Commun. Korean Math. Soc. **37** (2022), No. 2, pp. 551–566
https://doi.org/10.4134/CKMS.c210144
pISSN: 1225-1763 / eISSN: 2234-3024

DOUBLE SERIES TRANSFORMS DERIVED FROM FOURIER–LEGENDRE THEORY

John Maxwell Campbell and Wenchang Chu

Abstract. We apply Fourier–Legendre-based integration methods that had been given by Campbell in 2021, to evaluate new rational double hypergeometric sums involving $\frac{1}{\pi}$. Closed-form evaluations for dilogarithmic expressions are key to our proofs of these results. The single sums obtained from our double series are either inevaluable ${}_2F_1(\frac{4}{5})$- or ${}_2F_1(\frac{1}{2})$-series, or Ramanujan's ${}_3F_2(1)$-series for the moments of the complete elliptic integral $\mathbf{K}$. Furthermore, we make use of Ramanujan's finite sum identity for the aforementioned ${}_3F_2(1)$-family to construct creative new proofs of Landau's asymptotic formula for the Landau constants.

1. Introduction

Double series are often involved in important applications concerning many of the main subfields within mathematical analysis. Thus, there is a much in the way of effort that is put into research pursuits related to the computation of double infinite series, and for transforming such summations into simpler or more manageable expressions. In this article, we apply Fourier–Legendre-based methods from [6] in novel ways that require the use of dilogarithm identities and that provide us with new rational bivariate hypergeometric series for constants involving $\frac{1}{\pi}$ and the famous golden ratio constant $\phi = \frac{1+\sqrt{5}}{2}$, and for expressions such as $\frac{\sqrt{2}\ln^2(\sqrt{2}\pm 1)}{\pi}$; these new results are very much inspired by Ramanujan's famous rational series for constants involving $\frac{1}{\pi}$ [22] (cf. [3, pp. 352–364]). Furthermore, we cleverly apply double series rearrangements in conjunction with Ramanujan's finite sum evaluation for the moments of the elliptic integral $\mathbf{K}$ (see Section 2), providing us with a novel proof of Landau's asymptotic formula for the Landau constants.

Applications of orthogonal polynomials frequently are involved in computer algebra-related and symbolic computation-related topics. In this regard, the use of Fourier–Legendre (FL) expansions has been employed, over the years,

Received April 21, 2021; Accepted September 23, 2021.
2010 *Mathematics Subject Classification.* Primary 33C20, 33C75.
Key words and phrases. Bivariate hypergeometric series, central binomial coefficient, closed form, Landau's constants, dilogarithm.

©2022 Korean Mathematical Society

in many notable research contributions concerning the study of closed-form evaluations for infinite series, as in with the work of Wan et al. in this area [10, 25–27]. The investigation into the use of Legendre polynomials in the evaluation of bivariate hypergeometric series involving squared central binomial coefficients was initiated in [7] and greatly improved upon in [6], in which the moment formula for shifted Legendre polynomials is used in quite a variety of ways in conjunction with the integration strategy put forth in [7], in order to evaluate new families of series involving Ramanujan's S-function

$$S(m) = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \frac{\binom{2k}{k}^2}{k+m} = \frac{1}{m}\, {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, m \\ 1, m+1 \end{matrix} \,\middle|\, 1\right], \tag{1.1}$$

which admits the following identity (cf. [1]):

$$S(m) = \frac{16^m}{\pi m^2 \binom{2m}{m}^2} \sum_{k=0}^{m-1} \left(\frac{1}{16}\right)^k \binom{2k}{k}^2. \tag{1.2}$$

The double sums from [6] are all for constants involving $\frac{1}{\pi^2}$, especially the value $\frac{\zeta(3)}{\pi^2}$, letting $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ denote the famous Riemann zeta function. In contrast, within our current research contribution, our applications of the double sum transformations from [6] yield new rational approximations for constants involving $\frac{1}{\pi}$ but not $\frac{1}{\pi^2}$. The proofs for the series evaluations in [6] are given by direct applications of the double series transforms from [6]; in contrast, our article is based on:

(1) Applying the techniques from [6] in conjunction with identities for a special function that is not involved in [6] known as the *dilogarithm function*; and
(2) Applying results from or inspired by [6] in conjunction both with series rearrangements and Ramanujan's finite sum identity for the S-function in (1.1).

The double series that are put forth in Section 3 and that have, in large part, motivated this article are such that we obtain infinite sums involving exotic ${}_2F_1(\frac{4}{5})$- or ${}_2F_1(\frac{1}{2})$-series by summing over one of the two indices involved in the aforementioned double sums; in contrast, we always obtain ${}_3F_2(1)$-expressions by restricting the index sets in [6] to a single variable. Through our successful applications of the series transforms from [6], we have shown how our sums involving ${}_2F_1(\frac{4}{5})$- or ${}_2F_1(\frac{1}{2})$-functions can be expressed symbolically with expressions of the form

$$\pm \operatorname{Li}_2(z) \mp \operatorname{Li}_2(-z), \tag{1.3}$$

which, naturally, begs the question as to what values of z are such that (1.3) admits a closed-form evaluation, letting the dilogarithm function be defined as follows:

$$\operatorname{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}. \tag{1.4}$$

As noted in [9], one of the few known special values for the dilogarithm may be obtained directly from the first Rogers–Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \left(1-q^{5n+1}\right)^{-1} \left(1-q^{5n+4}\right)^{-1}, \tag{1.5}$$

as we later briefly review, in Section 3. Actually, out of all of the known special values for the dilogarithm [30, pp. 6–7], the only values z that are such that both $\operatorname{Li}_2(z)$ and $\operatorname{Li}_2(-z)$ are convergent according to the original series definition for Li_2 and also both admit closed-form evaluations are as follows: $z = \pm 1$ and $z = \pm\frac{1}{\phi}$. By applying techniques from [6] to reduce double series to expressions involving (1.3) for a variable z, and by then setting $z = \phi^{-1}$, this gives us new rational hypergeometric series for constants involving expressions as in $\frac{\ln(\phi)}{\pi}$. As it turns out, there is actually a value z whereby the difference in (1.3) admits a closed form but the separate terms in this difference do not [19]. In particular, the closed form

$$\operatorname{Li}_2\left(\sqrt{2}-1\right) - \operatorname{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} + \frac{1}{2}\ln^2\left(\sqrt{2}+1\right) - \ln^2\left(\sqrt{2}-1\right) \tag{1.6}$$

proved in [19] is to be of much utility in this article.

