



ITERATIVE METHOD FOR SOLVING FINITE FAMILIES OF VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS OF CERTAIN MULTI-VALUED MAPPINGS

Musa Adewale Olona¹ and Ojen Kumar Narain²

¹School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal, Durban, South Africa
e-mail: 219095783@stu.ukzn.ac.za

²School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal, Durban, South Africa
e-mail: naraino@ukzn.ac.za

Abstract. In this paper, we propose a viscosity iterative algorithm for approximating a common solution of finite family of variational inequality problem and fixed point problem for finite family of multi-valued type-one demicontractive mappings in real Hilbert spaces. A strong convergence result of the aforementioned problems were proved and some consequences of our result was also displayed. In addition, we discuss an application of our main result to convex minimization problem. The result presented in this article complements and extends many recent results in literature.

1. INTRODUCTION

Let C be a nonempty, closed and convex subset of a real Hilbert space H . Let $CB(C)$ and $K(C)$ denote the family of nonempty, closed and bounded subset and nonempty compact subset of C , respectively. The Hausdorff metric on $CB(C)$ is defined by

$$\mathcal{H}(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\} \text{ for } A, B \in CB(C),$$

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⁰Corresponding author: M. A .Olona(219095783@stu.ukzn.ac.za) .

where $d(x, C) = \inf\{\|x - y\| : y \in C\}$.

Let $T : C \rightarrow CB(C)$ be a multi-valued mapping. Then $P_Tx = \{u \in Tx : \|x - u\| = d(x, Tx)\}$. A point $x \in C$ is called a fixed point of T if $x \in Tx$. However, if $Tx = \{x\}$, then x is called a strict point of T . We denote the set of fixed points of T by $F(T)$. A multi-valued mapping T is said to be *L-Lipschitzian* if there exists $L > 0$ such that

$$\mathcal{H}(Tx, Ty) \leq L\|x - y\|, \quad x, y \in C. \quad (1.1)$$

In (1.1), if $L \in (0, 1)$, then T is called a contraction while T is called nonexpansive if $L = 1$.

A mapping $T : C \rightarrow CB(C)$ is said to be

(i) *of type-one*, if

$$\|u - v\| \leq \mathcal{H}(Tx, Ty), \quad \forall x, y \in C, \quad u \in P_Tx, \quad v \in P_Ty,$$

(ii) *λ -hybrid* (see [45]), if there exists $\lambda \in \mathbb{R}$ such that

$$(1 + \lambda)\mathcal{H}(Tx, Ty)^2 \leq (1 - \lambda)\|x - y\|^2 + \lambda d(y, Tx)^2 + \lambda d(x, Ty)^2, \quad \forall x, y \in C,$$

(iii) *quasi-nonexpansive*, if $F(T) \neq \emptyset$ and

$$\mathcal{H}(Tx, Ty) \leq \|x - y\|, \quad \forall x \in C, \quad y \in F(T),$$

(iv) *demictractive-type* in the sense of [24] if $F(T) \neq \emptyset$ and

$$\mathcal{H}^2(Tx, Ty) \leq \|x - y\|^2 + kd^2(x, Tx), \quad x \in C, \quad y \in F(T) \text{ and } k \in (0, 1).$$

Remark 1.1. Clearly, every multi-valued quasi-nonexpansive mapping is a multi-valued demictractive-type mapping. However, the following example shows that the converse of this statement is not always true.

Let C be a nonempty, closed and convex subset of a real Hilbert space H , the variational inequality problem (VIP) is to find $x \in C$ such that

$$\langle A(x), y - x \rangle \geq 0, \quad \forall y \in C, \quad (1.2)$$

where $A : C \rightarrow H$ is a nonlinear mapping. We denote by $VI(C, A)$ the solution set of (1.2).

Variational inequality theory introduced by Stampacchia and Fichera [19, 43] independently, in early sixties in mechanics and potential theory respectively provides the natural, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems arising in elasticity, economics, transportation, optimization, control theory and engineering sciences (see [1, 5, 2, 3, 4, 10, 11, 20, 22, 28, 33, 34, 36, 37, 39]).

The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. The first aspect reveals the fundamental facts on the qualitative behavior of solutions to important classes of problems. On the other hand, it allows us to develop highly efficient and powerful numerical methods to solve, for instance, obstacle, unilateral, free and moving boundary value problems.

In 1985, Pang [38] showed that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformly modelled as a VIP.

In 1976, Korpelevich [29] proposed the following extragradient method for solving VIP (1.2), when A is monotone and Lipschitz continuous in the finite-dimensional Euclidean space \mathbb{R}^N .

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrarily,} \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases}$$

for each $n \in \mathbb{N}$ under some suitable conditions, the sequence $\{x_n\}$ and $\{y_n\}$ converge to the some point $z \in VI(C, A)$.

In 2012, Censor et al. [14] introduced the general common solutions to variational inequality problem (CSVIP), which consists of finding common solutions to unrelated variational inequalities for finite number of sets. That is, find $x^* \in \cap_{i=1}^N C_i$ such that for each $i = 1, 2, \dots, N$,

$$\langle A_i(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in C_i, i = 1, 2, \dots, N, \tag{1.3}$$

where $A_i : H \rightarrow H$ is a nonlinear operator for each $i = 1, 2, \dots, N$ and C_i is a nonempty, closed and convex subset of H . They proved a weak convergence theorem for approximating a solution of (1.3) using the following algorithm

$$\begin{cases} x_0 \in H, \\ x_{k+1} = \Pi_{i=1}^N (P_{C_i}(I - \lambda A_i))(x_k). \end{cases}$$

For more information on research output on variational inequality problem, (see [1, 25, 46, 47] and the references contained in).

