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A POWER SERIES ASSOCIATED WITH THE GENERALIZED HYPERGEOMETRIC FUNCTIONS WITH THE UNIT ARGUMENT WHICH ARE INVOLVED IN BELL POLYNOMIALS

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Abstract. There have been provided a surprisingly large number of summation formulae for generalized hypergeometric functions and series incorporating a variety of elementary and special functions in their various combinations. In this paper, we aim to consider certain generalized hypergeometric function ${}_3F_2$ with particular arguments, through which a number of summation formulas for ${}_{p+1}F_p(1)$ are provided. We then establish a power series whose coefficients are involved in generalized hypergeometric functions with unit argument. Also, we demonstrate that the generalized hypergeometric functions with unit argument mentioned before may be expressed in terms of Bell polynomials. Further, we explore several special instances of our primary identities, among numerous others, and raise a problem that naturally emerges throughout the course of this investigation.

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1. Introduction and preliminaries

The Pochhammer symbol $(\alpha)_\nu$ $(\alpha,\, \nu \in \mathbb{C})$ is defined, in terms of Gamma function Γ (see, e.g., [35, p. 2 and p. 5]), by

$$\begin{aligned}(\alpha)_\nu &= \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)} \quad \big(\alpha+\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-,\, \nu \in \mathbb{C}\setminus\{0\};\, \alpha \in \mathbb{C}\setminus\mathbb{Z}_0^-,\, \nu=0\big)\\ &= \begin{cases} 1 & \big(\nu=0,\, \alpha \in \mathbb{C}\setminus\mathbb{Z}_0^-\big),\\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (\nu=n\in\mathbb{N},\, \alpha\in\mathbb{C}),\end{cases}\end{aligned} \tag{1.1}$$

it being accepted that $(0)_0 = 1$ (see, e.g., [8, 29, 30, 31, 32]). Here and throughout, let $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of complex numbers, integers, and positive integers, respectively, and also let $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}\setminus\mathbb{N}$. Among numerous identities involving the Pochhammer symbol, we recall

$$(\alpha)_{m+n} = (\alpha)_m\,(\alpha+m)_n \quad (m,\, n \in \mathbb{N}_0,\, \alpha\in\mathbb{C}) \tag{1.2}$$

and

$$(\alpha)_{mn} = m^{mn}\prod_{j=1}^{m}\left(\tfrac{\alpha+j-1}{m}\right)_n \quad (m\in\mathbb{N},\, n\in\mathbb{N}_0,\, \alpha\in\mathbb{C})\,. \tag{1.3}$$

The generalized hypergeometric series (or function) ${}_pF_q$ $(p,\, q\in\mathbb{N}_0)$, which is a natural generalization of the Gaussian hypergeometric series ${}_2F_1$, is defined by (see, e.g., [3, 4, 5, 17, 33, 35, 36, 37])

$$\begin{aligned}{}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p\,;\\ \beta_1,\,\ldots,\,\beta_q\,;\end{matrix}\; z\right] &= \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}\\ &= {}_pF_q(\alpha_1,\,\ldots,\,\alpha_p;\,\beta_1,\,\ldots,\,\beta_q;\,z).\end{aligned} \tag{1.4}$$

Here and elsewhere an empty product is interpreted as 1, and it is assumed that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1,\, \ldots,\, \beta_q$ take on complex values, provided that

$$(\beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-;\; j=1,\,\ldots,q). \tag{1.5}$$

Then, if a numerator parameter is a negative integer or zero, the ${}_pF_q$ series terminates.

With none of the numerator and denominator parameters being zero or a negative integer, the ${}_pF_q$ in (1.4)

(i) diverges for all $z\in\mathbb{C}\setminus\{0\}$, if $p > q+1$;

(ii) converges for all $z\in\mathbb{C}$, if $p\le q$;

(iii) converges for $|z|<1$ and diverges for $|z|>1$ if $p=q+1$;

(iv) converges absolutely for $|z|=1$, if $p=q+1$ and $\Re(\varpi)>0$;

(v) converges conditionally for $|z| = 1$ $(z \neq 1)$, if $p = q+1$ and $-1 < \Re(\varpi) \leqq 0$;

(vi) diverges for $|z| = 1$, if $p = q+1$ and $\Re(\varpi) \leqq -1$,

where

$$\varpi := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j. \tag{1.6}$$

Recall Pfaff-Kummer transformation (see, e.g., [3, p. 68], [4, p. 284], [33, p. 60], [35, p. 67], [37, p. 33])

$$\begin{gathered} {}_2F_1\,[\alpha,\, \beta\,;\, \gamma\,;\, z] = (1-z)^{-\alpha}\,{}_2F_1\left[\alpha,\, \gamma-\beta\,;\, \gamma\,;\, \tfrac{-z}{1-z}\right] \\ \left(\gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ |\arg(1-z)| \leq \pi-\epsilon\ (0<\epsilon<\pi);\ |z|<1,\ |z/(1-z)|<1\right). \end{gathered} \tag{1.7}$$

As in the restrictions in (1.7), throughout this paper, whenever a multiple valued function appears, its principal value is assumed to be taken. Recall the representations of $\arctan z$ and $\arcsin z$ in terms of ${}_2F_1$ (see, e.g., [3, p. 64], [23, p. 259], [33, p. 71], [35, p. 67]):

$$\begin{gathered} \tan^{-1}(\sqrt{z}) = \sqrt{z}\,{}_2F_1\left[1,\, \tfrac{1}{2}\,;\, \tfrac{3}{2}\,;\, -z\right] \\ \left(|z|<1,\ |\arg(z)|<\pi,\ |\arg(1\pm i\sqrt{z})|<\pi\right) \end{gathered} \tag{1.8}$$

and

$$\begin{gathered} \sin^{-1}(\sqrt{z}) = \sqrt{z}\,{}_2F_1\left[\tfrac{1}{2},\, \tfrac{1}{2}\,;\, \tfrac{3}{2}\,;\, z\right] \\ \left(|z|<1,\ |\arg(z)|<\pi,\ |\arg(1\pm \sqrt{z})|<\pi\right). \end{gathered} \tag{1.9}$$

