1}{2}\beta, & \frac{1}{2}(\beta-\gamma-1), & \lambda\\ \frac{1}{2}\beta, & \frac{1}{2}(3+\gamma+\beta), & 1-\mu+\alpha+\lambda & \end{array};\ 1\right].$$

$$F^{1:1;3}_{0:1;3}\left[\begin{array}{lll}\lambda: & \alpha; & \alpha, \beta-\epsilon, \delta-\gamma\\ -: & \mu; & \beta, \delta, 1-\mu+\lambda\end{array};1,\ 1\right] \tag{5.19}$$
$$= \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_4F_3\left[\begin{array}{llll}\alpha, & \beta-\epsilon, & \delta-\gamma, & \lambda\\ \beta, & \delta, & 1-\mu+\alpha+\lambda & \end{array};\ 1\right].$$

$$F^{1:1;4}_{0:1;4}\left[\begin{array}{lll}\lambda: & \alpha; & \alpha, \beta-\epsilon, \delta-\gamma-1, \delta-\gamma+2\\ -: & \mu; & \beta, \delta, \gamma-\delta+1, 1-\mu+\lambda\end{array};1,\ 1\right] \tag{5.20}$$
$$= \frac{\Gamma(\mu)\Gamma(\mu-\alpha-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\mu-\alpha)}$$
$$\times\ {}_5F_4\left[\begin{array}{lllll}\alpha, & \beta-\epsilon, & \delta-\gamma-1, & \delta-\gamma+2, & \lambda\\ \beta, & \delta, & \gamma-\delta+1 & 1-\mu+\alpha+\lambda & \end{array};\ 1\right].$$

Proof. The derivation of the reduction formulas (5.1) to (5.20) are quite straight forward. In order to establish the reduction formula (5.1), we observe that the left-hand side of the results (3.2) and (4.1) are the same, so equating the results (3.2) and (4.1), we immediately get the reduction formula (5.1).

In exactly the same manner, on equating the results (3.3) to (3.21) to the respectively results (4.2) to (4.20), we arrive at the reduction formulas (5.2) to (5.20). □

6. Concluding remark

In this paper, we have established a large number of reduction and transformation formulas for the Kampé de Fériet. The results are achieved with the help of the results recently obtained by Liu and Wang.

It is believed that the results established in this paper may be useful in the area of applied mathematics, engineering and mathematical physics.

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