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CO-UNIFORM AND HOLLOW S-ACTS OVER MONOIDS

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Abstract. In this paper, we first introduce the notions of superfluous and coessential subacts. Then hollow and co-uniform S-acts are defined as the acts that all proper subacts are superfluous and coessential, respectively. Also it is indicated that the class of hollow S-acts is properly between two classes of indecomposable and locally cyclic S-acts. Moreover, using the notion of radical of an S-act as the intersection of all maximal subacts, the relations between hollow and local S-acts are investigated. Ultimately, the notion of a supplement of a subact is defined to characterize the union of hollow S-acts.

1. Introduction

A submodule K of an R-module M is called superfluous (small), if the equality $N + K = M$ implies that $N = M$. The notion of small submodule plays a fundamental role in the category of modules over rings. According to [2], a non-zero module M is defined to be hollow if every submodule of M is small (superfluous). The classical notion of hollow modules has been studied extensively for a long time in many papers (see for example [3, 10]). In the category of S-acts the notions of small (coessential) and superfluous subacts are distinct which we define both as follows. For S-acts, first we refer the reader to [7] and for preliminaries and basic results related monoids and S-acts. A subact B_S of A_S is called *large* in A_S if any homomorphism $g : A_S \longrightarrow C_S$ such that $g|_B$ is a monomorphism is itself a monomorphism. An extension B of A with the embedding $f : A_S \longrightarrow B_S$ is called an *essential extension* of A if $\mathrm{Im} f$ is large in B.

The categorical dual of essential extension is called a *coessential* epimorphism which we recall as follows. Let S be a monoid. An act B_S is called a *cover* of an act A_S if there exists an epimorphism $f : B_S \to A_S$ such that for any proper subact C_S of B_S the restriction $f|_{C_S}$ is not an epimorphism. An epimorphism with this property is called a *coessential epimorphism*. Indeed it is defined in order to investigate $\mathcal{X}$-perfect monoids as monoids over which every right S-act has an $\mathcal{X}$-cover, where $\mathcal{X}$ is an act property which is preserved

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under coproduct. More information about various kinds of cover of acts one can see [4–6, 8].

As a dual of large subact, we call B_S a coessential (small) subact of A_S if A_S is a cover of the Rees factor act A_S/B_S. According to the notion of superfluous submodule, a subact B_S of an S-act A_S shall be called superfluous if the union of B_S with every proper subact of A_S is also a proper subact of A_S. In Section 2, We consider the properties of coessential and superfluous subacts. In [9], the authors investigated uniform acts over a semigroup S, as S-acts that all their non-zero subacts are large. In module theory, the dual notion of a uniform module is that of a hollow module. In fact hollow and co-uniform modules are equal. For S-acts, as we mentioned earlier, the notion of coessential and superfluous are distinct, so we define co-uniform as a dual of uniform S-acts and hollow S-acts with respect to the definition of hollow in module theory. In Section 3, we characterize the classes of co-uniform and hollow acts as the acts all proper subacts are coessential and superfluous respectively. In Section 4, we investigate radical of an S-acts and local S-acts, and consider the relationship between local and hollow S-acts. Finally, in Section 5, a supplement of a subact and supplemented S-acts are introduced and using these notions to characterize the union of hollow S-acts. The following lemma is clearly proved which is needed in the sequel.

Lemma 1.1. *If M is a maximal subact of a right S-act A_S, then A/M is finitely generated.*

2. Coessential or superfluous subacts

In this section we introduce the notions of coessential and superfluous subacts, and consider general properties of them.

Definition. A subact B_S of an S-act A_S is called

(i) *coessential* if the epimorphism $\pi : A_S \longrightarrow A_S/B_S$ is a coessential epimorphism; in other words, A_S is a cover of A_S/B_S. It is denoted by $B \ll A$.

(ii) *superfluous* if $B_S \cup C_S \neq A_S$ for each proper subact C_S of A_S, and it is denoted by $B \leq_s A$.

In the following lemma we present an equivalent condition for being coessential.

Lemma 2.1. *A subact B_S of an S-act A_S is coessential if and only if for each proper subact C_S of A_S, $C \cap B \neq \emptyset$ implies that $C \cup B \neq A$.*

Proof. Necessity. Let C_S be a proper subact of A_S and $C \cap B \neq \emptyset$. Since $\pi : A_S \longrightarrow A_S/B_S$ is a coessential epimorphism, $\pi|_{C_S}$ is not an epimorphism, which implies the existence of $a \in A_S$ such that $[a] \notin \pi(C)$. Now we claim that $a \notin C \cup B$. Otherwise, either $a \in C$ which means $[a] \in \pi(C)$ or $a \in B$ which implies $[a] = [b] \in \pi(C)$ for some $b \in C \cap B$. Thus $C \cup B \neq A$.

