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HALPERN TSENG'S EXTRAGRADIENT METHODS FOR SOLVING VARIATIONAL INEQUALITIES INVOLVING SEMISTRICTLY QUASIMONOTONE OPERATOR

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Abstract. In this paper, we study the strong convergence of new methods for solving classical variational inequalities problems involving semistrictly quasimonotone and Lipschitz-continuous operators in a real Hilbert space. Three proposed methods are based on Tseng's extragradient method and use a simple self-adaptive step size rule that is independent of the Lipschitz constant. The step size rule is built around two techniques: the monotone and the non-monotone step size rule. We establish strong convergence theorems for the proposed methods that do not require any additional projections or knowledge of an involved operator's Lipschitz constant. Finally, we present some numerical experiments that demonstrate the efficiency and advantages of the proposed methods.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. The strong converge of the sequence $\{x_n\}$ to an element x is denoted by

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$x_n \to x$. For a given closed and convex subset $\mathcal{C} \subset \mathcal{H}$, the variational inequality problem denoted by $VI(\mathcal{C}, \mathcal{G})$ is to find $x^* \in \mathcal{C}$ such that

$$\left\langle \mathcal{G}(x^*), y - x^* \right\rangle \geq 0, \ \forall \, y \in \mathcal{C}, \tag{VIP}$$

where $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is an operator. It is well known (see for details [7]) that the original problem is closely related to find a point $x^* \in \mathcal{C}$ such that

$$\left\langle \mathcal{G}(y), y - x^* \right\rangle \geq 0, \ \forall \, y \in \mathcal{C}. \tag{DVIP}$$

We will refer to problem (DVIP) as the dual variational inequality problem $DVI(\mathcal{C}, \mathcal{G})$ of (VIP) based on the paper [7]. For a closed and convex $\mathcal{C} \subset \mathcal{H}$, the metric projection $P_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ is described for all $x \in \mathcal{H}$ such that

$$P_{\mathcal{C}}(x) = \arg\min\{\|x - y\| : y \in \mathcal{C}\}.$$

Moreover, $\mathbb{R}$, $\mathbb{N}$ are the set of real numbers and natural numbers, respectively. It is useful to note that the problem (VIP) is equivalent to solve the following problem:

$$\text{Find } x^* \in \mathcal{C} \text{ such that } x^* = P_{\mathcal{C}}[x^* - \lambda \mathcal{G}(x^*)],$$

where λ is any positive real number.

The theory of variational inequalities has been used extensively in the investigation of a collection of topics, consisting of physics, engineering, economics, and optimization theory. This problem was first introduced by Stampacchia [33] in 1964, and it is also well established that the problem (VIP) is a pivotal problem in nonlinear analysis. It is a significant mathematical problem that incorporates several important topics of applied mathematics, such as network equilibrium problems, the necessary optimality conditions, complementarity problems, and systems of nonlinear equations (for more details [8, 13, 14, 15, 16, 17, 28, 29, 30, 32]). On the other hand, the projection methods are effective iterative methods to solve variational inequalities. Many iterative methods for solving variational inequalities have been designed and investigated previously (see for more details [1, 4, 5, 6, 11, 12, 18, 20, 21, 22, 23, 24, 25, 26, 27, 31, 34, 35, 37]).

Then, Korpelevich [18] and Antipin [2] were the first to propose the extragradient method. The method is of the following design:

$$\begin{cases} x_0 \in \mathcal{C}, \\ y_n = P_{\mathcal{C}}[x_n - \lambda \mathcal{G}(x_n)], \\ x_{n+1} = P_{\mathcal{C}}[x_n - \lambda \mathcal{G}(y_n)], \end{cases} \tag{1.1}$$

where $0 < \lambda < \frac{1}{L}$ is the Lipschitz constant of an operator $\mathcal{G}$ and L is the Lipschitz constant of an operator $\mathcal{G}$. In view of the above method, we use two projections on the underlying set $\mathcal{C}$ over each iteration. This, of course, can affect the computational effectiveness of the used method if the feasible set

$\mathcal{C}$ has a complicated structure. Here, we restrict in our interest to presenting some methods which can address this drawback. The first is the following subgradient extragradient method due to Censor et al. [4]. This method takes the form of

$$\begin{cases} x_0 \in \mathcal{C}, \\ y_n = P_{\mathcal{C}}[x_n - \lambda \mathcal{G}(x_n)], \\ x_{n+1} = P_{\mathcal{H}_n}[x_n - \lambda \mathcal{G}(y_n)], \end{cases} \tag{1.2}$$

where $0 < \lambda < \frac{1}{L}$ and

$$\mathcal{H}_n = \{z \in \mathcal{H} : \langle x_n - \lambda \mathcal{G}(x_n) - y_n, z - y_n \rangle \leq 0\}.$$

In this article, we focus on the Tseng's extragradient method [34] that need to calculate only one projection for each iteration:

$$\begin{cases} x_0 \in \mathcal{C}, \\ y_n = P_{\mathcal{C}}[x_n - \lambda \mathcal{G}(x_n)], \\ x_{n+1} = y_n + \lambda \big[\mathcal{G}(x_n) - \mathcal{G}(y_n)\big], \end{cases} \tag{1.3}$$

where $0 < \lambda < \frac{1}{L}$. The above-mentioned methods have two significant disadvantages. The first is the constant step size, which necessarily involves the knowledge or estimation of the Lipschitz constant of the involved operator and only converges weakly in Hilbert spaces. In certain cases, the Lipschitz constants are unknown since they are difficult to compute for an operator. Estimating the Lipschitz constant a priori may be difficult from a computational point of view, which may affect the method's convergence rate and applicability.

The primary objective of this paper is to investigate semistrictly quasimonotone variational inequalities in infinite-dimensional Hilbert spaces. To prove that the iterative sequence generated by Tseng's extragradient method for the solution of semistrictly quasimonotone variational inequalities converges strongly to some solutions. The proposed methods are inspired by the projection method [34] and the Halpern method [10]. The proposed methods only require solving one projection onto the feasible set per iteration. If suitable conditions are imposed on control parameters, the iterative sequences generated by our methods strongly converge to some solution to the problem. We also provide examples to demonstrate the computational performance of the new methods.