Recall the Kampé de Fériet series (see, e.g., [36, p. 27]) defined by

$$\begin{aligned} &F^{p:q;k}_{\ell:m;n}\left[\begin{matrix} (a_p): & (b_q); & (c_k); \\ (\alpha_\ell): & (\beta_m); & (\gamma_n); \end{matrix}\ x,\, y\right] \\ &= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\ \prod_{j=1}^{q}(b_j)_r\ \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\ \prod_{j=1}^{m}(\beta_j)_r\ \prod_{j=1}^{n}(\gamma_j)_s}\ \frac{x^r}{r!}\ \frac{y^s}{s!}, \end{aligned} \tag{1.10}$$

where, for convergence,

(i) $p+q < \ell+m+1$, $p+k < \ell+n+1$, $|x| < \infty$, $|y| < \infty$

or

(ii) $p+q = \ell+m+1$, $p+k = \ell+n+1$, and

$$\begin{cases} |x|^{1/(p-\ell)} + |y|^{1/(p-\ell)} < 1, & \text{if } p > \ell, \\ \max\{|x|, |y|\} < 1, & \text{if } p \leq \ell. \end{cases}$$

We recall Faà di Bruno's formula, which is a generalization of the chain rule to higher derivatives (see, e.g., [26, p. 5], [42]; for prehistory of this formula, [15], [21]):

$$\frac{d^n}{dx^n} f(g(x)) = \sum \frac{n!}{m_1! m_2! \cdots m_n!} \cdot f^{(k)}(g(x)) \cdot \prod_{j=1}^{n} \left(\frac{g^{(j)}(x)}{j!} \right)^{m_j}, \tag{1.11}$$

where the sum is taken over all nonnegative integers $m_1,\ m_2, \ldots, m_n$ that satisfy

$$m_1 + m_2 + \cdots + m_n = k$$

and

$$m_1 + 2m_2 + \cdots + nm_n = n.$$

We recall the partial or incomplete exponential Bell polynomials which are a triangular array of polynomials given by (see, e.g., [1, 9, 11, 16, 25, 43], [14, p. 133 et seq.])

$$\begin{aligned} &B_{n,k}\left(x_1, x_2, \ldots, x_{n-k+1}\right) \\ &= \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \end{aligned} \tag{1.12}$$

where the sum is taken over all sequences $j_1,\ j_2,\ j_3,\ \ldots,\ j_{n-k+1}$ of non-negative integers such that these two conditions are satisfied:

$$\begin{aligned} &j_1 + j_2 + \cdots + j_{n-k+1} = k, \\ &j_1 + 2j_2 + 3j_3 + \cdots + (n-k+1) j_{n-k+1} = n. \end{aligned}$$

The sum

$$B_n\left(x_1, \ldots, x_n\right) = \sum_{k=1}^{n} B_{n,k}\left(x_1, x_2, \ldots, x_{n-k+1}\right) \tag{1.13}$$

is called the nth complete exponential Bell polynomial.

Combining the terms with the same value of $m_1 + m_2 + \cdots + m_n = k$ and noticing that m_j has to be zero for $j > n-k+1$, the Faà di Bruno's formula (1.11) is expressed in terms of Bell polynomials $B_{n,k}\left(x_1, \ldots, x_{n-k+1}\right)$:

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k}\left(g'(x),\ g^{(2)}(x), \ldots, g^{(n-k+1)}(x)\right). \tag{1.14}$$

The Gaussian hypergeometric function ${}_2F_1$ and the generalized hypergeometric functions ${}_pF_q$ have been turned out to have a variety of applications in a wide range of research subjects. Also a number of elementary functions as well as diverse special functions and polynomials are found to be expressed in terms of ${}_2F_1$ and ${}_pF_q$ as (for example) (1.8) and (1.9). Accordingly these functions and their numerous generalizations, (for instance) (1.10), have been presented and investigated.

In this paper, by using a set of identities for ${}_3F_2$ with the argument in (2.1) and another similar one, we aim to present certain summation identities for ${}_{p+1}F_p(1)$ which are found to be Saalschützian. Then we establish a power series whose coefficients are involved in generalized hypergeometric functions with unit argument as in Theorem 4.1. We also show that the generalized hypergeometric functions with unit argument in the first equality of (4.2) can be expressed in terms of Bell polynomials as in Theorem 5.1. Further we consider some special cases of our main identities, among numerous ones, and pose a problem which naturally arises amid present investigation.

2. Certain identities for ${}_3F_2$ with special arguments

In the literature we choose to recall some identities for ${}_3F_2(\mu)$ with the argument μ given by

$$\mu = \tfrac{-27z}{4(1-z)^3} \tag{2.1}$$

and another very similar argument. Here we omit the detailed restrictions for each identity which can be easily recovered from those in Section 1.

We begin by recalling the following formula (see [28, p. 533, Entry 7.4.3-12]):

$${}_3F_2\left[\begin{matrix}\frac{1}{2},\ \frac{2}{3},\ \frac{4}{3}\,; \\ \frac{3}{2},\ \frac{3}{2}\,;\end{matrix}\quad -z\right] = \frac{3\sqrt{3}}{2\sqrt{z}}\,\tan^{-1} x, \tag{2.2}$$

where

$$4z = 27\,x^2\,(1+x)^2\,, \tag{2.3}$$

and presenting its several equivalent ones. Here we note that the condition (2.3) may be a misprint and is corrected as

$$4z = 27\,x^2\left(1+x^2\right)^2. \tag{2.4}$$

With this condition (2.4), expressing (2.2) in terms of the variable x and then replacing x by z gives

$$\begin{aligned}{}_3F_2\left[\begin{matrix}\frac{1}{2},\ \frac{2}{3},\ \frac{4}{3}\,; \\ \frac{3}{2},\ \frac{3}{2}\,;\end{matrix}\quad -\tfrac{27\,z^2(1+z^2)^2}{4}\right] &= \frac{\tan^{-1} z}{z\,(1+z^2)} \\ &= \frac{1}{z\,(1+z^2)}\sin^{-1}\left(\frac{z}{\sqrt{1+z^2}}\right) \\ &= \left(1+z^2\right)^{-1}\,{}_2F_1\left[\tfrac{1}{2},\,1\,;\,\tfrac{3}{2}\,;\,-z^2\right] \\ &= (1+z^2)^{-\frac{3}{2}}\,{}_2F_1\left[\tfrac{1}{2},\,\tfrac{1}{2}\,;\,\tfrac{3}{2}\,;\,\tfrac{z^2}{1+z^2}\right],\end{aligned} \tag{2.5}$$

where the third and fourth equalities follow from (1.8) and (1.7), respectively. Here the second equality follows from (1.7), (1.8) and (1.9).