In 2017, Ming Tian and Bing-Nan Jiang [47] proposed an iterative method for finding an element to solve a class of split variational inequality problems under weaker conditions and get a weak convergence theorem, Let H_1 be real Hilbert space. Let C be a nonempty closed convex subset of H_1 . Let $A : H_1 \rightarrow H_2$ be a bounded linear operator such that $A \neq 0, f : C \rightarrow H_1$ be a monotone and k - Lipschitz continuous mapping and $T : H_2 \rightarrow H_2$ be a nonexpansive mapping. Setting $\Gamma := \{z \in VI(C, f) : Az \in Fix(T)\}$, assume

that $\Gamma \neq \emptyset$, Let the sequences $\{x_n\}, \{y_n\}$ and t_n be generated by $x_1 = x \in C$ and

$$\begin{cases} y_n = P_C(x_n - \gamma_n A^*(I - T)Ax_n), \\ t_n = P_C(y_n - \lambda_n f(y_n)), \\ x(n+1) = P_C(y_n - \lambda_n f(t_n)). \end{cases}$$

Recently, Chinedu Izuchukwu [25] introduced a new iterative algorithm for approximating a common solution of certain class of multiplesets split variational inequality problems. Let H_1 and H_2 be real Hilbert spaces, and for each $i = 1, 2, \dots, N$, let C_i be a nonempty closed and convex subset of H_1 . Let $A : H_1 \rightarrow H_2$ be a bounded linear operator such that $A \neq 0$. Let $f_i : H_1 \rightarrow H_1$ be an α_i -inverse strongly monotone mapping and $S : H_2 \rightarrow H_2$ be k -strictly pseudo-contractive mapping. Assume that

$$\Gamma = \{z \in \bigcap_{i=1}^N VI(C_i, f_i) : Az \in F(S) \neq \emptyset\}$$

and the sequence $\{x_n\}$ is generated for arbitrary $x_1, u \in H_1$ by

$$\begin{cases} u_n = (1 - \beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - T_\beta)Au_n), \\ x_{n+1} = P_{C_N}(I - \lambda f_N) \circ \dots \circ P_{C_1}(I - \lambda f_1)y_n, \quad n \geq 1. \end{cases}$$

Motivated by the aforementioned results, we introduced a viscosity iterative method for approximating a common solution of finite families of variational inequality problem and fixed point problem for finite family of multi-valued demicontractive-type mappings in real Hilbert spaces. We prove a strong convergence result to a common solution of the aforementioned problems and state some consequences of our main results. We also give an application of our main result. The result present in this paper extends and complements many related results in literature.

Our contributions are as follows:

- (i) We were able to dispense for each $p \in \Gamma$, $S_i(p) = \{p\}$ for all $i \in \mathbb{N}$, see [3]. The type-one condition employed in this paper is weaker than the condition employed in [3].
- (ii) By taking $g = u$ for some $u \in H$, the algorithm (3.1) becomes the Halpern-type algorithm.
- (iii) We prove a strong convergence result without imposing a compactness condition, see [3]. The strong convergence result proved in this article is more desirable than the weak convergence result proved in [14, 47].

2. PRELIMINARIES

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "→" and "→" respectively.

Let C be a nonempty, closed and convex subset of a real Hilbert space H . A mapping $M : C \rightarrow C$ is said to be

(i) monotone, if

$$\langle Mx - My, x - y \rangle \geq 0, \forall x, y \in C,$$

(ii) α -inverse strongly monotone (ism), if there exists a constant $\alpha > 0$ such that

$$\langle Mx - My, x - y \rangle \geq \alpha \|Mx - My\|^2, \forall x, y \in C,$$

(iii) firmly nonexpansive, if

$$\langle Mx - My, x - y \rangle \geq \|Mx - My\|^2, \forall x, y \in C,$$

(iv) Lipschitz, if there exists a constant $L > 0$ such that

$$\|Mx - My\| \leq L \|x - y\|, \forall x, y \in C.$$

Remark 2.1. It is generally known that every α -ism mapping is $\frac{1}{\alpha}$ Lipschitz continuous (see [9]).

If M is a multi-valued mapping, that is, $M : H \rightarrow 2^H$, then M is called monotone, if

$$\langle x - y, u - v \rangle \geq 0, \forall x, y \in H, u \in M(x), v \in M(y)$$

and M is maximal monotone, if the graph $G(M)$ of M defined by

$$G(M) := \{(x, y) \in H \times H : y \in M(x)\}$$

is not properly contained in the graph of any other monotone mapping. It is generally known that M is maximal if and only if for $(x, u) \in H \times H, \langle x - y, u - v \rangle \geq 0$ for all $(y, v) \in G(M)$ implies $u \in M(x)$. A mapping $T : C \rightarrow C$ is said to be averaged nonexpansive if for all $x, y \in C, T = (1 - \beta)I + \beta S$ holds for a nonexpansive operator $S : C \rightarrow C$ and $\beta \in (0, 1)$. The term "averaged mapping" was first developed by Baillon *et al.* [8]. Recall that a mapping T is firmly nonexpansive if and only if T can be expressed as $T = \frac{1}{2}(I + S)$, where S is nonexpansive (see [35]). Thus, we make the following remark which can be easily verified.