Sufficiency. Let C_S be a proper subact of A_S. We show that for the epimorphism $\pi : A_S \longrightarrow A_S/B_S$, $\pi|_{C_S}$ is not an epimorphism. If $C \cap B = \emptyset$, clearly for each $b \in B$ we have $[b] \notin \pi(C)$. Otherwise, if $C \cap B \neq \emptyset$, by assumption $C \cup B \neq A$. So we have $[a] \notin \pi(C)$ for each $a \in A \setminus (C \cup B)$. Therefore, $\pi|_{C_S}$ is not an epimorphism. □

In view of the previous lemma, it is obvious that being a superfluous subact implies coessential. But the converse is not valid. For instance, let S be an arbitrary monoid and $A_S = \Theta \coprod \Theta = \{\theta_1, \theta_2\}$. Then $\{\theta_1\}$ is coessential but not superfluous.

Lemma 2.2. *A coessential subact of each indecomposable right S-act is superfluous.*

Proof. Suppose that B is a coessential subact of an indecomposable right S-act A_S and $B \cup C = A$ for a subact C of A. If $B \cap C = \emptyset$, then $A = B \coprod C$ which contradicts with being indecomposable. So $B \cap C \neq \emptyset$ and $B \cup C = A$ which imply that $C = A$. Therefore, B is superfluous. □

Lemma 2.3. *Suppose that A_S, B_S, C_S, D_S are S-acts such that $D_S \subseteq C_S \subseteq B_S \subseteq A_S$. The following hold.*

(i) *$B \leq_s A$ if and only if $C \leq_s A$ and $B/C \leq_s A/C$.*
(ii) *If $C \leq_s B$, then $C \leq_s A$.*
(iii) *$B \leq_s A$ if and only if for each S-act X_S and $h : X \longrightarrow A$, $\mathrm{Im}(h) \cup B = A$ implies $\mathrm{Im}(h) = A$.*
(iv) *$B/D \leq_s A/D$ if and only if $B/C \leq_s A/C$ and $C/D \leq_s A/D$.*

Proof. (i) Necessity. The first part is obvious. Let K be a subact of A/C with $B/C \cup K = A/C$. So $D = \{t \in A \mid [t] \in B/C\}$ is a subact of A_S and it is easily checked that $D \cup B = A$. By assumption, $D = A$, and thus $K = A/C$.

Sufficiency. Let D be a subact of A and $D \cup B = A$. So $B/C \cup (D \cup C)/C = A/C$ which implies $(D \cup C)/C = A/C$. Then $D \cup C = A$ implies that $D = A$, as desired.

Parts (ii) and (iii) are clear.

(iv) We only show the sufficiency. Suppose that $(B/D) \cup K = A/D$ for some subact K of A/D. Get $X = \{t \in A \mid [t] \in K\}$ which is clearly a subact of A_S. Then $(B/C) \cup ((X \cup C)/C) = A/C$. Since $B/C \leq_s A/C$, we have $X \cup C = A$. So $(C/D) \cup K = A/D$ and since $C/D \leq_s A/D$, $K = A/D$. Therefore $B/D \leq_s A/D$. □

Similar to the proof of the previous lemma, two following lemmas are easily checked.

Lemma 2.4. *The following hold for a monoid S.*

(i) *If $C_S \subseteq B_S \subseteq A_S$ and $C \ll B$, then $C \ll A$.*
(ii) *If $C_S \subseteq B_S \subseteq A_S$ and $B \ll A$, then $C \ll A$ and $B/C \ll A/C$.*

(ii) *If* $B \ll A$ $(B \leq_s A)$ *and* $f : A \longrightarrow C$ *is a monomorphism, then* $f(B) \ll C$ $(f(B) \leq_s C)$.

Lemma 2.5. *Let* B, C *be proper subacts of* A_S. *Then* $B \cup C \leq_s A$ *if and only if* $B \leq_s A$ *and* $C \leq_s A$.

Lemma 2.6. *Suppose that* B_i *is a proper subact of* A_i *for each* $i \in I$. *The following hold for a monoid* S.