Replacing z by $\sqrt{\frac{z}{1-z}}$ in (2.5), we obtain

$$\begin{aligned} {}_3F_2\left[\begin{matrix} \frac{1}{2},\ \frac{2}{3},\ \frac{4}{3}\,; \\ \frac{3}{2},\ \frac{3}{2}\,; \end{matrix}\ \frac{-27z}{4(1-z)^3}\right] &= \frac{(1-z)^{\frac{3}{2}}}{\sqrt{z}}\ \tan^{-1}\left(\sqrt{\tfrac{z}{1-z}}\right) \\ &= \frac{(1-z)^{\frac{3}{2}}}{\sqrt{z}}\ \sin^{-1}(\sqrt{z}) \\ &= (1-z)\,{}_2F_1\left[\tfrac{1}{2},\,1\,;\,\tfrac{3}{2}\,;\,\tfrac{-z}{1-z}\right] \\ &= (1-z)^{\frac{3}{2}}\,{}_2F_1\left[\tfrac{1}{2},\,\tfrac{1}{2}\,;\,\tfrac{3}{2}\,;\,z\right]. \end{aligned} \tag{2.6}$$

Note that the second equality of (2.6) is a known identity (see, e.g., [10, p. 590, Entry 8.1.2-12]).

Remark 2.1. We recall some works to deal with ${}_pF_q$ of a similar argument or the same argument as in the left member of (2.6):

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} a,\ a+\frac{1}{3},\ a+\frac{2}{3}\,; \\ b,\ 3a-b+\frac{3}{2}\,; \end{matrix}\ \frac{27z^2}{4(1-z)^3}\right] \\ &= (1-z)^{3a}\,{}_3F_2\left[\begin{matrix} 3a,\ b-\frac{1}{2},\ 3a-b+1\,; \\ 2b-1,\ 6a-2b+2\,; \end{matrix}\ 4z\right] \end{aligned} \tag{2.7}$$

in [6, Eq. (4.06)] (see also [10, p. 590, Entry 8.1.2-5]), [19, Eqs. (5.6) and (5.7)], [22, Eq. (3.27)], [27, Eq. (2.4)];

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} a,\ a+\frac{1}{3},\ a+\frac{2}{3}\,; \\ b,\ 3a-b+\frac{3}{2}\,; \end{matrix}\ \frac{-27z}{4(1-z)^3}\right] \\ &= (1-z)^{3a}\,{}_3F_2\left[\begin{matrix} 3a,\ -3a+2b-1,\ 3a-2b+2\,; \\ b,\ 3a-b+\frac{3}{2}\,; \end{matrix}\ \frac{z}{4}\right] \end{aligned} \tag{2.8}$$

in [6, Eq. (4.05)] (see also [19, Eqs. (5.3)], [22, Eq. (3.25)]);

$${}_3F_2\left[\begin{matrix} \frac{1}{2},\ \frac{5}{6},\ \frac{7}{6}\,; \\ \frac{3}{2},\ \frac{3}{2}\,; \end{matrix}\ \frac{-27z}{4(1-z)^3}\right] = (1-z)^{\frac{3}{2}}\,{}_2F_1\left[\tfrac{1}{2},\,\tfrac{1}{2}\,;\,\tfrac{3}{2}\,;\,\tfrac{z}{4}\right], \tag{2.9}$$

which is a special case of (2.8) when $a=\frac{1}{2}$ and $b=\frac{3}{2}$.

3. Certain summation formulas for ${}_pF_q(1)$

In this section, we offer certain interesting summation formulas for terminating generalized hypergeometric functions which are deducible from some identities in Section 2 as in the following theorem.

Theorem 3.1. *Let* $n,\ \ell \in \mathbb{N}_0$. *Then*

$$ {}_5F_4\left[\begin{matrix} -n,\ \frac{2}{3},\ \frac{4}{3},\ \frac{3+2n}{4},\ \frac{5+2n}{4}\,; \\ \frac{3}{2},\ \frac{3}{2},\ \frac{5}{6},\ \frac{7}{6}\,; \end{matrix}\ 1\right] = \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n} = \left(\frac{1}{1+2n}\right)^2; \tag{3.1}$$

$$\begin{aligned} &{}_5F_4\left[\begin{matrix} -n,\ -\frac{1}{2}-n,\ -\frac{1}{2}-n,\ \frac{1}{6}-n,\ -\frac{1}{6}-n\,; \\ \frac{1}{3}-n,\ -\frac{1}{3}-n,\ \frac{1}{4}-\frac{3}{2}n,\ -\frac{1}{4}-\frac{3}{2}n\,; \end{matrix}\ 1\right] \\ &= (-1)^n\,\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{5}{6}\right)_n\left(\frac{7}{6}\right)_n}{\left(\frac{2}{3}\right)_n\left(\frac{4}{3}\right)_n\left(\frac{3}{4}+\frac{n}{2}\right)_n\left(\frac{5}{4}+\frac{n}{2}\right)_n}; \end{aligned}\tag{3.2}$$

$$\begin{aligned} &{}_6F_5\left[\begin{matrix} -n,\ 1,\ \frac{2}{3}+\ell,\ \frac{4}{3}+\ell,\ \frac{3+6\ell+2n}{4},\ \frac{5+6\ell+2n}{4}\,; \\ 1+\ell,\ \frac{3}{2}+\ell,\ \frac{3}{2}+\ell,\ \frac{5}{6}+\ell,\ \frac{7}{6}+\ell\,; \end{matrix}\ 1\right] \\ &= \left(\frac{1}{(1+2\ell+2n)^2} - \sum_{j=0}^{\ell-1}\frac{(-n-\ell)_j\left(\frac{2}{3}\right)_j\left(\frac{4}{3}\right)_j\left(\frac{3+2\ell+2n}{4}\right)_j\left(\frac{5+2\ell+2n}{4}\right)_j}{j!\left(\frac{3}{2}\right)_j\left(\frac{3}{2}\right)_j\left(\frac{5}{6}\right)_j\left(\frac{7}{6}\right)_j}\right) \\ &\quad\times\frac{\ell!\left(\frac{3}{2}\right)_\ell\left(\frac{3}{2}\right)_\ell\left(\frac{5}{6}\right)_\ell\left(\frac{7}{6}\right)_\ell}{(-n-\ell)_\ell\left(\frac{2}{3}\right)_\ell\left(\frac{4}{3}\right)_\ell\left(\frac{3+2\ell+2n}{4}\right)_\ell\left(\frac{5+2\ell+2n}{4}\right)_\ell}; \end{aligned}\tag{3.3}$$