Remark 2.2. In a real Hilbert space, T is firmly nonexpansive if and only if it is averaged with $\beta = \frac{1}{2}$.

The metric projection P_C is a map defined on H onto C which assign to each $x \in H$, the unique point in C , denoted by $P_C x$ such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

It is well known that $P_C x$ is characterized by the inequality $\langle x - P_C x, z - P_C x \rangle \leq 0$, for all $z \in C$ and P_C is a firmly non-expansive mapping. We also know that if f is β -inverse strongly monotone mapping with $\lambda \in (0, 2\beta)$, then $P_C(I - \lambda f)$ is averaged nonexpansive (see [14], Lemma 2.9). Hence, from Remark 2.2 we obtain the following.

Remark 2.3. In a real Hilbert space, if f is β -inverse strongly monotone with $\lambda \in (0, 2\beta)$, then $P_C(I - \lambda f)$ is firmly nonexpansive.

For more information on metric projections, (see [14, 21]) and the references therein. Recall that the normal cone of C at the point $z \in H$ is define as

$$N_C z = \begin{cases} \{d \in H : \langle d, y - z \rangle \leq 0 \forall y \in C\}, & z \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Definition 2.4. Let H be a real Hilbert space and $T : H \rightarrow CB(H)$ a multi-valued mapping. Then, T is said to be demiclosed at the origin if for any sequence $\{x_n\} \subset H$ with $x_n \rightarrow x^*$, and $d(x_n, T(x_n)) \rightarrow 0$, we have $x^* \in T x^*$.

Lemma 2.5. ([15]) *Let H be a real Hilbert space. Then for all $x, y \in H$ and $\alpha \in (0, 1)$, we have*

- (i) $2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2$,
- (ii) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$,
- (iii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$.

Lemma 2.6. ([18]) *Let H be a real Hilbert space and $x_i \in H$, ($1 \leq i \leq m$) and $\{\alpha_i\}_{i=1}^m \subset (0, 1)$ such that $\sum_{i=1}^m \alpha_i = 1$, the following identity holds:*

$$\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 = \sum_{i=1}^m \alpha_i \|x\|^2 - \sum_{i,j=1, i \neq j}^m \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.7. ([42]) *Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, \quad n \geq 1.$$

If $\limsup_{k \rightarrow \infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k \rightarrow \infty} \{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then $\lim_{n \rightarrow \infty} a_n = 0$.

3. MAIN RESULTS

In this section, we propose a viscosity iterative algorithm for approximating a common solution of finite family of variational inequality problem and fixed point problem for finite family of multi-valued type-one demicontractive mappings in real Hilbert spaces. A strong convergence result of the aforementioned problems will be proved and some consequences of our result are also displayed.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $\{S_i\}_{i=1}^m : H \rightarrow CB(H)$ be a finite family of multi-valued type-one demicontractive type mappings with constant $k_i \in (0, 1)$ such that $I - S_i$ is demiclosed at zero. Let $f_j : H \rightarrow H$ for $j = 1, 2, \dots, N$ be an α_j -inverse strongly monotone mapping and $g : H \rightarrow H$ be a contractive mapping with constant $\theta \in (0, 1)$. Suppose that*

$$\Omega := \left\{ \bigcap_{i=1}^m F(S_i) \bigcap \bigcap_{j=1}^N VI(C, f_j) \right\} \neq \emptyset.$$

For $x_1 \in H$, let the sequence $\{x_n\}$ be defined by

$$\begin{cases} w_n = \gamma_n g(x_n) + (1 - \gamma_n)x_n, \\ u_n = \beta_{n,0}w_n + \sum_{i=1}^m \beta_{n,i}z_n^i, \quad n \geq 1, \\ x_{n+1} = P_C(I - \lambda f_N) \circ P_C(I - \lambda f_{N-1}) \circ \dots \circ P_C(I - \lambda f_1)u_n, \end{cases} \quad (3.1)$$

for $n \in \mathbb{N}$, where $z_n^i \in P_{S_i w_n}$, $P_{S_i w_n} = \{z_n^i \in S_i w_n : \|z_n^i - w_n\| = d(w_n, S_i w_n)\}$ and $\lambda \in (0, 2\alpha)$, $\alpha = \min\{\alpha_j, j = 1, 2, \dots, N\}$, and the sequences $\{\beta_{n,i}\}_{n=1}^\infty$ for all $i \geq 0$ and $\{\gamma_n\}_{n=1}^\infty$ satisfy the following conditions:

- (i) $\{\beta_{n,0}\} \in (k, 1)$, $\{\beta_{n,i}\} \in (0, 1)$ such that $\sum_{i=0}^m \beta_{n,i} = 1$, $k < a \leq \beta_{n,i} \leq b < 1$, $i = 1, 2, \dots, m$, $k := \sup_{i \geq 1} \{k_i\} < 1$;
- (ii) $\gamma_n \in (0, 1)$, $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\sum_{n=1}^\infty \gamma_n = \infty$.