2. Three related motivating examples

Since our present article is based on further applications of the series and methods from [6], it seems worthwhile to recall the main motivating example highlighted in [6]:

$$\frac{14\zeta(3)}{\pi^2} = \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^{m+n} \frac{\binom{2m}{m}^2\binom{2n}{n}^2}{m+n+1}. \tag{2.1}$$

It is unexpected that the above identity is equivalent to the following one:

$$\frac{7\zeta(3)}{2\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^{n} \binom{2n}{n}^2 \sum_{m=n}^{\infty} \frac{1}{(2m+1)^2}. \tag{2.2}$$

In fact, summing over m, we get from (2.1) that

$$\frac{14\zeta(3)}{\pi^2} = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^{n} \binom{2n}{n}^2 S(n+1).$$

Then applying (1.2) to $S(n+1)$, we can rewrite the double sum as

$$\frac{14\zeta(3)}{\pi^2} = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^{n} \binom{2n}{n}^2 \times \frac{16^{n+1}}{\pi(n+1)^2\binom{2n+2}{n+1}^2} \sum_{k=0}^{n} \left(\frac{1}{16}\right)^{k} \binom{2k}{k}^2$$

$$= \frac{4}{\pi} \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \binom{2k}{k}^2 \sum_{n=k}^{\infty} \frac{1}{(2n+1)^2},$$

which is clearly equivalent to (2.2).

Classical identities for reducing bivariate hypergeometric series (cf. Slater [23, §8]) cannot be applied in any obvious or direct way to prove (2.1) or any of the other closed-form formulas introduced in [6] or in our current article. We recall that *classical hypergeometric series* [2, §2.5] are of the form

$${}_{p+1}F_p\left[\begin{matrix} a_0, a_1, \dots, a_p \\ b_1, b_2, \dots, b_p \end{matrix} \,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_p)_n}{n! \, (b_1)_n (b_2)_n \cdots (b_p)_n} z^n,$$

making use of the usual notation/definitions for the Pochhammer symbol, the Γ-function, etc. As in [6], we find it worthwhile to state that what is meant by the phrase *bivariate hypergeometric series* dates back to Jacob Horn's 1889 article [14], and that this term refers to double power sums $\sum_{m,n\geq 0} a_{m,n} x^m y^n$ such that $a_{m+1,n}/a_{m,n}$ and $a_{m,n+1}/a_{m,n}$ form rational functions in m and n.

Before we proceed with our motivating examples in Sections 2.1–2.3 below, it is worthwhile to define the above referred *complete elliptic integral of the first kind*, which is to be later involved in our work. This function may be defined as follows:

$$\mathbf{K}(k) = \frac{\pi}{2} \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \,\middle|\, k^2\right].$$

The *complete elliptic integral of the second kind* may be defined as follows:

$$\mathbf{E}(k) = \frac{\pi}{2} \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix} \,\middle|\, k^2\right].$$

Although the proofs of the results from [6], which are the main source of inspiration behind this current article, heavily rely on FL theory, our current work does not directly involve FL expansions or Legendre polynomials.

2.1. A double series formula involving π and the golden ratio

Much of this article is inspired by the following result that we had discovered and proved using the methods from [6], together with special values for Li_2:

$$\frac{\sqrt{5}\pi}{3} - \frac{3\sqrt{5}\ln^2(\phi)}{\pi} = \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^m \left(\frac{1}{5}\right)^n \frac{\binom{2m}{m}^2 \binom{2n}{n}}{m+n+1}. \tag{2.3}$$

Setting aside our generalizations of this result and our applications inspired by this same result, the equality in (2.3) is of interest in its own right, providing a rational approximation to a constant involving $\frac{1}{\pi}$ that is of a noticeably different appearance compared to the evaluations for Ramanujan's series for $\frac{1}{\pi}$, which do not involve logarithmic expressions or the golden ratio constant, as in the following two equalities [22] (cf. [3, pp. 352–364]):

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{8n} \binom{2n}{n}^3 (6n+1),$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{12n} \binom{2n}{n}^3 (42n+5).$$

State-of-the-art computer algebra system (CAS) software cannot evaluate the difficult double sum in (2.3), and, more generally, cannot evaluate any of the series introduced in this article. By applying (1.2) to the sum over $m \in \mathbb{N}_0$ we obtain the inevaluable expression

$$\left(\frac{1}{5}\right)^n \binom{2n}{n} \frac{1}{n+1} {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, n+1 \\ 1, n+2 \end{matrix} \,\middle|\, 1\right]. \tag{2.4}$$

It seems impossible to use this classical ${}_3F_2(1)$ identity to evaluate (2.3). For example, by plugging in an equivalent form of the right-hand side of (1.2) into (2.4) and summing over $n \in \mathbb{N}_0$, the triangular double sum that we obtain seems to be just as intractable as the right-hand side of (2.3). By taking the summand in (2.3) and summing over $n \in \mathbb{N}_0$, we obtain

$$\left(\frac{1}{16}\right)^m \binom{2m}{m}^2 \frac{1}{m+1} {}_2F_1\left[\begin{matrix} \frac{1}{2}, m+1 \\ m+2 \end{matrix} \,\middle|\, \frac{4}{5}\right],$$

which also does not admit an explicit evaluation. There does not seem to be much known about ${}_2F_1\left(\frac{4}{5}\right)$-series, which further motivates our interest in (2.3). Also, the moment formula whereby

$$S(r) = \frac{2}{\pi} \int_0^1 z^{r-1} \mathbf{K}(z)\, dz$$

for $\Re(r) > 0$ (see [1]) cannot be used in any direct or obvious way to evaluate (2.3).