$$ {}_6F_5\left[\begin{matrix} -n,\ 1,\ \frac{5}{3},\ \frac{7}{3},\ \frac{9+2n}{4},\ \frac{11+2n}{4}\,; \\ 2,\ \frac{5}{2},\ \frac{5}{2},\ \frac{11}{6},\ \frac{13}{6}\,; \end{matrix}\ 1\right] = \frac{315\,(2+n)}{2\,(5+2n)\,(7+2n)\,(3+2n)^2}; \tag{3.4}$$

$$\begin{gathered} {}_3F_2\left[\begin{matrix} -\frac{n}{2},\ -\frac{n}{2}+\frac{1}{2},\ 3a+n\,; \\ b,\ 3a-b+\frac{3}{2}\,; \end{matrix}\ 1\right] = \frac{4^n\,\left(b-\frac{1}{2}\right)_n\,(3a-b+1)_n}{(2b-1)_n\,(6a-2b+2)_n} \\ \left(n\in\mathbb{N}_0;\ b,\,3a-b+\tfrac{3}{2}\in\mathbb{C}\setminus\mathbb{Z}_0^-\right); \end{gathered}\tag{3.5}$$

$$\begin{gathered} {}_3F_2\left[\begin{matrix} -n,\ \frac{3a+n}{2},\ \frac{3a+n+1}{2}\,; \\ b,\ 3a-b+\frac{3}{2}\,; \end{matrix}\ 1\right] = \frac{(-3a+2b-1)_n\,(3a-2b+2)_n}{4^n\,(b)_n\,\left(3a-b+\frac{3}{2}\right)_n} \\ \left(n\in\mathbb{N}_0;\ b,\,3a-b+\tfrac{3}{2}\in\mathbb{C}\setminus\mathbb{Z}_0^-\right); \end{gathered}\tag{3.6}$$

$$ {}_3F_2\left[\begin{matrix} -n,\ \frac{3+2n}{4},\ \frac{5+2n}{4}\,; \\ \frac{3}{2},\ \frac{3}{2}\,; \end{matrix}\ 1\right] = \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{4^n\,\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n} = \frac{1}{4^n}\left(\frac{1}{1+2n}\right)^2. \tag{3.7}$$

Proof. We choose to use the fourth equality of (2.6):

$$ {}_2F_1\left[\tfrac{1}{2},\ \tfrac{1}{2};\ \tfrac{3}{2};\ z\right] = (1-z)^{-\frac{3}{2}}\ {}_3F_2\left[\begin{array}{l} \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{4}{3};\\ \quad\ \ \tfrac{3}{2},\ \tfrac{3}{2}; \end{array} \frac{-27z}{4(1-z)^3}\right]. \tag{3.8} $$

Expanding both sides of (3.8) as Maclaurin series, we get

$$ \begin{aligned} \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_m}{m!\left(\frac{3}{2}\right)_m} z^m &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{4}{3}\right)_n}{n!\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n} \left(-\tfrac{27}{4}\right)^n z^n (1-z)^{-\frac{3}{2}-3n} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}+3n\right)_m}{m!} \frac{\left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{4}{3}\right)_n}{n!\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n} \left(-\tfrac{27}{4}\right)^n z^{m+n}. \end{aligned} $$

Using a series rearrangement technique in the last double series, we obtain

$$ \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_m}{m!\left(\frac{3}{2}\right)_m} z^m = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{\left(\frac{3}{2}+3n\right)_{m-n}}{(m-n)!} \frac{\left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{4}{3}\right)_n}{n!\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n} \left(-\tfrac{27}{4}\right)^n z^m, $$

which, upon equating the coefficients of z^m, gives

$$ \frac{\left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_m}{\left(\frac{3}{2}\right)_m} = \sum_{n=0}^{m} \frac{m!\left(\frac{3}{2}+3n\right)_{m-n}}{(m-n)!} \frac{\left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{4}{3}\right)_n}{n!\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n} \left(-\tfrac{27}{4}\right)^n. $$

Employing the following known identities ($k,\ n,\ N \in \mathbb{N}_0;\ \alpha \in \mathbb{C}$):

$$ (-n)_k = \begin{cases} \frac{(-1)^k\, n!}{(n-k)!} & (0 \leq k \leq n), \\ 0 & (k > n) \end{cases} \tag{3.9} $$

and (see, e.g., [34, p. 239, Eq. (I.11)])

$$ (\alpha + kn)_{N-n} = \frac{(\alpha)_N\,(\alpha+N)_{(k-1)n}}{(\alpha)_{kn}}, \tag{3.10} $$

we obtain

$$ \frac{\left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_m}{\left(\frac{3}{2}\right)_m \left(\frac{3}{2}\right)_m} = \sum_{n=0}^{m} (-m)_n \frac{\left(\frac{3}{2}+m\right)_{2n}}{\left(\frac{3}{2}\right)_{3n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{4}{3}\right)_n}{n!\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n} \left(\tfrac{27}{4}\right)^n. \tag{3.11} $$

Using (1.3) on the right member of (3.11), we have

$$ \frac{\left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_m}{\left(\frac{3}{2}\right)_m \left(\frac{3}{2}\right)_m} = \sum_{n=0}^{m} \frac{(-m)_n}{n!} \frac{\left(\frac{2}{3}\right)_n \left(\frac{4}{3}\right)_n}{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n} \frac{\left(\frac{3}{4}+\frac{m}{2}\right)_n \left(\frac{5}{4}+\frac{m}{2}\right)_n}{\left(\frac{5}{6}\right)_n \left(\frac{7}{6}\right)_n}. $$

In view of (1.4), interchanging the roles of m and n, the last identity is the same as (3.1). This completes the proof (3.1).

In view of the identity [37, p. 42, Eq. (3)], it is easy to see that the identities (3.1) and (3.2) are equivalent.