(i) $\coprod_{i\in I} B_i \leq_s \coprod_{i\in I} A_i$ *if and only if* $B_i \leq_s A_i$ *for each* $i \in I$.
(ii) *If* $\coprod_{i\in I} B_i \ll \coprod_{i\in I} A_i$, *then* $B_i \ll A_i$ *for each* $i \in I$.
(iii) *If* $B_i \leq_s A_i$ $(B_i \ll A_i)$ *for each* $i \in \{1, \ldots, n\}$, *then* $\cup_{i=1}^{i=n} B_i \leq_s \cup_{i=1}^{i=n} A_i$ $(\cup_{i=1}^{i=n} B_i \ll \cup_{i=1}^{i=n} A_i)$.

Proof. (i) Necessity. Suppose that $\coprod_{i\in I} B_i \leq_s \coprod_{i\in I} A_i$. Fix $j \in I$ and D_j a subact of A_j such that $B_j \cup D_j = A_j$. Then $D = (\coprod_{i\neq j} A_i) \coprod D_j$ is a subact of $\coprod_{i\in I} A_i$ and $\coprod_{i\in I} B_i \cup D = \coprod_{i\in I} A_i$. By assumption, $D = \coprod_{i\in I} A_i$ which implies that $D_j = A_j$.

Sufficiency. Suppose that $B_i \leq_s A_i$ for each $i \in I$. Let D be a subact of $\coprod_{i\in I} A_i$ such that $\coprod_{i\in I} B_i \cup D = \coprod_{i\in I} A_i$. Since B_i is a proper subact of A_i for each $i \in I$, $D = \coprod_{i\in I} D_i$ such that $D_i \neq \emptyset$ is a subact of A_i. Obviously, $B_i \cup D_i = A_i$ for every $i \in I$ and by assumption $D_i = A_i$ which gives that $D = \coprod_{i\in I} A_i$.

By a similar argument one can prove part (ii). Part (iii) is a straightforward consequence of Lemmas 2.3 and 2.5. □

3. Co-uniform and hollow S-acts

In this section we study the classes of co-uniform and hollow S-acts.

Definition. An S-act A_S is called *co-uniform* if all proper subacts of A_S are coessential, and A_S is said to be *hollow* if every its proper subact is superfluous.

Obviously, hollow implies co-uniform, but the converse is not valid. Let S be an arbitrary monoid. It is easily checked that, $\Theta \coprod \Theta$ is co-uniform but not hollow.

Proposition 3.1. *Every factor act of a* (*co-uniform*) *hollow act is also* (*co-uniform*) *hollow.*

Proof. Let A be a hollow S-act and $f : A \longrightarrow C$ an epimorphism. Let D be a proper subact of C. We show that $D \leq_s C$. Clearly, $B = f^{-1}(D)$ is also a proper subact of A. So $B \leq_s A$. Now, suppose that $D \cup E = C$. It is easily checked that $B \cup f^{-1}(E) = A$. So by assumption, $f^{-1}(E) = A$, and thus $E = C$. By a similar argument one could prove for co-uniform acts. □

Recall that an S-act A_S is called locally cyclic if for all $a, a' \in A_S$ there exists $a'' \in A$ such that $a, a' \in a''S$. Every locally cyclic S-act is indecomposable and every cyclic S-acts is locally cyclic.

Proposition 3.2. *Every locally cyclic right S-act is hollow, and consequently, every cyclic right S-act is hollow.*

Proof. Let A_S be a locally cyclic S-act. If A_S is simple, i.e., contains no proper subacts, the result follows. Otherwise, let B be a proper subact of A_S. If $C \cup B = A$ for some proper subact C of A, take $a \in A \setminus B$ and $a' \in A \setminus C$. So there exists $a'' \in A$ with $a, a' \in a''S$. Since $A = B \cup C$, we have $a'' \in B$ or $a'' \in C$ which implies that $a \in B$ or $a' \in C$, a contradiction. Thus $C = A$, and B is a superfluous subact of A_S. □

Theorem 3.3. *A right S-act A_S is hollow if and only if A_S is an indecomposable co-uniform right S-act.*

Proof. Necessity. Suppose that A_S is hollow, and B, C are proper subacts of A such that $A = B \coprod C$. Thus $A = B \cup C$ which means that B is not superfluous subact of A, a contradiction.

In view of Lemma 2.2, the following the sufficiency is deduced. □

In general being indecomposable does not imply being hollow. For instance, let A_S be a cyclic S-act with a proper subact B, then $A \coprod^B A$ is indecomposable but not hollow. In particular, for a proper right ideal I of a monoid S, $S \coprod^I S$ is indecomposable but not hollow. So we have the following strict implications,

$$\text{cyclic} \Longrightarrow \text{locally cyclic} \Longrightarrow \text{hollow} \Longrightarrow \text{indecomposable}.$$

In the following proposition we characterize co-uniform S-acts.