The paper is organized in the following manner. In Sect. 2, some preliminary results were presented. Sect. 3 provides new algorithms and their convergence study. Finally, Sect. 4 presents some numerical results to point out the practical efficiency of the proposed methods.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space. Given $x, y \in \mathcal{H}$, we interpret the closed line segment

$$[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}.$$

The segments $(x, y]$, $[x, y)$, and (x, y) are defined similarly. A *metric projection* $P_{\mathcal{C}}(x)$ of $x \in \mathcal{H}$ is defined by

$$P_{\mathcal{C}}(x) = \arg\min\{\|x - y\| : y \in \mathcal{C}\}.$$

For all $x, y \in \mathcal{H}$, we have

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2.$$

Lemma 2.1. *Assume $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ and $P_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ be a metric projection from $\mathcal{H}$ onto $\mathcal{C}$.*

(i) *Let $x \in \mathcal{C}$ and $y \in \mathcal{H}$, we have*

$$\|x - P_{\mathcal{C}}(y)\|^2 + \|P_{\mathcal{C}}(y) - y\|^2 \leq \|x - y\|^2.$$

(ii) *$z = P_{\mathcal{C}}(x)$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \ \forall\, y \in \mathcal{C}.$$

(iii) *For $y \in \mathcal{C}$ and $x \in \mathcal{H}$*

$$\|x - P_{\mathcal{C}}(x)\| \leq \|x - y\|.$$

Definition 2.2. Suppose that $\mathcal{C}$ is a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$, and let $\mathcal{G} : \mathcal{C} \to \mathcal{H}$ be a mapping. The mapping $\mathcal{G}$ is said to be:

(a) *strongly monotone* on $\mathcal{C}$ with constant $\gamma > 0$ if for each pair of points $x, y \in \mathcal{C}$, we have

$$\big\langle \mathcal{G}(x) - \mathcal{G}(y), x - y \big\rangle \geq \gamma \|x - y\|^2;$$

(b) *strictly monotone* on $\mathcal{C}$ if for all distinct $x, y \in \mathcal{C}$, we have

$$\big\langle \mathcal{G}(x) - \mathcal{G}(y), x - y \big\rangle > 0;$$

(c) *monotone* on $\mathcal{C}$ if for all distinct $x, y \in \mathcal{C}$, we have

$$\big\langle \mathcal{G}(x) - \mathcal{G}(y), x - y \big\rangle \geq 0;$$

(d) *pseudomonotone* on $\mathcal{C}$ if for all distinct $x, y \in \mathcal{C}$, $\big\langle \mathcal{G}(y), x - y \big\rangle \geq 0$, then

$$\big\langle \mathcal{G}(x), x - y \big\rangle \geq 0;$$

(e) *quasimonotone* on $\mathcal{C}$ if for all distinct $x, y \in \mathcal{C}$, $\big\langle \mathcal{G}(y), x - y \big\rangle > 0$, then

$$\big\langle \mathcal{G}(x), x - y \big\rangle \geq 0;$$

(f) *semistrictly quasimonotone* [9] on $\mathcal{C}$ if $\mathcal{G}$ is quasimonotone on $\mathcal{C}$ and for all distinct of points $x, y \in \mathcal{C}, \langle \mathcal{G}(y), x - y \rangle > 0$, thent

$$\langle \mathcal{G}(z), x - y \rangle \geq 0, \text{ for some } z \in \Big(\frac{x + y}{2}, x \Big).$$

Remark 2.3. The implications are as follows:

$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (e) \quad \text{and} \quad (f) \Longrightarrow (e).$$

However, in general, the inverse assertions are false.

Lemma 2.4. ([3]) *For any $x, y \in \mathcal{H}$ and $\ell \in \mathbb{R}$, we have*

(i) $\|\ell x + (1 - \ell) y\|^2 = \ell \|x\|^2 + (1 - \ell) \|y\|^2 - \ell (1 - \ell) \|x - y\|^2$.
(ii) $\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle$.

Definition 2.5. A mapping $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is said to be

(i) *weakly hemicontinuous* if $\mathcal{G}$ is upper semicontinuous from line segments in $\mathcal{H}$ to the weak topology of $\mathcal{H}$;
(ii) *weakly sequentially continuous* if $\{\mathcal{G}(x_n)\}$ converges weakly to $\mathcal{G}(x)$ for every sequence $\{x_n\}$ converges weakly to x.

Remark 2.6. It is easy to prove that if $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is weakly sequentially continuous, then $\mathcal{G}$ must be weakly hemicontinuous.

It is known that the following conclusion is true.

Lemma 2.7. ([7]) *A solution of problem* (DVIP) *is always a solution of problem* (VIP) *provided that the operator $\mathcal{G}$ is, say, weakly hemicontinuous.*

Lemma 2.8. ([36]) *Let $\{p_n\} \subset [0, +\infty)$ be a sequence such that*

$$p_{n+1} \leq (1 - q_n) p_n + q_n r_n, \ \forall \ n \in \mathbb{N}.$$

Moreover, two sequence $\{q_n\} \subset (0, 1)$ and $\{r_n\} \subset \mathbb{R}$ such that

$$\lim_{n \to +\infty} q_n = 0, \ \sum_{n=1}^{+\infty} q_n = +\infty \ \textit{ and } \ \limsup_{n \to +\infty} r_n \leq 0.$$

Then, $\lim_{n \to +\infty} p_n = 0$.

Lemma 2.9. ([19]) *Let a real number sequence $\{p_n\}$ and there exists a subsequence $\{n_i\}$ of $\{n\}$ such that*

$$p_{n_i} < p_{n_{i+1}}, \ \forall \ i \in \mathbb{N}.$$

Then, there exist a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to +\infty$ *as* $k \to +\infty$, *and satisfying the following inequality for* $k \in \mathbb{N}$:

$$p_{m_k} \leq p_{m_{k+1}} \quad \text{and} \quad p_k \leq p_{m_{k+1}}.$$

Indeed, $m_k = \max\{j \leq k : p_j \leq p_{j+1}\}$.

3. Main Results

In this section, we introduce different iterative methods for variational inequalities involving semistrictly quasimonotone based on Tseng's extragradient method and the Halpern method that does not involve knowledge of the operator's Lipschitz constant or additional projection.

Algorithm 1.