Replacing n by $n + \ell$ on both sides of (3.1), and simplifying the resulting identity, we obtain (3.3).

The equation (3.4) is a particular case of (3.3) when $\ell = 1$. Using (2.7), (2.8) and (2.9), respectively, the proofs of (3.5), (3.6) and (3.7) would run parallel with that of (3.1). Its details are omitted. □

Remark 3.2. It is interesting to observe, incidentally, that each of the left and leftmost members of the identities in Theorem 3.1 is Saalschützian. Any ${}_{p+1}F_p$ which the sum of its denominator parameters exceeds the sum of its numerator parameters by unity is called Saalschützian. A transformation between a nearly-poised ${}_4F_3(1)$ series and a Saalschützian ${}_5F_4(1)$ series, which is due to Whipple [39, Eq. (3.5)] (see also [40, Eqs. (6.4) and (6.5)], [6, Eq. (6.3)], [7, Eq. (2.12)]), is recorded and noted in [34, p. 65, Eq. (2.4.2.3)]. Yet it is found that the above-mentioned Whipple's transformation cannot be applied to (3.1). Gessel [18] made a systematic use of an extension of the WZ method [41], with the help of a Maple program, to provide a number of terminating hypergeometric series identities which includes ten ${}_5F_4(1)$ series identities. It is checked that any one of the ten ${}_5F_4(1)$ series identities cannot match or yield (3.1).

The equations (3.6) and (3.7) may be obtained by setting, respectively,

$$(\alpha,\, \beta,\, \gamma) = \left(\tfrac{3a+n}{2},\, \tfrac{3a+n+1}{2},\, b\right)$$

and

$$(\alpha,\, \beta,\, \gamma) = \left(\tfrac{3+2n}{4},\, \tfrac{5+2n}{4},\, \tfrac{3}{2}\right)$$

in the following well-known Saalschützian ${}_3F_2(1)$ series identity (see, e.g., [33, p. 87, Theorem 29]):

$$ {}_3F_2\left[\begin{matrix} -n,\ \alpha,\ \beta\,; \\ \gamma,\ 1-\gamma+\alpha+\beta-n\,; \end{matrix}\ 1\right] = \frac{(\gamma-\alpha)_n\,(\gamma-\beta)_n}{(\gamma)_n\,(\gamma-\alpha-\beta)_n}, \tag{3.12}$$

where $n \in \mathbb{N}_0$ and α, β, γ are independent of n.

4. Series involving terminating ${}_{q+1}F_q$

In Theorem 4.1, we establish a formula for certain series associated with terminating ${}_{p+\ell+1}F_{p+\ell}(1)$, which can be summed to a single ${}_{p+1}F_p(\nu)$ with the argument ν given by

$$\nu = \tfrac{-\ell^\ell z}{(\ell-1)^{\ell-1}(1-z)^\ell} \quad \left(\ell \in \mathbb{N} \setminus \{1\}\right). \tag{4.1}$$

The case $\ell = 3$ of (4.1) reduces to the argument (2.1). Then we consider some particular cases as in corollaries.

Theorem 4.1. *Let $\ell \in \mathbb{N}\setminus\{1\}$, $p \in \mathbb{N}_0$, $a_1, \ldots, a_{p+1}, \mu \in \mathbb{C}$, and $b_1, \ldots, b_p \in \mathbb{C}\setminus\mathbb{Z}_0^-$. Also let $|\arg(1-z)| < \pi$, $|z| < 1$, and $\frac{|z|}{|1-z|^\ell} < \frac{(\ell-1)^{\ell-1}}{\ell^\ell}$. Then*

$$\begin{aligned}
&(1-z)^{-\mu}\,{}_{p+1}F_p\left[\begin{matrix} a_1,\ a_2, \ldots,\ a_p,\ a_{p+1}; \\ b_1,\ b_2, \ldots, b_p\,; \end{matrix}\ \frac{-\ell^\ell z}{(\ell-1)^{\ell-1}(1-z)^\ell}\right] \\
&= \sum_{n=0}^{\infty} \frac{(\mu)_n z^n}{n!}\,{}_{p+\ell+1}F_{p+\ell}\left[\begin{matrix} -n,\ a_1,\ \ldots,\ a_{p+1},\ \Delta(\ell-1;\mu+n)\,; \\ b_1,\ \ldots,\ b_p, \Delta(\ell;\mu)\,; \end{matrix}\ 1\right] \\
&= \sum_{k=0}^{\ell-1} \frac{(\mu)_k\, z^k}{k!} F^{\ell:1;p+1}_{0:\ell;p+\ell} \\
&\qquad \left[\begin{matrix} \Delta(\ell;\mu+k): & 1; & (a_{p+1}); \\ -: & \Delta(\ell;1+k); & (b_p), \Delta(\ell;\mu); \end{matrix}\ z^\ell,\ -\frac{\ell^\ell z}{(\ell-1)^{(\ell-1)}}\right].
\end{aligned} \tag{4.2}$$

Here and elsewhere, conventionally, let (α_p) and $\Delta(\ell;\alpha)$ denote the horizontal arrays of p and ℓ parameters, respectively, given by

$$\alpha_1,\ \alpha_2\ \cdots,\ \alpha_p$$

and

$$\frac{\alpha}{\ell},\ \frac{\alpha+1}{\ell}, \cdots,\ \frac{\alpha+\ell-1}{\ell},$$

where $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{C}$.

Proof. Let

$$\Lambda := (1-z)^{-\mu}\,{}_{p+1}F_p\left[\begin{matrix} a_1,\ a_2, \ldots,\ a_p,\ a_{p+1}; \\ b_1,\ b_2, \ldots, b_p\,; \end{matrix}\ \frac{-\ell^\ell z}{(\ell-1)^{\ell-1}(1-z)^\ell}\right]. \tag{4.3}$$

Expanding the ${}_{p+1}F_p$ in the series as in (1.4), we have

$$\begin{aligned}
\Lambda &= (1-z)^{-\mu} \sum_{m=0}^{\infty} \frac{(a_1)_m\,(a_2)_m \cdots (a_p)_m (a_{p+1})_m}{(b_1)_m (b_2)_m \cdots (b_p)_m\, m!} \frac{(-1)^m \ell^{\ell m} z^m}{(\ell-1)^{(\ell-1)m\,(1-z)^{\ell m}}} \\
&= \sum_{m=0}^{\infty} \frac{(a_1)_m\,(a_2)_m \cdots (a_p)_m (a_{p+1})_m}{(b_1)_m (b_2)_m \cdots (b_p)_m\, m!} \frac{(-1)^m \ell^{\ell m} z^m}{(\ell-1)^{(\ell-1)m}}\,(1-z)^{-(\mu+\ell m)}.
\end{aligned}$$