Proposition 3.4. *Every co-uniform S-act A is indecomposable or $A = A_1 \coprod A_2$, where each A_i is simple.*

Proof. Suppose that A_S is a co-uniform decomposable S-act. Let $A = \coprod_{i \in I} A_i$. If $|I| > 2$, fix $k \neq j \in I$ and put $B = A_k \coprod A_j$. So $B \cup (\coprod_{i \neq j} A_i) = A$ and $B \cap (\coprod_{i \neq j} A_i) = A_k \neq \emptyset$. Then B is not coessential which is a contradiction. Thus $|I| = 2$. Now, suppose that $A = A_1 \coprod A_2$ such that A_1 is not simple. Let B_1 be a proper subact of A_1. Then $B = B_1 \coprod A_2$ is a proper subact of A such that $B \cap A_1 \neq \emptyset$ and $B \cup A_1 = A$ which means that B is not coessential, a contradiction. Then $A = A_1 \coprod A_2$ which A_1, A_2 are simple, as desired. □

Let S be an arbitrary monoid and $A = \Theta \coprod \Theta \coprod \Theta$. Using Proposition 3.4, A is not co-uniform. So for each arbitrary monoid S there exists a finitely generated S-act which is not hollow or co-uniform.

An S-act A is said to be a *uniserial* S-act if every two subacts of A are comparable with respect to inclusion. In the next theorem we characterize an S-act all its subacts are hollow.

Theorem 3.5. *For an S-act A_S the following statements are equivalent.*

(i) *A is a uniserial S-act.*
(ii) *Every subact of A is hollow.*
(iii) *Every subact of A generated by two elements is hollow.*

Proof. The implications (i)⇒(ii) and (ii)⇒(iii) are obvious.

(iii)⇒(i) Let B and C be subacts of A and let $B \nsubseteq C$. Then there exists an element $x \in B \backslash C$. To show that $C \subseteq B$, suppose that $y \in C$. Put $N = xS \cup yS$. If $N = yS$, then $xS \subseteq N = yS \subseteq C$. So $x \in C$, a contradiction. Hence yS is a proper subact of N, and since N is hollow, then $N = xS$. Therefore, $yS \subseteq N = xS \subset B$ which implies that $y \in B$, and so $C \subseteq B$. □

Proposition 3.6. *The following hold for a monoid S.*

(i) *Every hollow S-act with a minimal generating set is cyclic.*
(iii) *Every finitely generated hollow S-act is cyclic.*

Proof. It suffices to prove part (i). Let A_S be a right S-act with a minimal generating set $\{a_i \mid i \in I\}$. In contrary suppose that $|I| > 1$, and fix $i \in I$. Then $a_iS \cup (\cup_{j \neq i} a_jS) = A$, and since A_S is hollow, $A_S = \cup_{j \neq i} a_jS$, a contradiction. □

Recall that a monoid S satisfies condition (A) if all right S-acts satisfy the ascending chain condition for cyclic subacts. In [5] it is shown that a monoid S satisfies condition (A) if and only if every locally cyclic S-act is cyclic, equivalently, every right S-act contains a minimal generating set. Now, using this fact and the previous proposition we deduce the following result as a generalization of that result in [5].

Lemma 3.7. *A monoid S satisfies condition* (A) *if and only if every hollow S-act is cyclic.*

We conclude this section considering the cover of hollow S-acts. In [5], it is shown that a cover of a locally cyclic right S-act is indecomposable. Now, we extend this to the following result.

Lemma 3.8. *Each cover of a hollow S-act is indecomposable.*

Proof. Let A_S be a hollow S-act and $f : D_S \to A_S$ a coessential epimorphism. Suppose that $D = \coprod_{i \in I} D_i$ such that each D_i is indecomposable. In contrary, suppose that $|I| > 1$ and choose $i \neq j \in I$. Since $f\mid_{D \setminus D_i}$ is not an epimorphism, $f(D \setminus D_i)$ is a proper subact of A and $f(D \setminus D_i) \cup f(D \setminus D_j) = A$. Now since A_S is hollow, $f(D \setminus D_j) = A$, and so $f\mid_{D \setminus D_j}$ is an epimorphism, a contradiction. Therefore D is indecomposable. □

The following corollary is a straightforward result of the previous lemma.