Then $\{x_n\}$ converges strongly to an element in Ω .

Proof. Let $z \in \Omega$, $\Phi^N = P_C(I - \lambda f_N) \circ P_C(I - \lambda f_{N-1}) \circ \dots \circ P_C(I - \lambda f_1)$, where $\Phi^0 = I$. Then from (3.1), Lemma 2.5 and Lemma 2.6, and the fact that S_i is

of type-one demicontractive-type mapping for each i , we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|P_C(I - \lambda f_N)(\Phi^{N-1}u_n) - z\|^2 \\
&\leq \|\Phi^{N-1}u_n - z\|^2 \\
&\vdots \\
&\leq \|u_n - z\|^2 \\
&= \|\beta_{n,0}w_n + \sum_{i=1}^m \beta_{n,i}z_n^i - z\|^2 \\
&= \|\beta_{n,0}(w_n - z) + \sum_{i=1}^m \beta_{n,i}(z_n^i - z)\|^2 \\
&= \beta_{n,0}\|w_n - z\|^2 + \sum_{i=1}^m \beta_{n,i}\|z_n^i - z\|^2 - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|w_n - z_n^i\|^2 \\
&\leq \beta_{n,0}\|w_n - z\|^2 + \sum_{i=1}^m \beta_{n,i}\mathcal{H}^2(S_i w_n, S_i z) \\
&\quad - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|w_n - z_n^i\|^2 \\
&\leq \beta_{n,0}\|w_n - z\|^2 + \sum_{i=1}^m \beta_{n,i}[\|w_n - z\|^2 + k\|w_n - z_n^i\|^2] \\
&\quad - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}\|w_n - z_n^i\|^2 \\
&= \|w_n - z\|^2 + (k - \beta_{n,0}) \sum_{i=1}^m \beta_{n,i}\|w_n - z_n^i\|^2 \\
&\leq \|w_n - z\|^2. \tag{3.2}
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - z\| &\leq \|w_n - z\| \\
&= \|\gamma_n(g(x_n) - z) + (1 - \gamma_n)(x_n - z)\| \\
&\leq \gamma_n\|g(x_n) - z\| + (1 - \gamma_n)\|x_n - z\| \\
&\leq \gamma_n(\|g(x_n) - z\| + \|g(z) - z\|) + (1 - \gamma_n)\|x_n - z\| \\
&\leq \gamma_n(\theta\|x_n - z\| + \|g(z) - z\|) + (1 - \gamma_n)\|x_n - z\|
\end{aligned}$$

$$\begin{aligned}
&= (1 - \gamma_n(1 - \theta))\|x_n - z\| + \gamma_n(1 - \theta)\frac{\|g(z) - z\|}{1 - \theta} \\
&\leq \max \left\{ \|x_n - z\|, \frac{\|g(z) - z\|}{1 - \theta} \right\}.
\end{aligned}$$

By continuous taking this process, we obtain that

$$\|x_n - z\| \leq \max \left\{ \|x_1 - z\|, \frac{\|g(z) - z\|}{1 - \theta} \right\},$$

for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is bounded. Consequently, $\{u_n\}$ and $\{w_n\}$ are bounded.

Now, from (3.2) and Lemma 2.5, we obtain that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \|u_n - z\|^2 \\
&= \|w_n - z\|^2 + (k - \beta_{n,0}) \sum_{i=1}^{\infty} \beta_{n,i} \|w_n - z_n^i\|^2 \\
&= \|\gamma_n g(x_n) + (1 - \gamma_n)x_n - z\|^2 + (k - \beta_{(n),0}) \sum_{i=1}^m \beta_{n,i} \|w_n - z_n^i\|^2 \\
&\leq (1 - \gamma_n)^2 \|x_n - z\|^2 + 2\gamma_n \langle x_{n+1} - z, g(x_n) - z \rangle \\
&\quad + (k - \beta_{n,0}) \sum_{i=1}^m \beta_{n,i} \|w_n - z_n^i\|^2 \\
&\leq (1 - \gamma_n) \|x_n - z\|^2 + \gamma_n (2 \langle x_{n+1} - z, g(x_n) - z \rangle). \tag{3.3}
\end{aligned}$$

On substituting $d_n = \langle x_{n+1} - z, g(x_n) - z \rangle$ in view of Lemma 2.7, we need to prove that $\limsup_{k \rightarrow \infty} d_{n_k} \leq 0$ for every $\{\|x_{n_k} - z\|\}$ of $\{\|x_n - z\|\}$ satisfying the condition

$$\lim_{k \rightarrow \infty} \{\|x_{n_{k+1}} - z\| - \|x_{n_k} - z\|\} \geq 0. \tag{3.4}$$

To show this, suppose that $\{\|x_{n_k} - z\|\}$ is a subsequence of $\{\|x_n - z\|\}$ such that (3.4) holds. Then

$$\begin{aligned}
&\liminf_{k \rightarrow \infty} (\|x_{n_{k+1}} - z\|^2 - \|x_{n_k} - z\|^2) \\
&= \lim_{k \rightarrow \infty} \left((\|x_{k+1} - z\| - \|x_{n_k} - z\|)(\|x_{n_{k+1}} - z\| + \|x_{n_k} - z\|) \right) \\
&\geq 0.
\end{aligned}$$