Step 0: Choose $x_1 \in \mathcal{C}$, $0 < \lambda < \frac{1}{L}$, $\mu \in (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

$$\lim_{n\to+\infty} \alpha_n = 0 \;\text{ and }\; \sum_{n=1}^{+\infty} \alpha_n = +\infty.$$

Step 1: Compute

$$y_n = P_{\mathcal{C}}(x_n - \lambda \mathcal{G}(x_n)).$$

If $x_n = y_n$, then STOP and y_n is a solution. Otherwise, go to **Step 2**.

Step 2: Compute

$$z_n = y_n + \lambda\big[\mathcal{G}(x_n) - \mathcal{G}(y_n)\big].$$

Step 3: Compute

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) z_n.$$

Set $n = n + 1$ and go back to **Step 1**.

To prove the convergence analysis, it is assumed that the following conditions have been met:

($\mathcal{G}$1) The solution set of problem (VIP) is denoted by Ω is nonempty;

($\mathcal{G}$2) An operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is semistrictly quasimonotone, that is, if $\mathcal{G}$ is quasimonotone on $\mathcal{C}$ and $\big\langle \mathcal{G}(y), x - y \big\rangle > 0$, then

$$\big\langle \mathcal{G}(z), x - y \big\rangle \geq 0, \text{ for some } z \in \Big(\frac{x+y}{2}, x\Big); \tag{SQM}$$

($\mathcal{G}$3) An operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is *Lipschitz continuous* with constant $L > 0$, that is,

$$\|\mathcal{G}(x) - \mathcal{G}(y)\| \leq L\|x - y\|, \; \forall\, x, y \in \mathcal{C}; \tag{LC}$$

($\mathcal{G}4$) An operator $\mathcal{G}:\mathcal{H}\to\mathcal{H}$ is *sequentially weakly continuous*, that is, if $\{\mathcal{G}(x_n)\}$ converges weakly to $\mathcal{G}(x)$ for every sequence $\{x_n\}$ converges weakly to x.

Lemma 3.1. *Suppose that $\mathcal{G}:\mathcal{H}\to\mathcal{H}$ satisfies the conditions* ($\mathcal{G}1$)-($\mathcal{G}4$) *and sequence $\{x_n\}$ generated by Algorithm* 1. *Then, we have*

$$\|z_n - x^*\|^2 \le \left\|x_n - x^*\right\|^2 - \left(1-\lambda^2L^2\right)\left\|x_n - y_n\right\|^2.$$

Proof. Since $x^*\in\Omega$, we have

$$\begin{aligned}
&\left\|z_n - x^*\right\|^2\\
&= \left\|y_n + \lambda[\mathcal{G}(x_n)-\mathcal{G}(y_n)] - x^*\right\|^2\\
&= \left\|y_n - x^*\right\|^2 + \lambda^2\left\|\mathcal{G}(x_n)-\mathcal{G}(y_n)\right\|^2 + 2\lambda\langle y_n - x^*, \mathcal{G}(x_n)-\mathcal{G}(y_n)\rangle\\
&= \left\|y_n + x_n - x_n - x^*\right\|^2 + \lambda^2\left\|\mathcal{G}(x_n)-\mathcal{G}(y_n)\right\|^2 + 2\lambda\langle y_n - x^*, \mathcal{G}(x_n)-\mathcal{G}(y_n)\rangle\\
&= \left\|y_n - x_n\right\|^2 + \left\|x_n - x^*\right\|^2 + 2\langle y_n - x_n, x_n - x^*\rangle\\
&\quad + \lambda^2\left\|\mathcal{G}(x_n)-\mathcal{G}(y_n)\right\|^2 + 2\lambda\langle y_n - x^*, \mathcal{G}(x_n)-\mathcal{G}(y_n)\rangle\\
&= \left\|x_n - x^*\right\|^2 + \left\|y_n - x_n\right\|^2 + 2\langle y_n - x_n, y_n - x^*\rangle + 2\langle y_n - x_n, x_n - y_n\rangle\\
&\quad + \lambda^2\left\|\mathcal{G}(x_n)-\mathcal{G}(y_n)\right\|^2 + 2\lambda\langle y_n - x^*, \mathcal{G}(x_n)-\mathcal{G}(y_n)\rangle. \qquad (3.1)
\end{aligned}$$

It is given that

$$y_n = P_{\mathcal{C}}[x_n - \lambda\mathcal{G}(x_n)]$$

and it gives that

$$\langle x_n - \lambda\mathcal{G}(x_n) - y_n, y - y_n\rangle \le 0,\ \forall\, y\in\mathcal{C}. \qquad (3.2)$$

Thus, we have

$$\langle x_n - y_n, x^* - y_n\rangle \le \lambda\langle \mathcal{G}(x_n), x^* - y_n\rangle. \qquad (3.3)$$

Combining expressions (3.1) and (3.3), we have

$$\begin{aligned}
&\left\|z_n - x^*\right\|^2\\
&\le \left\|x_n - x^*\right\|^2 + \left\|y_n - x_n\right\|^2 + 2\lambda\langle \mathcal{G}(x_n), x^* - y_n\rangle - 2\langle x_n - y_n, x_n - y_n\rangle\\
&\quad + \lambda^2\left\|\mathcal{G}(x_n)-\mathcal{G}(y_n)\right\|^2 - 2\lambda\langle \mathcal{G}(x_n)-\mathcal{G}(y_n), x^* - y_n\rangle\\
&= \left\|x_n - x^*\right\|^2 - \left\|x_n - y_n\right\|^2 + \lambda^2\left\|\mathcal{G}(x_n)-\mathcal{G}(y_n)\right\|^2 - 2\lambda\langle \mathcal{G}(y_n), y_n - x^*\rangle. \qquad (3.4)
\end{aligned}$$

It is given that x^* is the solution of the problem (VIP) implies that

$$\langle \mathcal{G}(x^*), y - x^*\rangle \ge 0,\ \forall\, y\in\mathcal{C}.$$

It implies that

$$\langle \mathcal{G}(y), y - x^*\rangle \ge 0,\ \forall\, y\in\mathcal{C}.$$

Substituting $y = y_n \in \mathcal{C}$, we have

$$\langle \mathcal{G}(y_n), y_n - x^* \rangle \geq 0. \tag{3.5}$$

From expressions (3.4) and (3.5), we obtain

$$\begin{aligned}\|z_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + \lambda^2 L^2 \|x_n - y_n\|^2 \\ &= \|x_n - x^*\|^2 - \left(1 - \lambda^2 L^2\right)\|x_n - y_n\|^2. \end{aligned} \tag{3.6}$$