Corollary 3.9. *For a monoid S the following hold.*

(i) *Every projective cover of a hollow S-act is cyclic.*
(i) *Every strongly flat* (*condition* (P)) *cover of a hollow S-act is locally cyclic.*

4. The relation between hollow and radical of S-acts

In this section we consider local S-acts and the radical of an S-act. We also discuss the relationship between local and hollow S-acts.

Definition. A right S-act is called *local* if it contains exactly one maximal subact. A monoid S is also called *right* (*left*) *local* if it contains exactly one maximal right (left) ideal.

The set of maximal subacts of a right S-act A_S is denoted by $\mathrm{Max}(A)$.

Lemma 4.1. *Every cyclic right S-act is simple or local.*

Proof. Suppose that $A = aS$ is cyclic, and A_S is not simple. By using Zorn's Lemma, $\mathrm{Max}(A) \neq \emptyset$. Now, suppose that $M \neq N$ are maximal subacts of A. Then $M \cup N = A$ implies that $a \in M$ or $a \in N$, and so $N = A$ or $M = A$, a contradiction. Thus A is local. □

Now, we deduce the following remark which was also discussed in [1].

Remark 4.2. Every monoid S is a group or right local. Indeed the set

$$\{s \in S \mid s \text{ is not right invertible}\}$$

is either empty or the unique maximal right ideal of S. Then the local monoid property is left-right symmetric. Thus we briefly call it a local monoid.

The following theorem establishes a relation to hollow S-acts with local and cyclic S-acts.

Theorem 4.3. *Let A_S be a right S-act. Then the following are equivalent*:

(i) *A_S is a hollow right S-act and* $\mathrm{Max}(A) \neq \emptyset$;
(ii) *A_S is a cyclic and local right S-act*;
(iii) *A_S is a finitely generated local right S-act*;
(iv) *Every proper subact of A_S is contained in a maximal subact, and A_S is a local right S-act*;
(v) *A_S contains a maximal subact N such that $N \leq_s A$*;
(vi) *A_S contains the unique maximum subact N such that $N \leq_s A$.*

Proof. (i)⇒(ii) Let N be a maximal subact of A_S and let L be an arbitrary subact of A_S where $L \subsetneq N$. Since $N \cup L = A$, and A_S is a hollow right S-act, then $A = L$. Hence A_S has just one maximal subact. If $a \in A \setminus N$ and $L = aS$, then $A = aS$.

The implications (ii)⇒(iii) and (iii)⇒(iv) are obvious.

(iv)⇒(v) Let N be the unique maximal subact of A and let L be a proper subact of A. By assumption, $L \subseteq N$. Then $L \cup N = N \neq A$ and so $N \leq_s A$.

(v)⇒(vi) Let N be a maximal subact of A which $N \leq_s A$ and let B be a proper subact of A. So $N \cup B \neq A$ and by maximality of N we have $B \subseteq N$. So N is maximum.

(vi)⇒(i) Let N be the maximum subact of A which $N \leq_s A$. For each proper subact B of A we have $B \subseteq N \leq_s A$, we deduce that $B \leq_s A$. Therefore A_S is hollow. □

In general, every hollow (indecomposable co-uniform) S-act is not cyclic or local. For instance, take $S = (\mathbb{N}, \min) \cup \{\varepsilon\}$ where ε denotes the externally adjoined identity greater than each natural element. Then $A = \{1, 2, 3, \ldots\}$ is not cyclic act and $\mathrm{Max}(A) = \emptyset$. But all its subacts are $\{1\} \subseteq \{1, 2\} \subseteq \{1, 2, 3\} \subseteq \cdots$, and so A is hollow.

Let S be a monoid and A a right S-act. The radical of the act A is the intersection of all maximal subacts of A,

$$\mathrm{Rad}(A) = \cap\{N \mid N \text{ is a maximal subact of A}\}.$$

If A contains no a maximal subact, we put $\mathrm{Rad}(A) = A$. If $\mathrm{Rad}(A) \neq \emptyset$, the $\mathrm{Rad}(A)$ is a subact of A.

In module theory, the radical submodule is equal to the union of superfluous submodules. The next proposition demonstrates that it is also valid for S-acts. To reach that we need the following lemma.