Now, using (3.3), we have that

$$\begin{aligned}
& \limsup_{k \rightarrow \infty} \left(\sum_{i=1}^m \beta_{n_k, i} (\beta_{n_k, 0} - k) \|w_{n_k} - z_{n_k}^i\|^2 \right) \\
& \leq \limsup_{k \rightarrow \infty} \left((1 - \gamma_{n_k}) \|x_{n_k} - z\|^2 - \|x_{n_{k+1}} - z\|^2 \right. \\
& \quad \left. + \gamma_{n_k} (2 \langle x_{n_{k+1}} - z, g(x_{n_k} - z) \rangle) \right) \\
& \leq \limsup_{k \rightarrow \infty} \left(\|x_{n_k} - z\|^2 - \|x_{n_{k+1}} - z\|^2 \right) \\
& \quad + \limsup_{k \rightarrow \infty} \left(\gamma_{n_k} (2 \langle x_{n_{k+1}} - z, g(x_{n_k} - z) \rangle) \right) \\
& = - \liminf_{k \rightarrow \infty} \left(\|x_{n_{k+1}} - z\|^2 - \|x_{n_k} - z\|^2 \right) \\
& \leq 0.
\end{aligned} \tag{3.5}$$

Using condition (i) and (ii) of (3.1), we obtain that

$$\lim_{k \rightarrow \infty} \|w_{n_k} - z_{n_k}^i\| = 0, \quad i = 1, 2, \dots, m.$$

Hence, we have that

$$\lim_{k \rightarrow \infty} d(w_{n_k}, S_i w_{n_k}) = \lim_{k \rightarrow \infty} \|w_{n_k} - z_{n_k}^i\| = 0, \quad i = 1, 2, \dots, m. \tag{3.6}$$

From (3.1), we obtain that

$$\|w_{n_k} - x_{n_k}\| = \gamma_{n_k} \|g(x_{n_k}) - x_{n_k}\| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{3.7}$$

Also, from (3.1) and (3.6), we obtain that

$$\|u_{n_k} - w_{n_k}\| \leq \sum_{i=1}^m \beta_{n_k, i} \|z_{n_k}^i - w_{n_k}\| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{3.8}$$

Using (3.7) and (3.8), we get that

$$\lim_{k \rightarrow \infty} \|u_{n_k} - x_{n_k}\| = 0. \tag{3.9}$$

Applying the firmly nonexpansivity of $P_C(I - \lambda f_N)$ (See Remark 2.3), we obtain that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|P_C(I - \lambda f_N)\Phi^{N-1}u_n - z\|^2 \\
&\leq \langle x_{n+1} - z, \Phi^{N-1}u_n - z \rangle \\
&= \frac{1}{2} \left(\|x_{n+1} - z\|^2 + \|\Phi^{N-1}u_n - z\|^2 \right. \\
&\quad \left. - \|x_{n+1} - \Phi^{N-1}u_n\|^2 \right), \tag{3.10}
\end{aligned}$$

which implies from (3.10) that

$$\begin{aligned}
&\limsup_{k \rightarrow \infty} \left(\|x_{n_{k+1}} - \Phi^{N_{k-1}}u_{n_k}\|^2 \right) \\
&\leq \limsup_{k \rightarrow \infty} \left(\|\Phi^{N_{k-1}}u_{n_k} - z\|^2 - \|x_{n_{k+1}} - z\|^2 \right) \\
&\quad \vdots \\
&\leq \limsup_{k \rightarrow \infty} \left(\|u_{n_k} - z\|^2 - \|x_{n_{k+1}} - z\|^2 \right) \\
&\leq \limsup_{k \rightarrow \infty} \left(\|u_{n_k} - x_{n_k}\|^2 + 2\|u_{n_k} - x_{n_k}\| \|x_{n_k} - z\|^2 \right. \\
&\quad \left. + \|x_{n_k} - z\|^2 - \|x_{n_{k+1}} - z\|^2 \right) \\
&\leq \limsup_{k \rightarrow \infty} \left(\|u_{n_k} - x_{n_k}\|^2 + 2\|u_{n_k} - x_{n_k}\| \|x_{n_k} - z\|^2 \right) \\
&\quad + \limsup_{k \rightarrow \infty} \left(\|x_{n_k} - z\|^2 - \|x_{n_{k+1}} - z\|^2 \right) \\
&= -\liminf_{k \rightarrow \infty} \left(\|x_{n_{k+1}} - z\|^2 - \|x_{n_k} - z\|^2 \right) \\
&\leq 0. \tag{3.11}
\end{aligned}$$

Hence,

$$\lim_{k \rightarrow \infty} \|x_{n_{k+1}} - \Phi^{N_{k-1}}u_{n_k}\| = 0. \tag{3.12}$$