2.2. A curious rational series formula involving $\ln^2(\sqrt{2} \pm 1)/\pi$

We later prove the following evaluation using the techniques from [6] along with the dilogarithmic identity in (1.6):

$$\sum_{m,n \geq 0} \left(\frac{1}{2}\right)^{4m+3n} \frac{\binom{2m}{m}^2 \binom{2n}{n}}{m+n+1} \tag{2.5}$$

$$= \frac{2\sqrt{2} \ln^2\left(\sqrt{2}+1\right)}{\pi} - \frac{4\sqrt{2} \ln^2\left(\sqrt{2}-1\right)}{\pi} + \frac{\pi}{\sqrt{2}}. \tag{2.6}$$

This identity is new and nontrivial in much the same way as in with the motivating example shown in the preceding subsection. The above evaluation and its companion in (2.3) are of great interest to us, since these results show

how polylogarithmic identities and the advances in FL theory from [6, 7] can be applied together in novel, interesting ways. Taking the summand in (2.5) and summing over $m \in \mathbb{N}_0$, we obtain an expression involving Ramanujan's S-function, and by instead summing over $n \in \mathbb{N}_0$, we obtain an inevaluable ${}_2F_1(\frac{1}{2})$-series.

2.3. A new proof of Landau's asymptotic formula

As stated in [1], there is a great mathematical history concerning finite sums of the form in (1.2). Expressions of the form

$$G_n = \sum_{i=0}^{n} \left(\frac{1}{16}\right)^i \binom{2i}{i}^2$$

are known as *Landau's constants* [21, 29], which are used in many areas of analysis, in large part due to the following famous inequality due to Landau in 1913 [15]: If $f(z) = \sum_{k \geq 0} a_k z^k$ is a holomorphic function on the open unit disk such that $|f(z) < 1|$, then

$$\left| \sum_{k=0}^{n} a_k \right| \leq G_n.$$

As described in [21], there has been much research devoted to asymptotic formulas for the Landau constants, a subject that traces back to Landau's famous [29] result whereby

$$G_n \sim \frac{1}{\pi} \ln n; \tag{2.7}$$

see [29] for the original complex analysis-based proof of this result that involves Cauchy's theorem. Using the FL-based techniques from [6] together with a summation technique based on Ramanujan's S-function, we have come to construct a dramatically different proof of Landau's result in (2.7), which we have formulated in an equivalent way, so that

$$\frac{2}{\pi} = \lim_{n \to \infty} \frac{1}{O_n} \sum_{i=0}^{n} \left(\frac{1}{16}\right)^i \binom{2i}{i}^2, \tag{2.8}$$

where $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$ denotes the n^{th} *odd harmonic number*, with $O_n = H_{2n} - \frac{H_n}{2}$ for all $n \in \mathbb{N}_0$, and recalling the Euler–Mascheroni constant being given by $\gamma = \lim_{n \to \infty}(H_n - \ln n)$.

3. Main results and proofs

Hypergeometric functions have a role of much importance and utility in mathematical physics. For example, there are direct and recent applications of bivariate hypergeometric series within areas in physics such as conformal field theory [18]; since there are also many applications of dilogarithmic identities in conformal field theory [20, 24, 30], this motivates our interest in "combining" topics in the study of bivariate hypergeometric sums with that of Li_2 identities. The new results that we have discovered in this interdisciplinary endeavour, in our successfully applying the transformation methods from [6] in conjunction with dilogarithmic series and special values, are the emphasis of Sections 3.1–3.3.

3.1. The first transformation theorem and applications of polylogarithmic identities

Theorem 3.1 (Campbell [6], 2021)**.** *As in* [6]*, let the sequence* $(f_n)_{n\geq 0}$ *be such that the mapping* $g(x)$ *given by the ordinary generating function* (*g.f.*) *for this sequence is well-defined on* $(0,1)$*, and such that the conditions enumerated as follows are all satisfied*: *The definite integral*

$$\int_0^1 \mathbf{K}(\sqrt{x})g(x)\,dx \tag{3.1}$$

is well-defined, and if we replace $g(x)$ *in the integrand shown in* (3.1) *with* $\sum_{n=0}^{\infty} f_n x^n$*, and replace* $\mathbf{K}(\sqrt{x})$ *by either its Maclaurin series or its shifted FL series, summing over* $m \in \mathbb{N}_0$*, then the operators* $\int_0^1 \cdot\, dx$*,* $\sum_{n=0}^{\infty} \cdot$*, and* $\sum_{m=0}^{\infty} \cdot$ *commute in either case and are such that the following series are convergent. It then follows that*

$$\frac{\pi}{2}\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^m \frac{\binom{2m}{m}^2}{m+n+1}\cdot f_n \tag{3.2}$$

must be equal to the following:

$$2\sum_{\substack{m,n\in\mathbb{N}_0\\ n\geq m}} \frac{1}{2m+1}\frac{(n!)^2}{(n-m)!(n+m+1)!}\cdot f_n. \tag{3.3}$$

An application of this theorem, together with a dilogarithmic form for a bivariate hypergeometric series, as in (3.4), allow us to produce a full proof for the motivating example highlighted in Section 2.1.