□

Lemma 3.2. *Let $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ satisfying ($\mathcal{G}$1)–($\mathcal{G}$4). If there exists a weakly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ to $\hat{x}$ and $\lim_{k\to+\infty} \|x_{n_k} - y_{n_k}\| = 0$. Then, $\hat{x}$ is the solution of* (VIP)*.*

Proof. Indeed, we have

$$y_{n_k} = P_{\mathcal{C}}[x_{n_k} - \lambda_{n_k} \mathcal{G}(x_{n_k})],$$

that is equivalent to

$$\langle x_{n_k} - \lambda_{n_k} \mathcal{G}(x_{n_k}) - y_{n_k}, y - y_{n_k} \rangle \leq 0, \ \forall\, y \in \mathcal{C}. \tag{3.7}$$

The inequality mentioned above implies that

$$\langle x_{n_k} - y_{n_k}, y - y_{n_k} \rangle \leq \lambda_{n_k} \langle \mathcal{G}(x_{n_k}), y - y_{n_k} \rangle, \ \forall\, y \in \mathcal{C}. \tag{3.8}$$

Thus, we obtain

$$\frac{1}{\lambda_{n_k}} \langle x_{n_k} - y_{n_k}, y - y_{n_k} \rangle + \langle \mathcal{G}(x_{n_k}), y_{n_k} - x_{n_k} \rangle \leq \langle \mathcal{G}(x_{n_k}), y - x_{n_k} \rangle, \ \forall\, y \in \mathcal{C}. \tag{3.9}$$

Since $\min\left\{\frac{\mu}{L}, \lambda_1\right\} \leq \lambda \leq \lambda_1$ and $\{x_{n_k}\}$ is a bounded sequence, from the condition $\lim_{k\to+\infty} \|x_{n_k} - y_{n_k}\| = 0$ and $k \to +\infty$ in (3.9), we obtain

$$\liminf_{k\to+\infty} \langle \mathcal{G}(x_{n_k}), y - x_{n_k} \rangle \geq 0, \ \forall\, y \in \mathcal{C}. \tag{3.10}$$

Moreover, we have

$$\begin{aligned}&\langle \mathcal{G}(y_{n_k}), y - y_{n_k} \rangle \\ &= \langle \mathcal{G}(y_{n_k}) - \mathcal{G}(x_{n_k}), y - x_{n_k} \rangle + \langle \mathcal{G}(x_{n_k}), y - x_{n_k} \rangle + \langle \mathcal{G}(y_{n_k}), x_{n_k} - y_{n_k} \rangle.\end{aligned} \tag{3.11}$$

Since $\lim_{k\to+\infty} \|x_{n_k} - y_{n_k}\| = 0$ and $\mathcal{G}$ is L-Lipschitz continuity on $\mathcal{H}$ implies that

$$\lim_{k\to+\infty} \|\mathcal{G}(x_{n_k}) - \mathcal{G}(y_{n_k})\| = 0, \tag{3.12}$$

which together with (3.11) and (3.12), we obtain

$$\liminf_{k\to+\infty} \langle \mathcal{G}(y_{n_k}), y - y_{n_k} \rangle \geq 0, \ \forall\, y \in \mathcal{C}. \tag{3.13}$$

To prove further, let us take a positive sequence $\{\epsilon_k\}$ that is convergent to zero and decreasing. For each $\{\epsilon_k\}$, we denote by m_k the smallest positive integer such that

$$\langle \mathcal{G}(x_{n_i}), y - x_{n_i} \rangle + \epsilon_k > 0, \ \forall\, i \geq m_k, \tag{3.14}$$

where the existence of m_k follows from (3.13). Since $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{m_k\}$ is increasing.

Case I: If there is a subsequence $\{x_{n_{m_{k_j}}}\}$ of $\{x_{n_{m_k}}\}$ such that $\mathcal{G}(x_{n_{m_{k_j}}}) = 0$, for all j. Let $j \to +\infty$, we obtain

$$\langle \mathcal{G}(\hat{x}), y - \hat{x} \rangle = \lim_{j \to +\infty} \langle \mathcal{G}(x_{n_{m_{k_j}}}), y - \hat{x} \rangle = 0. \tag{3.15}$$

Thus, $\hat{x} \in \mathcal{C}$ and imply that $\hat{x} \in VI(\mathcal{C}, \mathcal{G})$.

Case II: If there exits $N_0 \in \mathbb{N}$ such that for all $n_{m_k} \geq N_0$, $\mathcal{G}(x_{n_{m_k}}) \neq 0$. Consider that

$$\digamma_{n_{m_k}} = \frac{\mathcal{G}(x_{n_{m_k}})}{\|\mathcal{G}(x_{n_{m_k}})\|^2}, \ \forall\, n_{m_k} \geq N_0. \tag{3.16}$$

Due to the above definition, we obtain

$$\langle \mathcal{G}(x_{n_{m_k}}), \digamma_{n_{m_k}} \rangle = 1, \ \forall\, n_{m_k} \geq N_0. \tag{3.17}$$

Moreover, expressions (3.14) and (3.17), for all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{G}(x_{n_{m_k}}), \ y + \epsilon_k \digamma_{n_{m_k}} - x_{n_{m_k}} \rangle > 0. \tag{3.18}$$

Since $\mathcal{G}$ is quasimonotone, then

$$\langle \mathcal{G}(y + \epsilon_k \digamma_{n_{m_k}}), \ y + \epsilon_k \digamma_{n_{m_k}} - x_{n_{m_k}} \rangle > 0. \tag{3.19}$$

For all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{G}(y), y - x_{n_{m_k}} \rangle \geq \langle \mathcal{G}(y) - \mathcal{G}(y + \epsilon_k \digamma_{n_{m_k}}), \ y + \epsilon_k \digamma_{n_{m_k}} - x_{n_{m_k}} \rangle - \epsilon_k \langle \mathcal{G}(y), \digamma_{n_{m_k}} \rangle. \tag{3.20}$$