Using the well-known binomial theorem

$$(1-z)^{-a} = \sum_{n=0}^{\infty} (a)_n \frac{z^n}{n!} = {}_1F_0\,[a\,; -; z] \quad (|z|<1,\ a \in \mathbb{C}), \tag{4.4}$$

we obtain

$$\Lambda = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\mu+\ell m)_n}{n!} \frac{(a_1)_m\,(a_2)_m \cdots (a_p)_m (a_{p+1})_m}{(b_1)_m (b_2)_m \cdots (b_p)_m\, m!} \frac{(-1)^m \ell^{\ell m}}{(\ell-1)^{(\ell-1)m}}\, z^{n+m}. \tag{4.5}$$

From (1.2) and (1.3), we find

$$(\mu+\ell m)_n = \frac{(\mu)_{n+\ell m}}{(\mu)_{\ell m}} = \frac{(\mu)_{n+\ell m}}{\ell^{\ell m} \prod_{j=1}^{\ell} \left(\frac{\mu+j-1}{\ell}\right)_m}. \tag{4.6}$$

Employing (4.6) in (4.5) gives

$$\Lambda = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a_1)_m\,(a_2)_m \cdots (a_p)_m (a_{p+1})_m}{(b_1)_m (b_2)_m \cdots (b_p)_m \prod_{j=1}^{\ell} \left(\frac{\mu+j-1}{\ell}\right)_m} \frac{(-1)^m\,(\mu)_{n+\ell m}}{(\ell-1)^{(\ell-1)m}} \frac{z^{n+m}}{n!\,m!}. \tag{4.7}$$

Using a series rearrange technique, we find

$$\Lambda = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(a_1)_m\,(a_2)_m \cdots (a_p)_m (a_{p+1})_m}{(b_1)_m (b_2)_m \cdots (b_p)_m \prod_{j=1}^{\ell} \left(\frac{\mu+j-1}{\ell}\right)_m} \times \frac{(-1)^m\,(\mu)_{n+(\ell-1)m}}{(\ell-1)^{(\ell-1)m}} \frac{z^n}{(n-m)!\,m!}.$$

Employing the identity (3.9), and the identities (1.2) and (1.3), we get

$$\Lambda = \sum_{n=0}^{\infty} \frac{(\mu)_n\, z^n}{n!} \sum_{m=0}^{n} \frac{(-n)_m\,(a_1)_m\,(a_2)_m \cdots (a_p)_m (a_{p+1})_m}{m!\,(b_1)_m (b_2)_m \cdots (b_p)_m} \frac{\prod_{j=1}^{\ell-1} \left(\frac{\mu+n+j-1}{\ell-1}\right)_m}{\prod_{j=1}^{\ell} \left(\frac{\mu+j-1}{\ell}\right)_m},$$

which, upon expressing in terms of (1.4), leads to the first equality of (4.2). Consider the following relation

$$\sum_{n=0}^{\infty} \phi(n) = \sum_{k=0}^{\ell-1}\sum_{n=0}^{\infty} \phi(\ell n+k), \tag{4.8}$$

where $\phi : \mathbb{N}_0 \to \mathbb{C}$ is a function. Then apply (4.8) to the index n in (4.7) and expand the resulting identity with the aid of (1.3) to express the last resulting identity in terms of (1.10) to give the second equality of (4.2). □

The case $p = 2$ and $\ell = 3$ of Theorem 4.1 is recorded in the following corollary.

Corollary 4.2. *Let $n \in \mathbb{N}_0$, a_1, a_2, a_3, $\mu \in \mathbb{C}$, and b_1, $b_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Also let $|\arg(1-z)| < \pi$, $|z| < 1$ and $|z| < \frac{4}{27}|1-z|^3$. Then*

$$\begin{aligned}
&(1-z)^{-\mu}{}_3F_2\left[\begin{matrix} a_1,\ a_2,\ a_3\,; \\ b_1,\ b_2\,; \end{matrix}\ \frac{-27z}{4(1-z)^3}\right] \\
&= \sum_{n=0}^{\infty} \frac{(\mu)_n z^n}{n!}\, {}_6F_5\left[\begin{matrix} -n,\ a_1,\ a_2,\ a_3,\ \frac{\mu+n}{2},\ \frac{\mu+n+1}{2}\,; \\ b_1,\ b_2,\ \frac{\mu}{3},\ \frac{\mu+1}{3},\ \frac{\mu+2}{3}\,; \end{matrix}\ 1\right].
\end{aligned} \tag{4.9}$$

Further, among numerous particular cases of (4.9) (see the identities in Section 2), we consider just one identity as in the following corollary.

Corollary 4.3. *Let $\mu \in \mathbb{C}$. Then*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\mu)_n \kappa^n}{n!}\, {}_6F_5\left[\begin{matrix} -n,\ \frac{1}{2},\ 1,\ 1,\ \frac{\mu+n}{2},\ \frac{\mu+n+1}{2}\,; \\ \frac{3}{2},\ \frac{3}{2},\ \frac{\mu}{3},\ \frac{\mu+1}{3},\ \frac{\mu+2}{3}\,; \end{matrix}\ 1\right] \\
&= (1-\kappa)^{-\mu}\left(\frac{\pi^2}{6} - 3\ln^2\left(\frac{\sqrt{5}-1}{2}\right)\right),
\end{aligned} \tag{4.10}$$

where

$$\kappa := 1 + \frac{3}{\sqrt[3]{2}}\left\{\left(-1+\sqrt{5}\right)^{\frac{1}{3}} - \left(1+\sqrt{5}\right)^{\frac{1}{3}}\right\}. \tag{4.11}$$