Recalling the Rogers–Ramanujan identity in (1.5), we let $q = e^{-t}$, and we then apply the operator $\lim_{t\to 0^+} t\ln(\cdot)$ to both sides of the resultant identity [9] to give us a closed form for Li_2 evaluated at the reciprocal of ϕ, namely $\frac{\pi^2}{10} - \ln^2(\phi)$; this is a perhaps prototypical instance of a *polylogarithm ladder*. As we shall see, what is of especial significance, for our purposes, about this particular dilogarithm value is due to the fact that the Li_2 function evaluated at the negative of the previously inputted constant $\frac{1}{\phi}$ also admits a closed form, with $\mathrm{Li}_2\left(-\phi^{-1}\right) = -\frac{\pi^2}{15} + \frac{1}{2}\ln^2(\phi)$ (cf. [30, p. 7]). We have discovered a way of applying the transformation methods from [6] so as to determine new rational hypergeometric series for expressions involving the difference $\mathrm{Li}_2(z) - \mathrm{Li}_2(-z)$ quotiented by π. However, despite the duplication formula $\mathrm{Li}_2(z^2) = 2(\mathrm{Li}_2(z) + \mathrm{Li}_2(-z))$ [30, p. 9], there is no such identity for differences of the form $\pm\mathrm{Li}_2(z) \mp \mathrm{Li}_2(-z)$. In order for both $\mathrm{Li}_2(z)$ and $\mathrm{Li}_2(-z)$ to admit a closed form, we are left with the cases whereby $z = \pm 1$ or $z = \pm\phi^{-1}$. This latter case, together with the above theorem, lead us toward the following proof for the first out of the motivating examples presented in Section 2.

Proof of (2.3): For a real parameter p, setting $f_n = p^n \binom{2n}{n}$ in the above hypergeometric transform from [6], we obtain that

$$\frac{\pi}{2} \sum_{m,n \geq 0} \left(\frac{1}{16}\right)^m p^n \frac{\binom{2m}{m}^2 \binom{2n}{n}}{m+n+1} \tag{3.4}$$

must equal

$$2 \sum_{m,n \geq 0} p^n \frac{\binom{2n}{n} (n!)^2}{(2m+1)(n-m)!(m+n+1)!}.$$

We may express

$$\sum_{n=m}^{\infty} p^n \frac{\binom{2n}{n} (n!)^2}{(2m+1)(n-m)!(m+n+1)!}$$

as

$$\frac{p^m}{(2m+1)^2} \, {}_2F_1\left[\begin{matrix} m+\frac{1}{2}, m+1 \\ 2m+2 \end{matrix} \,\middle|\, 4p\right].$$

The above ${}_2F_1$-function may be written as

$$-\frac{2^{2m-1}\left(\sqrt{1-4p}-1\right)\left(\sqrt{1-4p}+1\right)^{-2m}}{p},$$

which gives us that (3.4) equals

$$\frac{1-\sqrt{1-4p}}{p} \sum_{m=0}^{\infty} \left(\frac{2\sqrt{p}}{1+\sqrt{1-4p}}\right)^{2m} \frac{1}{(2m+1)^2}, \tag{3.5}$$

which we may symbolically compute by bisecting (1.4), giving us

$$\frac{-1}{\sqrt{p}} \times \left(\mathrm{Li}_2\left(-2\sqrt{\frac{p}{\left(\sqrt{1-4p}+1\right)^2}}\right) - \mathrm{Li}_2\left(2\sqrt{\frac{p}{\left(\sqrt{1-4p}+1\right)^2}}\right)\right),$$

noting the appearance of an expression of the form $\pm\mathrm{Li}_2(z) \mp \mathrm{Li}_2(-z)$. Setting $p = \frac{1}{5}$, this gives us a proof for (2.3), thanks to the values of the Li_2 function evaluated at $\pm\phi^{-1}$. $\square$

By mimicking this proof, we may obtain closed forms for the infinite family of double sums indicated below:

$$\sum_{m,n \geq 0} \left(\frac{1}{16}\right)^m \left(\frac{1}{5}\right)^n \frac{\binom{2m}{m}^2 \binom{2n}{n}}{(m+n+1)(2n-1)}$$
$$= \frac{9\ln^2(\phi)}{2\sqrt{5}\pi} - \frac{3\ln(\phi)}{2\pi} - \frac{1}{\sqrt{5}\pi} - \frac{\pi}{2\sqrt{5}},$$

$$\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^m\left(\frac{1}{5}\right)^n\frac{\binom{2m}{m}^2\binom{2n}{n}}{(m+n+1)(2n-3)}$$
$$=\frac{57\ln^2(\phi)}{20\sqrt{5}\pi}-\frac{33\ln(\phi)}{40\pi}-\frac{31}{60\sqrt{5}\pi}-\frac{19\pi}{60\sqrt{5}}.$$

Again, we come across inevaluable ${}_2F_1(\frac{4}{5})$-series or expressions involving the S-function by summing over a single index, for each member of the above family. Our evaluation of (3.4) as (3.5), together with a polylogarithmic identity from [19], give us a proof for the motivating example in Section 2.2.

Proof of the equality of (2.5) *and* (2.6): By setting $p=\frac{1}{8}$ in the equivalent expressions in (3.4) and (3.5), this gives us that the double series in (2.5) is equal to $\frac{\pi}{2}$ times

$$\frac{2\left(\sqrt{2}-2\right)\left(\mathrm{Li}_2\left(1-\sqrt{2}\right)-\mathrm{Li}_2\left(\sqrt{2}-1\right)\right)}{\sqrt{3-2\sqrt{2}}},$$

which proves the motivating example from Section 2.2, thanks to the dilogarithm identity (1.6) due to Lima [19]. ☐

By mimicking this approach, we may also evaluate the following double series:

$$\sum_{m,n\geq 0}\frac{\binom{2m}{m}^2\binom{2n}{n}}{2^{4m+3n}(m+n+1)(2n-1)}$$
$$=\frac{3\ln^2\left(\sqrt{2}+1\right)}{2\sqrt{2}\pi}-\frac{\ln\left(\sqrt{2}+1\right)}{\sqrt{2}\pi}-\frac{\ln\left(\sqrt{2}-1\right)}{\sqrt{2}\pi}+\frac{\ln\left(\sqrt{2}-1\right)}{\pi}-\frac{1}{\sqrt{2}\pi}-\frac{3\pi}{8\sqrt{2}},$$
$$\sum_{m,n\geq 0}\frac{\binom{2m}{m}^2\binom{2n}{n}}{2^{4m+3n}(m+n+1)(2n-3)}$$
$$=\frac{29\ln^2\left(\sqrt{2}+1\right)}{32\sqrt{2}\pi}+\frac{7\ln\left(\sqrt{2}-1\right)}{16\pi}-\frac{17}{48\sqrt{2}\pi}-\frac{29\pi}{128\sqrt{2}}.$$

By summing over a single index, for each of the members of the family of sums shown above, either inevaluable ${}_2F_1(\frac{1}{2})$-series or summands involving the S-function defined in (1.1) will show up.