Due to $\{x_{n_k}\}$ weakly converges to $\hat{x} \in \mathcal{C}$ through $\mathcal{G}$ is weakly sequentially continuous on the set $\mathcal{C}$, we get $\{\mathcal{G}(x_{n_k})\}$ weakly converges to $\mathcal{G}(\hat{x})$. Suppose that $\mathcal{G}(\hat{x}) \neq 0$, we have

$$\|\mathcal{G}(\hat{x})\| \leq \liminf_{k \to +\infty} \|\mathcal{G}(x_{n_k})\|. \tag{3.21}$$

Since $\{x_{n_{m_k}}\} \subset \{x_{n_k}\}$ and $\lim_{k \to +\infty} \epsilon_k = 0$, we have

$$0 \leq \lim_{k \to +\infty} \|\epsilon_k \digamma_{n_{m_k}}\| = \lim_{k \to +\infty} \frac{\epsilon_k}{\|\mathcal{G}(x_{n_{m_k}})\|} \leq \frac{0}{\|\mathcal{G}(\hat{x})\|} = 0. \tag{3.22}$$

Next, consider $k \to +\infty$ in (3.20), we obtain

$$\langle \mathcal{G}(y), y - \hat{x} \rangle \geq 0, \ \forall\, y \in \mathcal{C}. \tag{3.23}$$

Thus, we infer that $\hat{x} \in VI(\mathcal{C}, \mathcal{G})$. □

Theorem 3.3. *Assume that a mapping $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ satisfies the conditions ($\mathcal{G}$1)–($\mathcal{G}$4). Then, the sequence $\{x_n\}$ generated by the Algorithm 1 converges strongly to $x^* = P_\Omega(x_0)$.*

Proof. By using Lemma 3.1, we have

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \big(1 - \lambda^2 L^2\big)\|x_n - y_n\|^2. \tag{3.24}$$

Since $0 < \lambda < \frac{1}{L}$, we obtain

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2. \tag{3.25}$$

By the use of definition of $\{x_{n+1}\}$, we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\alpha_n x_0 + (1 - \alpha_n) z_n - x^*\| \\ &= \|\alpha_n [x_0 - x^*] + (1 - \alpha_n)[z_n - x^*]\| \\ &\le \alpha_n \|x_0 - x^*\| + (1 - \alpha_n)\|z_n - x^*\|. \end{aligned} \tag{3.26}$$

Combining expressions (3.25) and (3.26), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \alpha_n \|x_0 - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &\le \max\Big\{\|x_0 - x^*\|, \|x_n - x^*\|\Big\}. \end{aligned} \tag{3.27}$$

By induction, we have

$$\|x_n - x^*\| \le \|x_0 - x^*\|. \tag{3.28}$$

Thus, we conclude that $\{x_n\}$ is bounded sequence. By using Lemma 2.4 (i), we obtain

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &= \|\alpha_n x_0 + (1 - \alpha_n) z_n - x^*\|^2 \\ &= \|\alpha_n [x_0 - x^*] + (1 - \alpha_n)[z_n - x^*]\|^2 \\ &= \alpha_n \|x_0 - x^*\|^2 + (1 - \alpha_n)\|z_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|x_0 - z_n\|^2 \\ &\le \alpha_n \|x_0 - x^*\|^2 + (1 - \alpha_n)\Big[\|x_n - x^*\|^2 - \big(1 - \lambda^2 L^2\big)\|x_n - y_n\|^2\Big] \\ &\quad - \alpha_n(1 - \alpha_n)\|x_0 - z_n\|^2 \\ &\le \alpha_n \|x_0 - x^*\|^2 + \|x_n - x^*\|^2 - (1 - \alpha_n)\big(1 - \lambda^2 L^2\big)\|x_n - y_n\|^2. \end{aligned} \tag{3.29}$$

The above relationship implies that

$$(1 - \alpha_n)\big(1 - \lambda^2 L^2\big)\|x_n - y_n\|^2 \le \alpha_n \|x_0 - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{3.30}$$

From Lemma 2.4, we have

$$\begin{aligned}
&\left\|x_{n+1}-x^*\right\|^2\\
&=\left\|\alpha_n x_0+(1-\alpha_n)z_n-x^*\right\|^2\\
&=\left\|\alpha_n[x_0-x^*]+(1-\alpha_n)[z_n-x^*]\right\|^2\\
&\le(1-\alpha_n)^2\left\|z_n-x^*\right\|^2+2\alpha_n\langle x_0-x^*,(1-\alpha_n)[z_n-x^*]+\alpha_n[x_0-x^*]\rangle\\
&=(1-\alpha_n)^2\left\|z_n-x^*\right\|^2+2\alpha_n\langle x_0-x^*,x_{n+1}-x^*\rangle\\
&\le(1-\alpha_n)\left\|x_n-x^*\right\|^2+2\alpha_n\langle x_0-x^*,x_{n+1}-x^*\rangle. \qquad (3.31)
\end{aligned}$$

Case 1: Assume that there exists a fixed number $n_1 \in \mathbb{N}$ such that

$$\|x_{n+1}-x^*\| \le \|x_n-x^*\|,\ \forall\, n \ge n_1. \qquad (3.32)$$

Thus, above implies that $\lim_{n\to+\infty}\|x_n-x^*\|$ exists and let $\lim_{n\to+\infty}\|x_n-x^*\|=l$. From (3.30), we obtain

$$\begin{aligned}
&(1-\alpha_n)\big(1-\lambda^2L^2\big)\|x_n-y_n\|^2\\
&\le\alpha_n\|x_0-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2. \qquad (3.33)
\end{aligned}$$

The existence of $\lim_{n\to+\infty}\|x_n-x^*\|=l$ and $\alpha_n\to 0$, we can deduce that

$$\lim_{n\to+\infty}\|x_n-y_n\|=0. \qquad (3.34)$$

Thus, we have

$$\lim_{n\to+\infty}\|y_n-x^*\|=l. \qquad (3.35)$$

It follows that

$$\|z_n-y_n\|=\|y_n+\lambda[\mathcal{G}(x_n)-\mathcal{G}(y_n)]-y_n\|\le\lambda L\|x_n-y_n\|.$$

The above expression implies that

$$\lim_{n\to+\infty}\|z_n-y_n\|=0. \qquad (3.36)$$

It follows that

$$\lim_{n\to+\infty}\|x_n-z_n\|\le\lim_{n\to+\infty}\|x_n-y_n\|+\lim_{n\to+\infty}\|y_n-z_n\|=0. \qquad (3.37)$$