Proof. Let $\frac{-27z}{4(1-z)^3} = \omega$. Then we find the cubic equation

$$(z-1)^3 - \frac{27}{4\omega}(z-1) - \frac{27}{4\omega} = 0. \tag{4.12}$$

If $\omega \in \mathbb{R} \setminus \{0\}$ and $1 - 1/\omega \geq 0$, then one of the three solutions of the cubic equation (4.12) is given as

$$z = 1 + \frac{3}{2}\left\{\left(\frac{1}{\omega} + \frac{1}{|\omega|}\sqrt{1-\frac{1}{\omega}}\right)^{\frac{1}{3}} + \left(\frac{1}{\omega} - \frac{1}{|\omega|}\sqrt{1-\frac{1}{\omega}}\right)^{\frac{1}{3}}\right\}. \tag{4.13}$$

Setting $\omega = -\frac{1}{4}$ in (4.13) gives the κ in (4.11). Recall the identity [28, p. 551, Entry 2]

$${}_3F_2\left[\begin{matrix} \frac{1}{2},\ 1,\ 1\,; \\ \frac{3}{2},\ \frac{3}{2}\,; \end{matrix}\ -\frac{1}{4}\right] = \frac{\pi^2}{6} - 3\ln^2\left(\frac{\sqrt{5}-1}{2}\right). \tag{4.14}$$

Considering (4.11), (4.14), and $a_1 = \frac{1}{2}$, $a_2 = a_3 = 1$ and $b_1 = b_2 = \frac{3}{2}$ in (4.9), we obtain (4.10). $\square$

5. An expression in terms of Bell polynomials

We show that the generalized hypergeometric functions with the unit argument in the right member of the first equality in (4.2) are expressed in terms of Bell polynomials, asserted in Theorem 5.1.

Theorem 5.1. *Let $\ell \in \mathbb{N}\setminus\{1\}$, $p,\ n \in \mathbb{N}_0$, $a_1,\ \ldots,\ a_{p+1},\ \mu \in \mathbb{C}$ and $b_1,\ \ldots,\ b_p \in \mathbb{C}\setminus\mathbb{Z}_0^-$. Then*

$$\begin{aligned}
&{}_{p+\ell+1}F_{p+\ell}\left[\begin{matrix} -n,\ a_1,\ \ldots,\ a_{p+1},\ \Delta(\ell-1;\mu+n)\,; \\ b_1,\ \ldots,\ b_p, \Delta(\ell;\mu)\,; \end{matrix}\ 1\right] \\
&= \sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^r}{(1-\mu-n)_r}\sum_{\eta=1}^{r}\frac{\prod\limits_{j=1}^{p+1}(a_j)_\eta}{\prod\limits_{j=1}^{p}(b_j)_\eta}\left(\tfrac{-\ell^\ell}{(\ell-1)^{\ell-1}}\right)^{\eta} \\
&\quad\times B_{r,\eta}\left(1,2\ell,\ldots,(r-\eta+1)(\ell)_{r-\eta}\right) \\
&= \sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^r}{(1-\mu-n)_r}\mathcal{A}_r\left(\ell;a_1,\ \ldots,\ a_{p+1};b_1,\ b_2,\ldots,b_p\right),
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
&\mathcal{A}_r\left(\ell;a_1,\ \ldots,\ a_{p+1};b_1,\ b_2,\ldots,b_p\right) := \mathcal{A}_r\left(\ell;(a)_{p+1};(b)_p\right) \\
&= \sum\left(\frac{r!}{m_1!m_2!\cdots m_r!}\right)\prod_{\nu=1}^{r}\left(\frac{(\ell)_{\nu-1}}{(\nu-1)!}\right)^{m_\nu}\left(\tfrac{-\ell^\ell}{(\ell-1)^{\ell-1}}\right)^{k}\frac{\prod\limits_{j=1}^{p+1}(a_j)_k}{\prod\limits_{j=1}^{p}(b_j)_k},
\end{aligned} \tag{5.2}$$

where the sum is taken over all nonnegative integers $m_1,\ m_2,\ldots,m_r$ that satisfy $m_1+2m_2+\cdots+rm_r=r$ and $k=m_1+m_2+\cdots+m_r$.

Proof. Let

$$f(z) := {}_{p+1}F_p\left[\begin{matrix} a_1,\ a_2,\ldots,\ a_p,\ a_{p+1}; \\ b_1,\ b_2,\ldots,b_p\,; \end{matrix}\ z\right]$$

and

$$g(z) := \tfrac{-\ell^\ell z}{(\ell-1)^{\ell-1}(1-z)^\ell}.$$

Let

$$F(z) := f(g(z)) = {}_{p+1}F_p\left[\begin{matrix} a_1,\ a_2,\ldots,\ a_p,\ a_{p+1}; \\ b_1,\ b_2,\ldots,b_p\,; \end{matrix}\ \tfrac{-\ell^\ell z}{(\ell-1)^{\ell-1}(1-z)^\ell}\right].$$

Then we find that $F(z)$ is analytic in a nonempty neighborhood of the origin. Therefore we have

$$F(z)=\sum_{n=0}^{\infty}\frac{F^{(n)}(0)}{n!}z^n$$

in the neighborhood of the origin. We can obtain

$$g^{(r)}(0)=\tfrac{-\ell^{\ell}}{(\ell-1)^{\ell-1}}\,r\,(\ell)_{r-1}\quad (r\in\mathbb{N}_0)$$

and (see, e.g., [10, p. 86, Entry 1.30.1-1])

$$f^{(r)}(z)=\frac{\prod\limits_{j=1}^{p+1}(a_j)_r}{\prod\limits_{j=1}^{p}(b_j)_r}\,{}_{p+1}F_p\left[\begin{array}{r}a_1+r,\ a_2+r,\ldots,\ a_p+r,\ a_{p+1}+r;\\ b_1+r,\ b_2+r,\ldots,b_p+r\,;\end{array}\ z\right],$$

where $r\in\mathbb{N}_0$.