3.2. The second transformation theorem and applications to double series whose values involve the golden ratio

We now turn our attention to another transformation theorem.

Theorem 3.2 (Campbell [6], 2021)**.** *Let* $f\colon \mathbb{N}_0\to\mathbb{C}$ *and* $g(x)$ *be the ordinary g.f. for* f*. Assume that* g *is well-defined on* $(0,1)$ *such that*

$$\int_0^1 \mathbf{E}(\sqrt{x})g(x)\,dx \tag{3.6}$$

is well-defined and satisfies the following: *If we replace $g(x)$ in the above integrand in* (3.6) *with the summation $\sum_{n=0}^{\infty} f_n x^n$, and if we replace $\mathbf{E}(\sqrt{x})$ by either its Maclaurin series or its shifted FL series, summing over $m \in \mathbb{N}_0$, then the operators $\int_0^1 \cdot\, dx$, $\sum_{n=0}^{\infty} \cdot$, and $\sum_{m=0}^{\infty} \cdot$ commute in either case and are such that the following series are convergent. It then follows that the expression*

$$-\frac{\pi}{2} \sum_{m,n\geq 0} \left(\frac{1}{16}\right)^m \frac{\binom{2m}{m}^2}{(2m-1)(m+n+1)} \cdot f_n$$

must equal the following:

$$-4 \sum_{\substack{m,n\in\mathbb{N}_0 \\ n\geq m}} \frac{1}{(2m-1)(2m+1)(2m+3)} \frac{(n!)^2}{(n-m)!(n+m+1)!} \cdot f_n.$$

Applying this transform together with the special values for Li_2 evaluated at $\pm\phi^{-1}$, we can derive

$$\begin{aligned} &\sum_{m,n\geq 0} \left(\frac{1}{16}\right)^m \left(\frac{1}{5}\right)^n \frac{\binom{2m}{m}^2\binom{2n}{n}}{(2m-1)(m+n+1)} \\ &= \frac{3\sqrt{5}\ln^2(\phi)}{2\pi} - \frac{15\ln(\phi)}{4\pi} - \frac{\sqrt{5}\pi}{6} + \frac{\sqrt{5}}{2\pi}, \end{aligned} \tag{3.7}$$

and closed forms for the below series:

$$\begin{aligned} &\sum_{\substack{n\geq 0 \\ m\geq 0}} \frac{\binom{2m}{m}^2\binom{2n}{n}}{16^m 5^n (2m-1)(m+n+1)(2n-1)} \\ &= -\frac{3\ln^2(\phi)}{\sqrt{5}\pi} + \frac{21\ln(\phi)}{16\pi} + \frac{\pi}{3\sqrt{5}} + \frac{\sqrt{5}}{8\pi}, \\ &\sum_{\substack{n\geq 0 \\ m\geq 0}} \frac{\binom{2m}{m}^2\binom{2n}{n}}{16^m 5^n (2m-1)(m+n+1)(2n-3)} \\ &= -\frac{69\ln^2(\phi)}{40\sqrt{5}\pi} + \frac{61\ln(\phi)}{80\pi} + \frac{23\pi}{120\sqrt{5}} + \frac{89}{360\sqrt{5}\pi}. \end{aligned}$$

Similarly, by applying the above theorem together with Lima's polylogarithmic identity in (1.6), we can further evaluate not only the double series

$$\sum_{m,n\geq 0} \left(\frac{1}{2}\right)^{4m+3n} \frac{\binom{2m}{m}^2\binom{2n}{n}}{(2m-1)(m+n+1)}$$

in the following closed-form evaluation:

$$\frac{2\sqrt{2}\ln^2\left(\sqrt{2}-1\right)}{\pi} - \frac{\sqrt{2}\ln^2\left(\sqrt{2}+1\right)}{\pi} + \frac{2\sqrt{2}\ln\left(\sqrt{2}+1\right)}{\pi}$$

$$+\frac{2\left(2+\sqrt{2}\right)\ln\left(2-\sqrt{2}\right)}{\pi}-\frac{\left(2+\sqrt{2}\right)\ln(2)}{\pi}+\frac{2\sqrt{2}}{\pi}-\frac{\pi}{2\sqrt{2}},$$

but also an infinite family of generalizations of this result, in much the same way as in with our above generalizations of (3.7).