Furthermore, we obtain

$$\begin{aligned}
\left\|x_{n+1}-x_n\right\|&=\left\|\alpha_n x_0+(1-\alpha_n)z_n-x_n\right\|\\
&=\left\|\alpha_n[x_0-x_n]+(1-\alpha_n)[z_n-x_n]\right\|\\
&\le\alpha_n\left\|x_0-x_n\right\|+(1-\alpha_n)\left\|z_n-x_n\right\|. \qquad (3.38)
\end{aligned}$$

It follows that

$$\lim_{n\to+\infty}\|x_{n+1}-x_n\|=0. \qquad (3.39)$$

It is given that $x^* = P_\Omega(x_0)$, we have

$$\langle x_0 - x^*, y - x^* \rangle \le 0, \ \forall\, y \in \Omega. \tag{3.40}$$

Indeed, since $\{x_n\}$ is bounded, we assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in \mathcal{H}$. By Lemma 3.2, we have

$$\begin{aligned}\limsup_{n\to+\infty}\langle x_0 - x^*, x_n - x^* \rangle &= \limsup_{k\to+\infty}\langle x_0 - x^*, x_{n_k} - x^* \rangle \\ &= \langle x_0 - x^*, \hat{x} - x^* \rangle \\ &\le 0. \end{aligned}\tag{3.41}$$

By the use of $\lim_{n\to+\infty}\|x_{n+1} - x_n\| = 0$. We can deduce that

$$\begin{aligned}\limsup_{n\to+\infty}\langle x_0 - x^*, x_{n+1} - x^* \rangle &\le \limsup_{n\to+\infty}\langle x_0 - x^*, x_{n+1} - x_n \rangle \\ &\quad + \limsup_{n\to+\infty}\langle x_0 - x^*, x_n - x^* \rangle \\ &\le 0. \end{aligned}\tag{3.42}$$

By the use of expressions (3.31), (3.42) and Lemma 2.8, we can derive that

$$\|x_n - x^*\| \to 0 \quad \text{as} \quad n \to +\infty.$$

Case 2: Assume there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| \le \|x_{n_{i+1}} - x^*\|, \ \forall\, i \in \mathbb{N}.$$

Thus, by Lemma 2.9, there exists a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to +\infty$, such that

$$\|x_{m_k} - x^*\| \le \|x_{m_{k+1}} - x^*\| \quad \text{and} \quad \|x_k - x^*\| \le \|x_{m_{k+1}} - x^*\|, \ \text{for all } k \in \mathbb{N}. \tag{3.43}$$

As similar to Case 1, expression (3.30) provides that

$$\begin{aligned}&(1 - \alpha_{m_k})\big(1 - \lambda^2 L^2\big)\|x_{m_k} - y_{m_k}\|^2 \\ &\le \alpha_{m_k}\|x_0 - x^*\|^2 + \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2. \end{aligned}\tag{3.44}$$

Due to $\alpha_{m_k} \to 0$, we deduce the following:

$$\lim_{k\to+\infty}\|x_{m_k} - y_{m_k}\| = 0. \tag{3.45}$$

It follows that

$$\|z_{m_k} - y_{m_k}\| = \|y_{m_k} + \lambda[\mathcal{G}(x_{m_k}) - \mathcal{G}(y_{m_k})] - y_{m_k}\| \le \lambda L\|x_{m_k} - y_{m_k}\|.$$

The above expression implies that

$$\lim_{k\to+\infty}\|z_{m_k} - y_{m_k}\| = 0. \tag{3.46}$$

It follows that

$$\lim_{k\to+\infty}\|x_{m_k}-z_{m_k}\|\le\lim_{k\to+\infty}\|x_{m_k}-y_{m_k}\|+\lim_{k\to+\infty}\|y_{m_k}-z_{m_k}\|=0. \tag{3.47}$$

Furthermore, we obtain

$$\begin{aligned}\big\|x_{m_k+1}-x_{m_k}\big\| &= \big\|\alpha_{m_k}x_0+(1-\alpha_{m_k})z_{m_k}-x_{m_k}\big\|\\ &= \big\|\alpha_{m_k}[x_0-x_{m_k}]+(1-\alpha_{m_k})[z_{m_k}-x_{m_k}]\big\|\\ &\le \alpha_{m_k}\big\|x_0-x_{m_k}\big\|+(1-\alpha_{m_k})\big\|z_{m_k}-x_{m_k}\big\|.\end{aligned} \tag{3.48}$$

It follows that

$$\lim_{k\to+\infty}\|x_{m_k+1}-x_{m_k}\|=0. \tag{3.49}$$

We use the same argument as in Case 1, which is as follows:

$$\limsup_{k\to+\infty}\langle x_0-x^*,x_{m_k+1}-x^*\rangle\le 0. \tag{3.50}$$

Now, using expressions (3.31), we have

$$\begin{aligned}\big\|x_{m_k+1}-x^*\big\|^2 &\le (1-\alpha_{m_k})\big\|x_{m_k}-x^*\big\|^2+2\alpha_{m_k}\langle x_0-x^*,x_{m_k+1}-x^*\rangle\\ &\le (1-\alpha_{m_k})\big\|x_{m_k+1}-x^*\big\|^2+2\alpha_{m_k}\langle x_0-x^*,x_{m_k+1}-x^*\rangle.\end{aligned} \tag{3.51}$$

It continues from that

$$\big\|x_{m_k+1}-x^*\big\|^2\le 2\langle x_0-x^*,x_{m_k+1}-x^*\rangle. \tag{3.52}$$

Thus, (3.42) and (3.52) we obtain

$$\|x_{m_k+1}-x^*\|^2\to 0,\ \text{as } k\to+\infty. \tag{3.53}$$

It implies that

$$\lim_{k\to+\infty}\|x_k-x^*\|^2\le\lim_{k\to+\infty}\|x_{m_k+1}-x^*\|^2\le 0. \tag{3.54}$$

Consequently, $x_n\to x^*$. This completes the proof. □

Next, we introduce the first variant of Algorithm 1 in which the constant step size λ is chosen adaptively and thus produced a sequence $\{\lambda_n\}$ that does not require the knowledge of the Lipschitz-type constant L.