Lemma 4.4. *If $a \in A$ and $C \leq A$ such that $aS \cup C = A$, then $C = A$ or there exists a maximal subact M of A such that $C \subseteq M$ and $a \notin M$.*

Proof. Let $C \neq A$. Take $B = \{D \,|\, D \subsetneqq A \text{ and } C \subseteq D\}$. Clearly $C \in B \neq \emptyset$ and B is a partially ordered set. Let $\{D_i\}_{i \in I}$ be a chain in B, so $D_i \subsetneqq A$ and $C \subseteq D_i$. Let $D = \cup_{i \in I} D_i$. If $D \subsetneqq A$, then D is an upper bound. Otherwise, if $D = A$, $a \in A$ implies $a \in D$, and there exists $i \in I$ such that $a \in D_i$. Then $aS \subseteq D_i$ which implies that $aS \cup D_i = D_i = A$, a contradiction. Then by Zorn's Lemma, B has a maximal element M. So M is a maximal subact of A such that $C \subseteq M$, $a \notin M$. □

As we know, $A \leq_s A$ if and only if A is simple.

Proposition 4.5. *Let A_S be a right S-act. Then*

$$\mathrm{Rad}(A) = \cup\{B \,|\, B \leq_s A\}.$$

Proof. Suppose that $\Gamma = \cup\{B \,|\, B \leq_s A\}$. First we show that $\Gamma \subseteq \mathrm{Rad}(A)$. If $\mathrm{Max}(A) = \emptyset$, clearly $\Gamma \subseteq \mathrm{Rad}(A) = A$. Otherwise, let $B \leq_s A$ and N be an arbitrary maximal subact of A. If $B \nsubseteq N$, being maximal of N implies that $B \cup N = A$. Since $B \leq_s A$, $N = A$, a contradiction. Thus $B \subseteq N$, and so $\Gamma \subseteq \mathrm{Rad}(A)$. To show the converse, let $a \in \mathrm{Rad}(M)$. First we show that $aS \leq_s A$. If $aS = A$, then $A = \mathrm{Rad}(A)$ and by Lemma 4.1 A is simple. So $aS = A \leq_s A$. Now, let aS be a proper subact of A and $aS \cup C = A$. If $C \neq A$ by previous lemma there exists a maximal subact M of A such that $C \subseteq M$ and $a \notin M$, but $a \in \mathrm{Rad}(M)$ implies $a \in M$, a contradiction. Then $C = A$ which means that $aS \leq_s A$. We deduce $aS \subseteq \cup\{B \,|\, B \leq_s A\}$, and therefore $\mathrm{Rad}(A) \subseteq \Gamma$. □

Using the previous proposition, the following result is immediately deduced.

Corollary 4.6. *For a monoid S the following statements hold.*

(i) *Let A_S be a right S-act. Then for each element $a \in \mathrm{Rad}(A)$, $aS \leq_s A$.*
(ii) *Let A and B be right S-acts and let $f : A \longrightarrow B$ be an S-monomorphism. Then $f(\mathrm{Rad}(A)) \subseteq \mathrm{Rad}(B)$.*
(iii) $\mathrm{Rad}(A) = A$ *if and only if all finitely generated subact of A are superfluous in A.*

Corollary 4.7. *Let A_S be a right S-act. Then each non-cyclic hollow subact B of A is contained in* $\mathrm{Rad}(A)$.

Proof. Assume that B is a hollow subact of A and $b \in B$. So bS is a proper subact of B and $bS \leq_s B$, and by Lemma 2.3, $bS \leq_s A$. Using the previous proposition, $bS \subseteq \mathrm{Rad}(A)$ which implies that $B \subseteq \mathrm{Rad}(A)$. □

Now, we give an equivalent condition for an S-act which its radical is superfluous.

Theorem 4.8. *For a right S-act A the following statements are equivalent.*

(i) $\mathrm{Rad}(A) \leq_s A$.
(ii) *Every proper subact of A is contained in a maximal subact.*

Proof. (i)⇒(ii) Let C be a proper subact of A. Since $\mathrm{Rad}(A) \leq_s A$, $\mathrm{Rad}(A) \cup C \neq A$. Suppose $\{M_i \mid i \in I\}$ is the family of all maximal subacts of A. So $(\cap_{i\in I} M_i) \cup C \neq A$, which implies that $\cap_{i\in I}(M_i \cup C) \neq A$. Then there exists $j \in I$ such that $M_j \cup C \neq A$. Now, maximality of M_j implies that $C \subseteq M_j$, and the result follows.

(ii)⇒(i) Suppose that C is an arbitrary proper subact of A. There exists a maximal subact M of A with $C \subseteq M$. Then we have $C \cup \mathrm{Rad}(A) \subseteq M \cup \mathrm{Rad}(A) = M \neq A$. Thus, $\mathrm{Rad}(A) \leq_s A$. □

Proposition 4.9. *An S-act A is finitely generated if and only if $A/\mathrm{Rad}(A)$ is finitely generated and* $\mathrm{Rad}(A) \leq_s A$.