3.3. Applications concerning Bonnet's recursion identity

The primary hypergeometric transform from [6] that is derived from the Bonnet recursion may be summarized as follows: For a reasonably well-behaved function f from $\mathbb{N}_0$ to $\mathbb{C}$ that satisfies appropriate analogues of the conditions in Theorems 3.1 and 3.2, we can show that

$$\frac{\pi}{4}\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^m\frac{\binom{2m}{m}^2}{(m+1)(m+n+2)}\cdot f_n \tag{3.8}$$

equals

$$4\sum_{m,n\geq 0}\frac{2m+1}{(2m-1)^2(2m+3)^2}\frac{(n!)^2}{(n-m)!(n+m+1)!}\cdot f_n,$$

i.e., by using the Maclaurin series and the shifted FL expansion for $\mathbf{E}(\sqrt{x})-\mathbf{K}(\sqrt{x})+x\mathbf{K}(\sqrt{x})$; see [6, §3] for details. This can be used to prove the formula

$$\begin{aligned}&\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^m\left(\frac{1}{5}\right)^n\frac{\binom{2m}{m}^2\binom{2n}{n}}{(m+1)(m+n+2)}\\ &= -\frac{9\sqrt{5}\ln^2(\phi)}{4\pi}+\frac{15\ln(\phi)}{4\pi}+\frac{\sqrt{5}\pi}{4}-\frac{3\sqrt{5}}{2\pi}.\end{aligned}$$

Similarly, by the hypergeometric identity from [6, §3.1] concerning the g.f. for the sequence of squared Catalan numbers, we can show that

$$\begin{aligned}&\sum_{\substack{n\geq 0\\ m\geq 0}}\frac{\binom{2m}{m}^2\binom{2n}{n}}{16^m5^n(m+1)^2(m+n+2)}\\ &= -\frac{21\sqrt{5}\ln^2(\phi)}{2\pi}+\frac{45\ln(\phi)}{2\pi}+\frac{7\sqrt{5}\pi}{6}-\frac{5\sqrt{5}}{\pi}+2\sqrt{5}-10,\end{aligned}$$

and from the power series identity that is such that

$$\sum_{m=0}^{\infty}\left(\frac{1}{16}\right)^m\frac{\binom{2m}{m}^2x^m}{(2m-1)^2}=\frac{2(x-1)\mathbf{K}(\sqrt{x})}{\pi}+\frac{4\mathbf{E}(\sqrt{x})}{\pi},$$

we may also evaluate series as in

$$\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^m\left(\frac{1}{5}\right)^n\frac{\binom{2m}{m}^2\binom{2n}{n}}{(2m-1)^2(m+n+1)}.$$

We may also apply the equality in (1.6) together with the above identities derived from Bonnet's recursion to prove new rational bivariate hypergeometric series for constants involving $\frac{\sqrt{2}\ln^2(\sqrt{2}\pm 1)}{\pi}$; for the sake of brevity, we leave this to the reader.

3.4. Novel proofs of Landau's asymptotic formula

Setting $f_n = \frac{1}{n+1}$ for $n \in \mathbb{N}_0$ in (3.8), by restricting the resultant summand to $m \in \mathbb{N}_0$, we obtain that the sum of

$$\frac{\Gamma^2\left(m+\frac{1}{2}\right)\left(\psi_0(m+2)+\gamma\right)}{4\Gamma^2(m+2)}$$

for $m \geq 0$ must equal the sum of all series given by

$$\frac{\pi}{4(n+1)(n+2)}\,{}_3F_2\!\left[\begin{matrix}\frac{1}{2},\frac{1}{2},n+2\\ 2,n+3\end{matrix}\,\middle|\,1\right]. \tag{3.9}$$

Rewriting this former sum as

$$\frac{\pi}{4}\sum_{m=0}^{\infty}\left(\frac{1}{16}\right)^m\binom{2m}{m}^2\frac{H_{m+1}}{(m+1)^2},$$

and then splitting it into two with respect to $H_{m+1} = H_m + \frac{1}{m+1}$, we can evaluate it as $\frac{\pi}{4}(\frac{48}{\pi} - \frac{64\ln 2}{\pi})$, taking into account the two known summation formulae (cf. [5]):

$$\sum_{m=0}^{\infty}\left(\frac{1}{16}\right)^m\binom{2m}{m}^2\frac{H_m}{(m+1)^2} = 16 - 16\ln 2 + \frac{32G}{\pi} - \frac{64\ln 2}{\pi},$$

$${}_4F_3\!\left[\begin{matrix}\frac{1}{2},\frac{1}{2},1,1\\ 2,2,2\end{matrix}\,\middle|\,1\right] = \frac{48}{\pi} - \frac{32G}{\pi} + 16\ln 2 - 16.$$

Equating the previously obtained value to (3.9) gives us that

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{m=0}^{\infty}\left(\frac{1}{16}\right)^m\binom{2m}{m}^2\frac{1}{(m+1)(m+n+2)} = \frac{48}{\pi} - \frac{64\ln 2}{\pi}.$$

Applying partial fraction decomposition to the rational function factor of the summand of the inner sum shown above, we can show that

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\sum_{m=0}^{\infty}\left(\frac{1}{16}\right)^m\binom{2m}{m}^2\frac{1}{m+n+2} = \frac{64\ln 2}{\pi} - \frac{48}{\pi} + \frac{2\pi}{3}, \tag{3.10}$$

where Ramanujan's S-function emerges again. By means of (1.2), we may reformulate (3.10) as

$$\sum_{n=0}^{\infty}\frac{16^n}{\binom{2n}{n}^2(2n+1)^2(2n+3)^2}\sum_{k=0}^{n}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2 = 2\ln 2 - \frac{\pi^2}{8}. \tag{3.11}$$

By interchanging the summation order, we may rewrite the last sum as

$$\frac{1}{9}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k^2}{(k!)^2}\sum_{n=k}^{\infty}\frac{(n!)^2}{(\frac{5}{2})_n^2}=2\ln 2-\frac{\pi^2}{8}. \tag{3.12}$$