By a similar argument as in (3.11) and applying (3.8), we obtain that

$$\begin{aligned}
& \limsup_{k \rightarrow \infty} \left(\|\Phi^{N_{k-1}} u_{n_k} - \Phi^{N_{k-2}} u_{n_k}\|^2 \right) \\
& \leq \limsup_{k \rightarrow \infty} \left(\|\Phi^{N_{k-2}} u_{n_k} - z\|^2 - \|\Phi^{N_{k-1}} u_{n_k} - z\|^2 \right) \\
& \quad \vdots \\
& \leq \limsup_{k \rightarrow \infty} \left(\|u_{n_k} - z\|^2 - \|\Phi^{N_{k-1}} u_{n_k} - z\|^2 \right) \\
& \leq \limsup_{k \rightarrow \infty} \left(\|u_{n_k} - z\|^2 - \|x_{n_{k+1}} - z\|^2 \right) \\
& \leq \limsup_{k \rightarrow \infty} \left(\|u_{n_k} - x_{n_k}\|^2 + 2\|u_{n_k} - x_{n_k}\| \|x_{n_k} - z\|^2 \right. \\
& \quad \left. + \|x_{n_k} - z\|^2 - \|x_{n_{k+1}} - z\|^2 \right) \\
& \leq \limsup_{k \rightarrow \infty} \left(\|u_{n_k} - x_{n_k}\|^2 + 2\|u_{n_k} - x_{n_k}\| \|x_{n_k} - z\|^2 \right) \\
& \quad + \limsup_{k \rightarrow \infty} \left(\|x_{n_k} - z\|^2 - \|x_{n_{k+1}} - z\|^2 \right) \\
& = - \liminf_{k \rightarrow \infty} \left(\|x_{n_{k+1}} - z\|^2 - \|x_{n_k} - z\|^2 \right) \\
& \leq 0. \tag{3.13}
\end{aligned}$$

Hence,

$$\lim_{k \rightarrow \infty} \|\Phi^{N_{k-1}} u_{n_k} - \Phi^{N_{k-2}} u_{n_k}\| = 0. \tag{3.14}$$

Continuing in the same manner, we obtain that

$$\begin{aligned}
\lim_{k \rightarrow \infty} \|\Phi^{N_{k-2}} u_{n_k} - \Phi^{N_{k-3}} u_{n_k}\| &= \dots = \lim_{k \rightarrow \infty} \|\Phi^2 u_{n_k} - \Phi^1 u_{n_k}\| \\
&= \lim_{k \rightarrow \infty} \|\Phi^1 u_{n_k} - u_{n_k}\| = 0. \tag{3.15}
\end{aligned}$$

From (3.12), (3.14) and (3.15), we conclude that

$$\lim_{k \rightarrow \infty} \|\Phi^j u_{n_k} - \Phi^{j-1} u_{n_k}\| = 0, \quad j = 1, 2, \dots, N. \tag{3.16}$$

By Remark 2.1, we have that f_j is Lipschitz continuous for each $j = 1, 2, \dots, N$. Thus,

$$\lim_{k \rightarrow \infty} \|f_j \Phi^j u_{n_k} - f_j \Phi^{j-1} u_{n_k}\| = 0, \quad j = 1, 2, \dots, N. \tag{3.17}$$

Also,

$$\begin{aligned} \|x_{n_{k+1}} - u_{n_k}\| &\leq \|\Phi^N u_{n_k} - \Phi^{N-1} u_{n_k}\| + \|\Phi^{N-1} u_{n_k} - \Phi^{N-2} u_{n_k}\| \\ &\quad + \cdots + \|\Phi^1 u_{n_k} - u_{n_k}\|, \end{aligned} \quad (3.18)$$

which implies from (3.16) that

$$\lim_{k \rightarrow \infty} \|x_{n_{k+1}} - u_{n_k}\| = 0. \quad (3.19)$$

By applying (3.9) and (3.19), we have that

$$\lim_{k \rightarrow \infty} \|x_{n_{k+1}} - x_{n_k}\| = 0. \quad (3.20)$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_t}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_t}}\}$ converges weakly to p . By (3.7) and (3.9), we have that there exist subsequences $\{w_{n_{k_t}}\}$ of $\{w_{n_k}\}$ and $\{u_{n_{k_t}}\}$ of $\{u_{n_k}\}$ that converges weakly to p respectively. Thus, by the demi-closedness of S_i at zero and (3.6), we obtain that $p \in F(S_i)$ for each $i = 1, 2, \dots, m$.

We next show that $p \in \bigcap_{j=1}^N VI(C, f_j)$.

Let

$$B_j v = \begin{cases} f_j(v) + N_C v, & \forall v \in C; \\ \phi, & \forall v \notin C. \end{cases}$$

Then, B_j is maximal monotone for each $j = 1, 2, \dots, N$. Let $(v, w) \in G(B_j)$. Then we have

$$w \in B_j v = f_j(v) + N_C v.$$

Hence

$$w - f_j(v) \in N_C v.$$

For $\Phi^j u_{n_{k_t}} \in C$, we obtain

$$\langle v - \Phi^j u_{n_{k_t}}, w - f_j v \rangle \geq 0, \quad j = 1, 2, \dots, N. \quad (3.21)$$

From $\Phi^j u_{n_{k_t}} = P_C(I - \lambda f_j)\Phi^{j-1} u_{n_{k_t}}$, we have

$$\langle v - \Phi^j u_{n_{k_t}}, \Phi^j u_{n_{k_t}} - (\Phi^{j-1} u_{n_{k_t}} - \lambda f_j \Phi^{j-1} u_{n_{k_t}}) \rangle \geq 0, \quad j = 1, 2, \dots, N,$$

which implies that

$$\langle v - \Phi^j u_{n_{k_t}}, \frac{\Phi^j u_{n_{k_t}} - \Phi^{j-1} u_{n_{k_t}}}{\lambda} + f_j \Phi^{j-1} u_{n_{k_t}} \rangle \geq 0, \quad \text{for each } j = 1, 2, \dots, N.$$