Proof. Let A be finitely generated, clearly $A/\mathrm{Rad}(A)$ is finitely generated. Let $C \leq A$, $\mathrm{Rad}(A) \cup C = A$, by Proposition 4.5, $\mathrm{Rad}(A) = \cup\{B \mid B \leq_s A\}$, so $\cup\{B \mid B \leq_s A\} \cup C = A$. Since A is finitely generated, there exist $B_1, \dots, B_m \leq_s A$ such that $B_1 \cup B_2 \cup \cdots \cup B_m \cup C = A$. Since $B_1 \leq_s A$ and $B_1 \cup (B_2 \cup \cdots \cup B_m \cup C) = A$, we imply that $B_2 \cup \cdots \cup B_m \cup C = A$. Since $B_2, \dots, B_m \leq_s A$, we continue this manner to imply $C = A$. Thus $\mathrm{Rad}(A) \leq_s A$.

Sufficiency. Suppose that $A/\mathrm{Rad}(A) = \cup_{i=1}^{i=n}[a_i]S$. So $\mathrm{Rad}(A) \cup (\cup_{i=1}^{i=n} a_i S) = A$. Now, since $\mathrm{Rad}(A) \leq_s A$, $\cup_{i=1}^{i=n} a_i S = A$. Thus A is finitely generated. □

5. Supplemented acts

In this section we introduce the notions of a supplement of a subact and supplemented S-acts, and general properties of them are discussed. Our aim is to use the notion of a supplement of a subact to investigate the union of hollow S-acts.

Definition. Let B, C be proper subacts of a right S-act A. We call C is a *supplement* of B in A, or B has a supplement C in A if the following two conditions are satisfied.

(i) $B \cup C = A$.
(ii) If $D \subseteq C$ and $B \cup D = A$, then $D = C$.

If every proper subact of A has a supplement in A, then A is called a *supplemented* S-act.

Clearly, if an S-act $A = B \coprod C$, then C is a supplement of B. We first begin with elementary properties for being supplement.

Lemma 5.1. *Let $A = B \cup C$. If $B \cap C \neq \emptyset$, then C is a supplement of B in A if and only if $C \cap B = \emptyset$ or $C \cap B \leq_s C$.*

Proof. Let E be a subact of C. Then $(C \cap B) \cup E = C$ is equivalent to $A = B \cup E$ and so the result is easily checked. □

The following result presents that co-uniform implies supplemented.

Proposition 5.2. *Every co-uniform S-act is supplemented.*

Proof. Let A be a right S-act and B be a proper subact of A. First suppose that A is indecomposable. By Theorem 3.3, A is hollow. Then $B \cup A = A$ and $(B \cap A) = B \leq_s A$ imply that A is a supplemented S-act. In the case that A is not indecomposable, by Proposition 3.4, $A = B \coprod C$ where B, C are simple acts. Thus C is a supplement of B. □

The converse of Proposition 5.2 is not valid. For instance, let S be an arbitrary monoid and $A = \Theta \coprod \Theta \coprod \Theta$. Using Proposition 3.4, A is not co-uniform. But, as all subsets of A are also subacts, for each subact B of A we have $A \setminus B$ is a supplement of B.

Let C be a proper subact of an S-act A. By Lemma 2.3, each superfluous subact of C is also superfluous in A. So clearly $\mathrm{Rad}(C) \subseteq C \cap \mathrm{Rad}(A)$.

Proposition 5.3. *Suppose that C is a proper subact of an S-act A such that C is a supplement of a proper subact B of A. Then the following hold.*

(i) *If $D \cup C = A$ for some $D \subset B$, then C is a supplement of D.*
(ii) *If A is finitely generated, then C is also finitely generated.*
(iii) *If E is a subact of C such that $E \leq_s A$, then $E \leq_s C$.*
(iv) *If $N \leq_s A$, then $N \cap C \leq_s C$.*
(v) *If $N \leq_s A$, then C is a supplement of $N \cup B$.*
(vi) $\mathrm{Rad}(C) = C \cap \mathrm{Rad}(A)$.

Proof. (i) It is easily proved by using Lemmas 5.1 and 2.3.

(ii) Let A be finitely generated. Since $B \cup C = A$, there is a finitely generated subact $X \subseteq C$ such that $B \cup X = A$. By the minimality of C, we imply that $C = X$.