According to $1=2(n+\frac{3}{2})-2(n+1)$, we can split the inner sum into two and then evaluate them by telescoping:

$$\begin{aligned}\sum_{n=k}^{\infty}\frac{(n!)^2}{(\frac{5}{2})_n^2}&=3\sum_{n=k}^{\infty}\frac{(n!)^2}{(\frac{3}{2})_n(\frac{5}{2})_n}-2\sum_{n=k}^{\infty}\frac{n!(n+1)!}{(\frac{5}{2})_n^2}\\&=9\sum_{n=k}^{\infty}\left\{\frac{((n+1)!)^2}{(\frac{1}{2})_{n+1}(\frac{3}{2})_{n+1}}-\frac{(n!)^2}{(\frac{1}{2})_n(\frac{3}{2})_n}\right\}\\&\quad+18\sum_{n=k}^{\infty}\left\{\frac{(n+1)!(n+2)!}{(\frac{3}{2})_{n+1}^2}-\frac{n!(n+1)!}{(\frac{3}{2})_n^2}\right\}\\&=9\left\{\frac{\pi}{2}-\frac{(k!)^2}{(\frac{1}{2})_k(\frac{3}{2})_k}\right\}+18\left\{\frac{\pi}{4}-\frac{k!(k+1)!}{(\frac{3}{2})_k^2}\right\}\\&=9\pi-9(4k+3)\frac{(k!)^2}{(\frac{3}{2})_k^2}.\end{aligned}$$

By substituting this into (3.12), we get the equation

$$\sum_{k=0}^{\infty}\left\{\pi\frac{(\frac{1}{2})_k^2}{(k!)^2}-\frac{2}{2k+1}+\left(\frac{2}{2k+1}-\frac{4k+3}{(2k+1)^2}\right)\right\}=2\ln 2-\frac{\pi^2}{8}.$$

Factorizing the difference in the parentheses into $\frac{-1}{(2k+1)^2}$, and evaluating with respect to k by $\frac{-\pi^2}{8}$, we find the simplified equation

$$\sum_{k=0}^{\infty}\left\{\frac{(\frac{1}{2})_k^2}{(k!)^2}-\frac{2}{\pi(2k+1)}\right\}=\frac{2}{\pi}\ln 2,$$

noting that the series obtained by splitting the above summand according to its two terms are non-convergent. Equivalently,

$$\frac{2}{\pi}\ln 2=\lim_{m\to\infty}\left\{\sum_{k=0}^{m-1}\left(\frac{1}{16}\right)^k\binom{2k}{k}^2-\frac{2}{\pi}O_m\right\},$$

which is easily seen to give us the desired result in (2.8). □

Remark 3.3. Alternatively, starting from the formula (cf. [4] and [28])

$$\sum_{n=1}^{\infty}\left(\frac{1}{16}\right)^n\binom{2n}{n}^2\frac{H_n}{2n-1}=\frac{8\ln 2-4}{\pi}$$

we may rewrite it by interchanging the summation order as

$$\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=k}^{\infty}\left(\frac{1}{16}\right)^{n}\binom{2n}{n}^{2}\frac{1}{2n-1}=\frac{8\ln 2-4}{\pi}.$$

Then by following the same procedure as for (3.10), the above equality can be used to provide another proof of the asymptotic relation in (2.8).

4. Conclusion

The manipulations of the Kampé de Fériet function due very recently to Li [17] produce bivariate hypergeometric series that, informally, bear a resemblance to many of our results, and it seems reasonable to suggest that there is potential in the application of our methods with those from [17]. Holdeman's Legendre polynomial expansions of ${}_2F_1$-functions, as given in [13], would likely lead to interesting results if applied in conjunction with our techniques.

Since much of our article concerns the FL-derived techniques from [6] for constructing rational approximations to constants involving $\frac{1}{\pi^2}$, it seems worthwhile to build upon Levrie's work [16] (see also [12]) on FL expansions in the determination of rational sums for $\frac{1}{\pi}$ and $\frac{1}{\pi^2}$, e.g., by using variants of the techniques from [16] so as to form double series, and to then mimic the evaluation strategies from Sections 3. More specifically, by making use of the Maclaurin and the shifted FL expansions for $\left(\sqrt{1-x}\sqrt{x}\right)^{2k-1}$ in the case whereby $k\in\mathbb{N}$, we may devise analogues and variants of Theorems 3.1 and 3.2, by letting the odd, positive, integer powers of $\sqrt{1-x}\sqrt{x}$ be used in place of the complete elliptic integrals.

Mimicking the setup and the proof of Theorem 3.1 given in [6], by starting with an integral of the form

$$\int_0^1 \mathbf{K}(x)\left(\sum_{n=0}^{\infty}x^{2n}f_n\right)dx$$

and by setting $x=\sqrt{y}$, we can show, under suitable conditions on the sequence $(f_n : n\in\mathbb{N}_0)$, that the double series

$$\frac{\pi}{2}\sum_{m,n\geq 0}\left(\frac{1}{16}\right)^{m}\binom{2m}{m}^{2}\frac{1}{2m+2n+1}\cdot f_n$$

equals

$$\sum_{m,n\geq 0}\frac{\Gamma^2\left(n+\frac{1}{2}\right)}{(2m+1)\Gamma\left(n-m+\frac{1}{2}\right)\Gamma\left(n+m+\frac{3}{2}\right)}\cdot f_n,$$

which seems to be applicable in much the same way as Theorems 3.1 and 3.2, and we encourage the exploration of applications of this transform. For example, by setting $f_n=\frac{1}{(n+1)^2}$, and by then using binomial-harmonic evaluations introduced in [8] and [11], we obtain the following evaluation, where G denote Catalan's constant:

$$16\ln(2) - 8G - \frac{\pi^2}{6} = \sum_{m,n\geq 0} \frac{\Gamma^2\left(n+\frac{1}{2}\right)}{(2m+1)(n+1)^2\Gamma\left(n-m+\frac{1}{2}\right)\Gamma\left(n+m+\frac{3}{2}\right)}.$$

References

[1] V. S. Adamchik, *A certain series associated with Catalan's constant*, Z. Anal. Anwendungen **21** (2002), no. 3, 817–826. https://doi.org/10.4171/ZAA/1110

[2] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 32, Stechert-Hafner, Inc., New York, 1964.