From (3.21), we have

$$\begin{aligned}
& \langle v - \Phi^j u_{n_{k_t}}, w \rangle \\
& \geq \langle v - \Phi^j u_{n_{k_t}}, f_j v \rangle \\
& \geq \langle v - \Phi^j u_{n_{k_t}}, f_j v \rangle - \langle v - \Phi^j u_{n_{k_t}}, \frac{\Phi^j u_{n_{k_t}} - \Phi^{j-1} u_{n_{k_t}}}{\lambda} + f_j \Phi^{j-1} u_{n_{k_t}} \rangle \\
& = \langle v - \Phi^j u_{n_{k_t}}, f_j z - f_j \Phi^{j-1} u_{n_{k_t}} - \frac{\Phi^j u_{n_{k_t}} - \Phi^{j-1} u_{n_{k_t}}}{\lambda} \rangle \\
& = \langle v - \Phi^j u_{n_{k_t}}, f_j z - f_j \Phi^{j-1} u_{n_{k_t}} \rangle + \langle v - \Phi^j u_{n_{k_t}}, f_j \Phi^j u_{n_{k_t}} - f_j \Phi^{j-1} u_{n_{k_t}} \rangle \\
& \quad - \langle v - \Phi^j u_{n_{k_t}}, \frac{\Phi^j u_{n_{k_t}} - \Phi^{j-1} u_{n_{k_t}}}{\lambda} \rangle \\
& \geq \langle v - \Phi^j u_{n_{k_t}}, f_j \Phi^j u_{n_{k_t}} - f_j \Phi^{j-1} u_{n_{k_t}} \rangle \\
& \quad - \langle v - \Phi^j u_{n_{k_t}}, \frac{\Phi^j u_{n_{k_t}} - \Phi^{j-1} u_{n_{k_t}}}{\lambda} \rangle. \tag{3.22}
\end{aligned}$$

By applying (3.16), (3.17) and (3.19) on (3.22), we obtain that

$$\langle v - p, w \rangle \geq 0.$$

Now, we conclude that since $B_j, j = 1, 2, \dots, N$ is maximal monotone, thus $p \in B_j^{-1}(0)$, which implies that $0 \in B_j(p)$. Hence $p \in \bigcap_{j=1}^{\infty} VI(C, f_j)$. Therefore, we conclude that $p \in \Omega$.

We next show that $\limsup_{k \rightarrow \infty} \langle x_{n_{k+1}} - z, g(x_{n_k}) - z \rangle \leq 0$. Indeed, let $\{x_{n_{k_t}}\}$ be a sequence such that $\{x_{n_k}\}$ converges weakly to p and

$$\limsup_{k \rightarrow \infty} \langle x_{n_{k+1}} - z, g(x_{n_k}) - z \rangle = \lim_{t \rightarrow \infty} \langle x_{n_{k_t+1}} - z, g(x_{n_{k_t}}) - z \rangle.$$

Using (3.20), we obtain that

$$\begin{aligned}
\limsup_{k \rightarrow \infty} \langle x_{n_{k+1}} - z, g(x_{n_k}) - z \rangle &= \lim_{t \rightarrow \infty} \langle x_{n_{k_t+1}} - z, g(x_{n_{k_t}}) - z \rangle \\
&\leq \langle p - z, g(p) - z \rangle \\
&\leq 0. \tag{3.23}
\end{aligned}$$

On substituting (3.23) in (3.3) and applying Lemma 2.7, we obtain that $\{x_{n_k}\}$ converges strongly to p . This completes the proof. \square

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $\{S_i\}_{i=1}^m : H \rightarrow CB(H)$ be a finite family of multi-valued type-one demicontractive-type mappings with constant $k_i \in (0, 1)$ such that $I - S_i$ is demiclosed at zero. Let $f : H \rightarrow H$ be an α -inverse strongly monotone*

mapping and $g : H \rightarrow H$ be a contractive mapping with constant $\theta \in (0, 1)$. Suppose that

$$\Omega := \left\{ \bigcap_{i=1}^m F(S_i) \bigcap VI(C, f) \right\} \neq \emptyset.$$

For $x_1 \in H$, let the sequence $\{x_n\}$ be defined by

$$\begin{cases} w_n = \gamma_n g(x_n) + (1 - \gamma_n)x_n, \\ u_n = \beta_{n,0}w_n + \sum_{i=1}^m \beta_{n,i}z_n^i, \quad n \geq 1, \\ x_{n+1} = P_C(I - \lambda f)u_n, \quad n \in \mathbb{N}, \end{cases} \quad (3.24)$$

where $z_n^i \in P_{S_i w_n}$, $P_{S_i w_n} = \{z_n^i \in S_i w_n : \|z_n^i - w_n\| = d(w_n, S_i w_n)\}$ and $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $\lambda \in (0, 2\alpha)$, and the sequences $\{\beta_{n,i}\}_{n=1}^\infty$ for all $i \geq 0$ and $\{\gamma_n\}_{n=1}^\infty$ satisfy the following conditions:

- (i) $\{\beta_{n,0}\} \in (k, 1)$, $\{\beta_{n,i}\} \in (0, 1)$ such that $\sum_{i=0}^m \beta_{n,i} = 1$, $k < a \leq \beta_{n,i} \leq b < 1$, $i = 1, 2, \dots, m$, $k := \sup_{i \geq 1} \{k_i\} < 1$;
- (ii) $\gamma_n \in (0, 1)$, $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\sum_{n=1}^\infty \gamma_n = \infty$.