(iii) Let X be a subact of C with $E \cup X = C$. Since $B \cup C = A$, we have $B \cup E \cup X = A$. Now, since $E \leq_s A$, $B \cup X = A$ and so $X = C$.

(iv) Using part (iii) and Lemma 2.3, it is clearly checked.

(v) Let $N \leq_s A$. We have $(N \cup B) \cup C = A$. Let $X \subseteq C$ with $(N \cup B) \cup X = A$. Then $N \leq_s A$ implies that $B \cup X = A$, and hence $X = A$.

(vi) We have $\text{Rad}(C) \subseteq C \cap \text{Rad}(A)$. To show the converse, if $N \leq_s A$, by part (iv), $E = N \cap C \leq_s C$, and $E \subseteq \text{Rad}(C)$. Therefore, $C \cap Rad(A) = C \cap (\cup\{N \mid N \leq_s A\}) = \cup\{N \cap C \mid N \leq_s A\} \subseteq \text{Rad}(C)$. □

Now, we turn our attention to the concept of supplement in a projective S-act.

Proposition 5.4. *Let P be a projective S-act, and C be a supplement of B in P. Then C is projective or there exists an epimorphism $f : P \longrightarrow C$ such that $f(B) \leq_s C$.*

Proof. Let C be a supplement of B in P. So $P = B \cup C$. If $B \cap C = \emptyset$, then $P = B \coprod C$, and C is projective. Now, suppose that $B \cap C \neq \emptyset$. Let $\pi_1 : C \longrightarrow C/(B \cap C)$ be the canonical epimorphism, and define $\pi_2 : P \longrightarrow C/(B \cap C)$ by $\pi_2(p) = \begin{cases} [p], & p \in C \\ \theta, & p \in B. \end{cases}$ So since P is projective, there exists a homomorphism $f : P \longrightarrow C$ with $\pi_1 f = \pi_2$. It is easily checked that $\text{Im} f \cup B = P$, and by assumption, $\text{Im} f = C$. Moreover, since $f(B) \subseteq B \cap C \leq_s C$, by Lemma 2.3, $f(B) \leq_s C$. □

Finally, we conclude this paper by considering the union of hollow acts.

Theorem 5.5. *Let A be a right S-act such that* $\text{Rad}(A) \leq_s A$. *The following statements are equivalent.*

(i) *A is a union of hollow acts.*
(ii) *Each proper subact B of A whose A/B is finitely generated has a supplement.*
(iii) *Every maximal subact of A has a supplement.*

Proof. (i)⇒(ii) Suppose $A = \cup_{i \in I} L_i$ such that each L_i is hollow S-act. Let B be a proper subact of A such that A/B is finitely generated. Then $A/B = \cup_{i \in I}(L_i \cup B)/B$. Since A/B is finitely generated, $A = B \cup L_1 \cup L_2 \cup \cdots \cup L_n$ for some hollow S-acts $L_1, L_2, \ldots, L_n$ with $B \cap L_i \neq L_i$ for each $1 \leq j \leq n$. Take $L = L_1 \cup L_2 \cup \cdots \cup L_n$. To show that L is a supplement of B, let X be a proper subact L. There exists $1 \leq j \leq n$ such that $X \cap L_j$ is a proper subact of L_j. Now, since L_j is hollow, $(B \cap L_j) \cup (X \cap L_j) \neq L_j$. Thus $B \cup X \neq A$, and the result follows.

(ii)⇒(iii) follows by Lemma 1.1. (iii)⇒(i) Let B be the union of all hollow subacts of A. In contrary, suppose that B is a proper subact of A. So there exists a maximal subact N of A with $B \subseteq N$. Let L be a supplement of N in A. If L is simple, then $L \subseteq B$. Otherwise, let X be a proper subact of L. So $N \cup X \neq A$, and maximality of N implies that X is contained in N. So by Lemma 5.1, $N \cap L \leq_s L$, and using Lemma 2.3, $X \subseteq N \cap L \subseteq L$ implies $X \leq_s L$. Then L is a hollow act. Therefore L is contained in B, and so $A = L \cup N \subseteq B \cup N = N$, a contradiction. Therefore, $B = A$. Now suppose that C is an arbitrary proper subact of A. There exists a maximal subact M of A with $C \subseteq M$. Then we have $C \cup \mathrm{Rad}(A) \subseteq M \cup \mathrm{Rad}(A) = M \neq A$. Thus, $\mathrm{Rad}(A) \leq_s A$. □

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