[3] B. C. Berndt, *Ramanujan's Notebooks. Part IV*, Springer-Verlag, New York, 1994. https://doi.org/10.1007/978-1-4612-0879-2

[4] J. M. Campbell, *Ramanujan-like series for $\frac{1}{\pi}$ involving harmonic numbers*, Ramanujan J. **46** (2018), no. 2, 373–387. https://doi.org/10.1007/s11139-018-9995-9

[5] J. M. Campbell, *New series involving harmonic numbers and squared central binomial coefficients*, Rocky Mountain J. Math. **49** (2019), no. 8, 2513–2544. https://doi.org/10.1216/RMJ-2019-49-8-2513

[6] J. M. Campbell, *New families of double hypergeometric series for constants involving $1/\pi^2$*, Ann. Polon. Math. **126** (2021), no. 1, 1–20. https://doi.org/10.4064/ap200709-4-1

[7] J. M. Campbell, J. D'Aurizio, and J. Sondow, *On the interplay among hypergeometric functions, complete elliptic integrals, and Fourier-Legendre expansions*, J. Math. Anal. Appl. **479** (2019), no. 1, 90–121. https://doi.org/10.1016/j.jmaa.2019.06.017

[8] J. M. Campbell and A. Sofo, *An integral transform related to series involving alternating harmonic numbers*, Integral Transforms Spec. Funct. **28** (2017), no. 7, 547–559. https://doi.org/10.1080/10652469.2017.1318874

[9] E. R. Canfield, *From recursions to asymptotics: Durfee and dilogarithmic deductions*, Adv. in Appl. Math. **34** (2005), no. 4, 768–797. https://doi.org/10.1016/j.aam.2004.08.008

[10] H. H. Chan, J. Wan, and W. Zudilin, *Legendre polynomials and Ramanujan-type series for $1/\pi$*, Israel J. Math. **194** (2013), no. 1, 183–207. https://doi.org/10.1007/s11856-012-0081-5

[11] H. Chen, *Interesting series associated with central binomial coefficients, Catalan numbers and harmonic numbers*, J. Integer Seq. **19** (2016), no. 1, Article 16.1.5, 11 pp.

[12] W. Chu, *π-formulas implied by Dougall's summation theorem for ${}_5F_4$-series*, Ramanujan J. **26** (2011), no. 2, 251–255. https://doi.org/10.1007/s11139-010-9274-x

[13] J. T. Holdeman, Jr., *Legendre polynomial expansions of hypergeometric functions with applications*, J. Math. Phys. **11** (1970), 114–117. https://doi.org/10.1063/1.1665035

[14] J. Horn, *Ueber die Convergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen*, Math. Ann. **34** (1889), no. 4, 544–600. https://doi.org/10.1007/BF01443681

[15] E. Landau, *Abschtzung der Koeffiziententensumme einer Potenzreihe*, Arch. Math. Phys. **21** (1913), 42–50.

[16] P. Levrie, *Using Fourier-Legendre expansions to derive series for $\frac{1}{\pi}$ and $\frac{1}{\pi^2}$*, Ramanujan J. **22** (2010), no. 2, 221–230. https://doi.org/10.1007/s11139-010-9222-9

[17] L. Li, *Double series expansions for π*, AIMS Math. **6** (2021), no. 5, 5000–5007. https://doi.org/10.3934/math.2021294

[18] W. Li, *Lightcone expansions of conformal blocks in closed form*, J. High Energy Phys. **2020**, no. 6, 105, 22 pp. https://doi.org/10.1007/jhep06(2020)105

[19] F. M. S. Lima, *New definite integrals and a two-term dilogarithm identity*, Indag. Math. (N.S.) **23** (2012), no. 1-2, 1–9. https://doi.org/10.1016/j.indag.2011.08.008

[20] W. Nahm, A. Recknagel, and M. Terhoeven, *Dilogarithm identities in conformal field theory*, Modern Phys. Lett. A **8** (1993), no. 19, 1835–1847. https://doi.org/10.1142/S0217732393001562

[21] G. Nemes and A. Nemes, *A note on the Landau constants*, Appl. Math. Comput. **217** (2011), no. 21, 8543–8546. https://doi.org/10.1016/j.amc.2011.03.058

[22] S. Ramanujan, *Modular equations and approximations to* π, Q. J. Math. **45** (1914), 350–372.

[23] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge, 1966.

[24] M. Terhoeven, *Dilogarithm identities, fusion rules and structure constants of CFTs*, Modern Phys. Lett. A **9** (1994), no. 2, 133–141. https://doi.org/10.1142/S0217732394000149

[25] J. G. F. Wan, *Random walks, elliptic integrals and related constants*, PhD Thesis, University of Newcastle, 2013.

[26] J. G. F. Wan, *Series for* $1/\pi$ *using Legendre's relation*, Integral Transforms Spec. Funct. **25** (2014), no. 1, 1–14. https://doi.org/10.1080/10652469.2013.809072

[27] J. Wan and W. Zudilin, *Generating functions of Legendre polynomials: a tribute to Fred Brafman*, J. Approx. Theory **164** (2012), no. 4, 488–503. https://doi.org/10.1016/j.jat.2011.12.001

[28] X. Wang and W. Chu, *Further Ramanujan-like series containing harmonic numbers and squared binomial coefficients*, Ramanujan J. **52** (2020), no. 3, 641–668. https://doi.org/10.1007/s11139-019-00140-5

[29] G. N. Watson, *The constants of Landau and Lebesgue*, Quart. J. Math. Oxford Ser. 1 **2** (1930), no. 3, 310–318.

[30] D. Zagier, *The dilogarithm function*, in Frontiers in number theory, physics, and geometry. II, 3–65, Springer, Berlin, 2007. https://doi.org/10.1007/978-3-540-30308-4_1

John Maxwell Campbell
Department of Mathematics and Statistics
York University
Toronto, Canada
Email address: jmaxwellcampbell@gmail.com

Wenchang Chu
Department of Mathematics and Physics
University of Salento
Lecce, Italy
Email address: chu.wenchang@unisalento.it