Now we use Faà Di Bruno's Formula (see, e.g., [26, p. 5] to find

$$\begin{aligned}F^{(r)}(0)&=\frac{d^r}{dx^r}f(g(z))\Big|_{z=0}\\&=\sum\left(\frac{r!}{m_1!m_2!\cdots m_r!}\right)f^{(k)}(g(0))\\&\qquad\times\left(\frac{g'(0)}{1!}\right)^{m_1}\left(\frac{g''(0)}{2!}\right)^{m_2}\cdots\left(\frac{g^{(r)}(0)}{r!}\right)^{m_r},\end{aligned}$$

where the sum is taken over all nonnegative integers $m_1,\ m_2,\ldots,m_r$ that satisfy $m_1+2m_2+\cdots+rm_r=r$ and $k=m_1+m_2+\cdots+m_r$. Using the above higher-order derivative formulas, we obtain

$$\begin{aligned}F^{(r)}(0)&=\sum\left(\frac{r!}{m_1!m_2!\cdots m_r!}\right)f^{(k)}(0)\prod_{\nu=1}^{r}\left(\frac{g^{(\nu)}(0)}{\nu!}\right)^{m_\nu}\\&=\sum\left(\frac{r!}{m_1!m_2!\cdots m_r!}\right)\left(\tfrac{-\ell^{\ell}}{(\ell-1)^{\ell-1}}\right)^{k}\frac{\prod\limits_{j=1}^{p+1}(a_j)_k}{\prod\limits_{j=1}^{p}(b_j)_k}\prod_{\nu=1}^{r}\left(\frac{(\ell)_{\nu-1}}{(\nu-1)!}\right)^{m_\nu}\\&:=\mathcal{A}_r\left(\ell;(a)_{p+1};(b)_p\right).\end{aligned}$$

Use the notation in (4.3) and a series rearrangement technique to give

$$\Lambda=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}z^n\sum_{r=0}^{\infty}\frac{F^{(r)}(0)}{r!}z^r=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(\mu)_{n-r}}{(n-r)!\,r!}F^{(r)}(0)\,z^n.$$

Employing (4.2), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\mu)_n z^n}{n!}\,{}_{p+\ell+1}F_{p+\ell}\left[\begin{matrix}-n,\ a_1,\ \ldots,\ a_{p+1},\ \Delta(\ell-1;\mu+n)\,;\\ b_1,\ \ldots,\ b_p,\Delta(\ell;\mu)\,;\end{matrix}\ 1\right]\\ &=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(\mu)_{n-r}}{(n-r)!\,r!}F^{(r)}(0)\,z^n,\end{aligned}$$

on both sides of which, upon equating the coefficients of z^n and simplifying the resulting identity with the aid of

$$(\mu)_{n-r}=\frac{(-1)^r\,(\mu)_n}{(1-\mu-n)_r}$$

and using (1.14), yields the identity (5.1). □

We provide simple examples of (5.1), in which the corresponding restrictions are adjusted from those in Theorem 5.1 and omitted.

Example 5.2. The case $(p,\ell)=(0,2)$.

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}-n,\ a_1,\ \mu+n\,;\\ \frac{\mu}{2},\ \frac{\mu+1}{2}\,;\end{matrix}\ 1\right]\\ &=\sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^r}{(1-\mu-n)_r}\sum_{\eta=1}^{r}(a_1)_\eta\,(-4)^\eta\\ &\quad\times B_{r,\eta}\left(1,4,\ldots,(r-\eta+1)\cdot(r-\eta+1)!\right).\end{aligned}\tag{5.3}$$

Example 5.3. The case $(p,\ell,a_1)=\left(0,2,-\frac{1}{2}\right)$. The ${}_3F_2$ of (5.3) in this case is Saalschützian in (3.12) when $(\alpha,\beta,\gamma)=\left(-\frac{1}{2},\mu+n,\frac{\mu}{2}\right)$.

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^r}{(1-\mu-n)_r}\sum_{\eta=1}^{r}(-1)^{\eta+1}\,(\eta)_{\eta-1}\\ &\times B_{r,\eta}\left(1,4,\ldots,(r-\eta+1)\cdot(r-\eta+1)!\right)=\frac{\mu+2n}{2\mu}.\end{aligned}\tag{5.4}$$

Example 5.4. The case $(p,\ell,a_1)=(0,2,0)$.

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}\frac{(-1)^r}{(1-\mu-n)_r}\sum_{\eta=1}^{r}(-4)^\eta\\ &\times B_{r,\eta}\left(1,4,\ldots,(r-\eta+1)\cdot(r-\eta+1)!\right)=1.\end{aligned}\tag{5.5}$$

Example 5.5. The case $(p, \ell, \mu) = \left(2, 3, \frac{3}{2}\right)$, $(a_1, a_2, a_3) = \left(\frac{1}{2}, \frac{2}{3}, \frac{4}{3}\right)$, and $b_1 = b_2 = \frac{3}{2}$. Using (3.1), we obtain

$$\begin{aligned} &\sum_{r=0}^{n} \frac{\binom{n}{r}}{\left(n+\frac{3}{2}-r\right)_r} \sum_{\eta=1}^{r} \frac{(-1)^\eta}{\left(\frac{3}{2}\right)_\eta} (2\eta+2)_\eta \\ &\times B_{r,\eta}\left(1, 6, \ldots, (r-\eta+1)\cdot(3)_{r-\eta}\right) = \left(\frac{1}{1+2n}\right)^2. \end{aligned} \tag{5.6}$$

6. Concluding remarks and posing problem

A remarkably large number of summation formulas for ${}_pF_q$ and series containing a variety of elementary and special functions and their diverse combinations have been provided (see, e.g., [2, 3, 4, 5, 10, 12, 13, 17, 20, 24, 28, 33, 34, 35, 37, 38]). In this paper, by starting some identities for ${}_3F_2$ with particular arguments, we gave a number of summation formulas for ${}_pF_q(1)$, each of which is, interestingly, found to be Saalschützian. We also provided a power series involving generalized hypergeometric functions with the unit argument, which can be summed to a single generalized hypergeometric function with the argument in (4.1). Further we showed that the above-mentioned generalized hypergeometric functions with the unit argument are expressed in terms of Bell polynomials. Further, certain particular instances of our main findings were illustrated.

Finally we pose one problem which naturally arises under investigation:
As in Theorem 3.1, find summation formulas for the

$${}_{p+\ell+1}F_{p+\ell}\left[\begin{matrix} -n,\ a_1,\ \ldots,\ a_{p+1},\ \Delta(\ell-1;\mu+n)\,; \\ b_1,\ \ldots,\ b_p, \Delta(\ell;\mu)\,; \end{matrix}\ 1\right] \tag{6.1}$$

in (4.2) by suitably choosing to adjust the involved parameters. Then, whenever a closed-form for (6.1) is found, setting it in (4.2) may provide an interesting series representation of the generalized hypergeometric function corresponding to its left member.

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