Then $\{x_n\}$ converges strongly to an element in Ω .

Corollary 3.3. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $\{S_i\}_{i=1}^m : H \rightarrow CB(H)$ be a finite family of multi-valued quasi-nonexpansive mappings. Let $f : H \rightarrow H$ be an α -inverse strongly monotone mapping and $g : H \rightarrow H$ be a contractive mapping with constant $\theta \in (0, 1)$. Suppose that

$$\Omega := \left\{ \bigcap_{i=1}^m F(S_i) \bigcap VI(C, f) \right\} \neq \emptyset.$$

For $x_1 \in H$, let the sequence $\{x_n\}$ be defined by

$$\begin{cases} w_n = \gamma_n g(x_n) + (1 - \gamma_n)x_n, \\ u_n = \beta_{n,0}w_n + \sum_{i=1}^m \beta_{n,i}z_n^i, \quad n \geq 1, \\ x_{n+1} = P_C(I - \lambda f)u_n, \quad n \in \mathbb{N}, \end{cases} \quad (3.25)$$

where $z_n^i \in S_i u_n$, $\lambda \in (0, 2\alpha)$, and the sequences $\{\beta_{n,i}\}_{n=1}^\infty$ for all $i \geq 0$ and $\{\gamma_n\}_{n=1}^\infty$ satisfy the following conditions:

- (i) $\{\beta_{n,0}\} \in (0, 1)$, $\{\beta_{n,i}\} \in (0, 1)$ such that $\sum_{i=0}^m \beta_{n,i} = 1$;
- (ii) $\gamma_n \in (0, 1)$, $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\sum_{n=1}^\infty \gamma_n = \infty$.

Then $\{x_n\}$ converges strongly to an element in Ω .

4. APPLICATION TO CONVEX MINIMIZATION PROBLEM

Let $F : C \rightarrow \mathbb{R}$ be a convex and differentiable function. We know that if ∇F is $\frac{1}{\alpha}$ -Lipschitz continuous, then it is α -inverse strongly monotone, where ∇F is the gradient of F . Moreover,

$$p = \operatorname{argmin}_{x \in C} F(x) \Leftrightarrow p \in VI(C, \nabla F). \quad (4.1)$$

Suppose the solution set of (4.1) is Γ . Then the setting $f_j = \nabla F_j$ for each $j = 1, 2, \dots, N$ in (3.1), we obtain the following result.

Theorem 4.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $\{S_i\}_{i=1}^m : H \rightarrow CB(H)$ be a finite family of multi-valued type-one demicontractive-type mappings with constant $k_i \in (0, 1)$ such that $I - S_i$ is demiclosed at zero. Let $F_j : H \rightarrow \mathbb{R}$ for $j = 1, 2, \dots, N$ be an α_j -inverse strongly monotone mapping and $g : H \rightarrow H$ be a contractive mapping with constant $\theta \in (0, 1)$. Suppose that*

$$\Omega := \left\{ \bigcap_{i=1}^m F(S_i) \bigcap \bigcap_{j=1}^N VI(C, \nabla F_j) \right\} \neq \emptyset.$$

For $x_1 \in H$, let the sequence $\{x_n\}$ be defined by

$$\begin{cases} w_n = \gamma_n g(x_n) + (1 - \gamma_n)x_n, \\ u_n = \beta_{n,0}w_n + \sum_{i=1}^m \beta_{n,i}z_n^i, \quad n \geq 1, \\ x_{n+1} = P_C(I - \lambda \nabla F_N) \circ P_C(I - \lambda \nabla F_{N-1}) \circ \dots \circ P_C(I - \lambda \nabla F_1)u_n, \end{cases} \quad (4.2)$$

for all $n \in \mathbb{N}$, where $z_n^i \in P_{S_i w_n}$, $P_{S_i w_n} = \{z_n^i \in S_i w_n : \|z_n^i - w_n\| = d(w_n, S_i w_n)\}$ and $\lambda \in (0, 2\alpha)$, $\alpha = \min\{\alpha_j, j = 1, 2, \dots, N\}$, and the sequences $\{\beta_{n,i}\}_{n=1}^\infty$ for all $i \geq 0$ and $\{\gamma_n\}_{n=1}^\infty$ satisfy the following conditions:

- (i) $\{\beta_{n,0}\} \in (k, 1)$, $\{\beta_{n,i}\} \in (0, 1)$ such that $\sum_{i=0}^m \beta_{n,i} = 1$, $k < a \leq \beta_{n,i} \leq b < 1$, $i = 1, 2, \dots, m$, $k := \sup_{i \geq 1} \{k_i\} < 1$;
- (ii) $\gamma_n \in (0, 1)$, $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\sum_{n=1}^\infty \gamma_n = \infty$.

Then $\{x_n\}$ converges strongly to an element in Ω .

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