Algorithm 2.

Step 0: Choose $x_1\in\mathcal{C}$, $\lambda_1>0$, $\mu\in(0,1)$ and $\{\alpha_n\}\subset(0,1)$ satisfies the following conditions:

$$\lim_{n\to+\infty}\alpha_n=0\ \text{ and }\ \sum_{n=1}^{+\infty}\alpha_n=+\infty.$$

Step 1: Compute

$$y_n = P_{\mathcal{C}}(x_n - \lambda \mathcal{G}(x_n)).$$

If $x_n = y_n$, then STOP and y_n is a solution. Otherwise, go to **Step 2**.

Step 2: Compute

$$z_n = y_n + \lambda\big[\mathcal{G}(x_n) - \mathcal{G}(y_n)\big].$$

Step 3: Compute

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) z_n.$$

Step 4: Compute

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|x_n - y_n\|}{\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\|}\right\} & \text{if} \quad \mathcal{G}(x_n) - \mathcal{G}(y_n) \neq 0, \\ \lambda_n & \quad\textit{otherwise.} \end{cases} \tag{3.55}$$

Set $n = n + 1$ and go back to **Step 1**.

Lemma 3.4. *The sequence $\{\lambda_n\}$ generated by* (3.55) *is decreasing monotonically and converges to $\lambda > 0$.*

Proof. It is given that $\mathcal{G}$ is Lipschitz-continuous with constant $L > 0$. Let $\mathcal{G}(x_n) \neq \mathcal{G}(y_n)$ such that

$$\begin{aligned} \frac{\mu\|x_n - y_n\|}{\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\|} &\geq \frac{\mu\|x_n - y_n\|}{L\|x_n - y_n\|} \\ &\geq \frac{\mu}{L}. \end{aligned} \tag{3.56}$$

The above expression implies that $\lim_{n\to+\infty} \lambda_n = \lambda$. □

Lemma 3.5. *Suppose that $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ satisfies conditions* $(\mathcal{G}1)$-$(\mathcal{G}4)$ *and sequence $\{x_n\}$ generated by Algorithm 2. Then, we have*

$$\|x_{n+1} - x^*\|^2 \leq \big\|x_n - x^*\big\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\big\|x_n - y_n\big\|^2.$$

Next, we introduce the second variant of Algorithm 1 in which the constant step size λ is chosen adaptively and thus produced a sequence $\{\lambda_n\}$ that does not require the knowledge of the Lipschitz-like constants L. In this case, the step size sequence $\{\lambda_n\}$ is not monotone.

Algorithm 3.

Step 0: Choose $x_1 \in \mathcal{C}$, $\lambda_1 > 0$, $\mu \in (0, 1)$ and select a nonnegative real sequence $\{\varphi_n\}$ such that $\sum_{n=1}^{+\infty} \varphi_n < +\infty$. Moreover, choose $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions:

$$\lim_{n\to+\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{+\infty} \alpha_n = +\infty.$$

Step 1: Compute

$$y_n = P_{\mathcal{C}}(x_n - \lambda \mathcal{G}(x_n)).$$

If $x_n = y_n$, then STOP and y_n is a solution. Otherwise, go to **Step 2**.

Step 2: Compute

$$z_n = y_n + \lambda\big[\mathcal{G}(x_n) - \mathcal{G}(y_n)\big].$$

Step 3: Compute

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n) z_n.$$

Step 4: Compute

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n + \varphi_n, \frac{\mu\|x_n - y_n\|}{\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\|}\right\} & \text{if} \quad \mathcal{G}(x_n) - \mathcal{G}(y_n) \neq 0, \\ \lambda_n + \varphi_n & \textit{otherwise.} \end{cases} \tag{3.57}$$

Set $n = n+1$ and go back to **Step 1**.

Lemma 3.6. *The sequence* $\{\lambda_n\}$ *generated by* (3.57) *is convergent to* λ *and also satisfy the following inequality*

$$\min\left\{\frac{\mu}{L}, \lambda_0\right\} \leq \lambda \leq \lambda_0 + P, \quad \textit{where} \quad P = \sum_{n=1}^{+\infty} \varphi_n.$$

Proof. Due to the Lipschitz continuity of a mapping $\mathcal{G}$ there exists a fixed number $L > 0$. Consider that $\mathcal{G}(x_n) - \mathcal{G}(y_n) \neq 0$ such that

$$\begin{aligned} \frac{\mu\|x_n - y_n\|}{\|\mathcal{G}(x_n) - \mathcal{G}(y_n)\|} &\geq \frac{\mu\|x_n - y_n\|}{L\|x_n - y_n\|} \\ &\geq \frac{\mu}{L}. \end{aligned} \tag{3.58}$$

By using mathematical induction on the definition of λ_{n+1}, we have

$$\min\left\{\frac{\mu}{L}, \lambda_0\right\} \leq \lambda_n \leq \lambda_0 + P.$$

Let

$$[\lambda_{n+1} - \lambda_n]^+ = \max\big\{0, \lambda_{n+1} - \lambda_n\big\}$$

and

$$[\lambda_{n+1} - \lambda_n]^- = \max\big\{0, -(\lambda_{n+1} - \lambda_n)\big\}.$$

From the definition of $\{\lambda_n\}$, we have

$$\sum_{n=1}^{+\infty}[\lambda_{n+1} - \lambda_n]^+ = \sum_{n=1}^{+\infty} \max\big\{0, \lambda_{n+1} - \lambda_n\big\} \leq P < +\infty. \tag{3.59}$$

That is, the series $\sum_{n=1}^{+\infty}[\lambda_{n+1} - \lambda_n]^+$ is convergent.

Next we need to prove the convergence of $\sum_{n=1}^{+\infty}(\lambda_{n+1}-\lambda_n)^-$.

Let $\sum_{n=1}^{+\infty}[\lambda_{n+1}-\lambda_n]^- = +\infty$. Due to the reason that

$$\lambda_{n+1}-\lambda_n = (\lambda_{n+1}-\lambda_n)^+ - (\lambda_{n+1}-\lambda_n)^-,$$

we have

$$\lambda_{k+1}-\lambda_0 = \sum_{n=0}^{k}(\lambda_{n+1}-\lambda_n) = \sum_{n=0}^{k}[\lambda_{n+1}-\lambda_n]^+ - \sum_{n=0}^{k}[\lambda_{n+1}-\lambda_n]^-. \tag{3.60}$$

By allowing $k \to +\infty$ in (3.60), we have $\lambda_k \to -\infty$ as $k \to +\infty$. This is a contradiction. Due to the convergence of the series $\sum_{n=0}^{k}[\lambda_{n+1}-\lambda_n]^+$ and $\sum_{n=0}^{k}[\lambda_{n+1}-\lambda_n]^-$ taking $k \to +\infty$ in (3.60), we obtain $\lim_{n\to+\infty}\lambda_n = \lambda$. This completes the proof. □

Remark 3.7. (i) Three extragradient-type methods are established to find an approximate solution of variational inequalities involving semistrictly quasimonotone and Lipschitz-continuous operators in a real Hilbert space.

(ii) A strongly convergent result, corresponding to the proposed algorithms have been proved.

(iii) It is important to note that these methods have been used fixed, monotonic and non-monotonic step size rules that use operator value rather than the Lipschitz constant of an operator.

4. Numerical Illustrations

The computational results of the proposed schemes are described in this section, in contrast to some related work in the literature and also in the analysis of how variations in control parameters affect the numerical effectiveness of the proposed algorithms. All computations are done in MATLAB R2018b and run on HP i- 5 Core(TM)i5-6200 8.00 GB (7.78 GB usable) RAM laptop.

Example 4.1. Consider that $\mathcal{H} = l_2$ is a real Hilbert space with sequences of real numbers satisfying the following condition

$$\|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2 + \cdots < +\infty. \tag{4.1}$$

Assume that $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ is defined by

$$G(x) = (5 - \|x\|)x, \ \forall\, x \in \mathcal{H},$$

where $C = \{x \in \mathcal{H} : \|x\| \leq 3\}$. It is easy to see that $\mathcal{G}$ is weakly sequentially continuous on $\mathcal{H}$ and $VI(\mathcal{C}, \mathcal{G}) = \{0\}$. For any $x, y \in \mathcal{H}$, we have

$$\begin{aligned}\big\|\mathcal{G}(x) - \mathcal{G}(y)\big\| &= \big\|(5 - \|x\|)x - (5 - \|y\|)y\big\| \\ &= \big\|5(x-y) - \|x\|(x-y) - (\|x\| - \|y\|)y\big\| \\ &\leq 5\|x-y\| + \|x\|\|x-y\| + \big|\|x\| - \|y\|\big|\|y\| \\ &\leq 5\|x-y\| + 3\|x-y\| + 3\|x-y\| \\ &\leq 11\|x-y\|. \end{aligned} \tag{4.2}$$

Hence $\mathcal{G}$ is L-Lipschitz continuous with $L = 11$. For any $x, y \in \mathcal{H}$, let $\big\langle \mathcal{G}(x), y - x\big\rangle > 0$ such that

$$(5 - \|x\|)\big\langle x, y - x\big\rangle > 0.$$

Since $\|x\| \leq 3$ implies that

$$\big\langle x, y - x\big\rangle > 0.$$

Thus, we have

$$\begin{aligned}\big\langle \mathcal{G}(y), y - x\big\rangle &= (5 - \|y\|)\big\langle y, y - x\big\rangle \\ &\geq (5 - \|y\|)\big\langle y, y - x\big\rangle - (5 - \|y\|)\big\langle x, y - x\big\rangle \\ &\geq 2\|x - y\|^2 \geq 0. \end{aligned} \tag{4.3}$$

Thus, we shown that $\mathcal{G}$ is quasimonotone on $\mathcal{C}$. Let $x = (\frac{5}{2}, 0, 0, \cdots, 0, \cdots)$ and $y = (3, 0, 0, \cdots, 0, \cdots)$ such that

$$\big\langle \mathcal{G}(x) - \mathcal{G}(y), x - y\big\rangle = (2.5 - 3)^2 < 0.$$

A projection on the set C is computed explicitly as follows:

$$P_C(x) = \begin{cases} x, & \text{if } \|x\| \leq 3, \\ \frac{3x}{\|x\|}, & \text{otherwise.} \end{cases}$$

The control conditions have been taken as follows:

$$\lambda = \frac{1}{2L}, \quad \lambda_0 = \frac{5}{11}, \quad \mu = 0.44, \quad \varphi_n = \frac{100}{(n+3)^2}.$$

TABLE 1. Numerical results values for Example 4.1.

	Number of Iterations		
x_0	**Algorithm 3**	**Algorithm 3**	**Algorithm 3**
$(3, 3, \cdots, 3_{10000}, 0, 0, \cdots)$	102	91	89
$(1, 2, \cdots, 10000, 0, 0, \cdots)$	112	94	99
$(7, 7, \cdots, 7_{100000}, 0, 0, \cdots)$	117	101	110
$(20, 20, \cdots, 20_{100000}, 0, 0, \cdots)$	129	109	115

Table 2. Numerical results values for Example 4.1.

	Elapsed time in seconds		
x_0	**Algorithm 3**	**Algorithm 3**	**Algorithm 3**
$(3, 3, \cdots, 3_{10000}, 0, 0, \cdots)$	15.658362	13.575637	14.0057768
$(1, 2, \cdots, 10000, 0, 0, \cdots)$	16.274758	14.019125	15.0084828
$(7, 7, \cdots, 7_{100000}, 0, 0, \cdots)$	18.564383	17.476847	17.5384939
$(20, 20, \cdots, 20_{100000}, 0, 0, \cdots)$	22.657362	19.475593	21.5701981

Remark 4.2. (i) We observe from numerical results of Example 4.1 that our proposed Algorithm 3 is efficient and easy to implement for both finite and infinite dimensional spaces, see Tables 1 and 2.

(ii) The performance is better both in CPU time and the number of iterations for Example 4.1 (see Tables 1 and 2).

(iii) We also observe that different choice of initial points x_0 has significant effect of the CPU (time) and number of iterations.

Conclusion

In this study, we considered three strong convergence results for variational inequalities problem involving semistrictly quasimonotone and Lipschitz continuous monotone operator, but the Lipschitz constant is unknown. We modify the extragradient method with a natural step size rule. The strong convergence result is demonstrated without any provision of additional projections.

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