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NUMERICAL SOLUTIONS OF NONLINEAR VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS BY USING MADM AND VIM

Ayoob M. Abed[1], Muhammed F. Younis[2] and Ahmed A. Hamoud[3]

[1]Department of Mathematics, Thi Qar Directorates of Education
Ministry of Education, Thi Qar, Iraq
e-mail: ayob.moha75@gmail.com

[2]Department of Mathematics, Thi Qar Directorates of Education
Ministry of Education, Thi Qar, Iraq
e-mail: muhammedfaleh@yahoo.com

[3]Department of Mathematics, Faculty of Education and Science
Taiz University, Taiz-380 015, Yemen
e-mail: ahmed.hamoud@taiz.edu.ye

Abstract. The aim of the current work is to investigate the numerical study of a nonlinear Volterra-Fredholm integro-differential equation with initial conditions. Our approximation techniques modified adomian decomposition method (MADM) and variational iteration method (VIM) are based on the product integration methods in conjunction with iterative schemes. The convergence of the proposed methods have been proved. We conclude the paper with numerical examples to illustrate the effectiveness of our methods.

1. Introduction

Integral and integro-differential equations are important in many applications of applied mathematics. Since many physical problems are modeled by integro-differential equations, the numerical solutions of such integro-differential equations have been highly studied by many authors. In recent

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[0]Corresponding author: Ahmed A. Hamoud (ahmed.hamoud@taiz.edu.ye).

years, numerous works have been focusing on the development of more advanced and efficient methods for integral equations and integro-differential equations, for example [2, 9, 13, 16, 19, 20, 21, 23].

In recent years, many authors focus on the development of numerical and analytical techniques for integro-differential equations. For instance, we can remember the following works. Abbasbandy and Elyas [1] studied some applications on VIM for solving system of nonlinear Volterra integro-differential equations, Alao et al. [3] used ADM and VIM for solving integro-differential equations, Salih and Mehmet [28] applied the Taylor method to solve the linear Volterra-Fredholm integro-differential equations, Mittal and Nigam [25] applied the ADM to approximate solutions for fractional integro-differential equations, and Behzadi et al. [5] solved some class of nonlinear Volterra-Fredholm integro-differential equations by HAM. Moreover, several authors have applied the ADM and VIM to find the approximate solutions of various types of integro-differential equations [4, 6, 7, 8, 14, 15, 25].

In this paper, we consider nonlinear Volterra-Fredholm integro-differential equation of the form:

$$\begin{aligned}\sum_{j=0}^{k}\xi_j(x)u^{(j)}(x) &= f(x)+\lambda_1\int_a^x K_1(x,t)G_1(u(t))dt \\ &\quad +\lambda_2\int_a^b K_2(x,t)G_2(u(t))dt, \qquad (1.1)\end{aligned}$$

with the initial conditions

$$u^{(r)}(a)=b_r, \quad r=0,1,2,\cdots,(k-1), \qquad (1.2)$$

where $u^{(j)}(x)$ is the j^{th} derivative of the unknown function $u(x)$ that will be determined, $K_i(x,t), i=1,2$ are the kernels of the equation, $f(x)$ and $\xi_j(x)$ are analytic functions, G_1 and G_2 are nonlinear functions of u and $a, b, \lambda_1, \lambda_2$, and b_r are real finite constants.

To obtain the approximate solution, we integrating (k)-times in the interval $[a,x]$ with respect to x we obtain,

$$\begin{aligned}u(x) &= L^{-1}\left(\frac{f(x)}{\xi_k(x)}\right)+\sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r+\lambda_1 L^{-1}\left(\int_a^x\frac{K_1(x,t)}{\xi_k(x)}G_1(u(t))dt\right) \\ &\quad +\lambda_2 L^{-1}\left(\int_a^b\frac{K_2(x,t)}{\xi_k(x)}G_1(u(t))dt\right)-\sum_{j=0}^{k-1}L^{-1}\left(\frac{\xi_j(x)}{\xi_k(x)}u_n^{(j)}(x)\right), \qquad (1.3)\end{aligned}$$

where L^{-1} is the multiple integration operator given as follows:

$$L^{-1}(\cdot) = \int_a^x \int_a^x \cdots \int_a^x (\cdot) dt dt \cdots dt \quad (k - times).$$

2. Analysis of the methods

Some powerful methods have been focusing on the development of more advanced and efficient methods for solving integro-differential equations such as the MADM [2, 4, 6, 8, 14, 18, 22, 24, 25, 26, 29] and VIM [3, 7, 11, 12, 10].

2.1. **Modified Adomian Decomposition Method (MADM).** The modified decomposition method will facilitate the computational process and further accelerate the convergence of the series solution. The MADM depends mainly on splitting the function $f(x)$ into two parts.

$$f(x) = f_1(x) + f_2(x). \tag{2.1}$$

From Eq. (1.3) and Eq. (2.1), we obtain

$$\begin{aligned} u(x) &= L^{-1}\left(\frac{f_1(x)}{\xi_k(x)}\right) + L^{-1}\left(\frac{f_2(x)}{\xi_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r \\ &+\lambda_1 L^{-1}\left(\int_a^x \frac{K_1(x,t)}{\xi_k(x)} G_1(u(t))dt\right) \\ &+\lambda_2 L^{-1}\left(\int_a^b \frac{K_2(x,t)}{\xi_k(x)} G_1(u(t))dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{\xi_j(x)}{\xi_k(x)} u_n^{(j)}(x)\right). \end{aligned}$$

We assume

$$G_1(u(x)) = \sum_{n=0}^{\infty} A_n,\ G_2(u(x)) = \sum_{n=0}^{\infty} B_n, \tag{2.2}$$

where $A_n,\ B_n,\ n \geq 0$ are the adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}\Big[\frac{d^n}{d\mu^n} G_1(\sum_{i=0}^{\infty} \mu^i u_i)\Big]\Big|_{\mu=0},\ B_n = \frac{1}{n!}\Big[\frac{d^n}{d\mu^n} G_2(\sum_{i=0}^{\infty} \mu^i u_i)\Big]\Big|_{\mu=0}. \tag{2.3}$$

The adomian polynomials were introduced in [17, 18, 29] as:

$$\begin{aligned} A_0 &= G_1(u_0); \\ A_1 &= u_1 G_1^{'}(u_0); \\ A_2 &= u_2 G_1^{'}(u_0) + \frac{1}{2!} u_1^2 G_1^{''}(u_0); \\ A_3 &= u_3 G_1^{'}(u_0) + u_1 u_2 G_1^{''}(u_0) + \frac{1}{3!} u_1^3 G_1^{'''}(u_0), \ldots \end{aligned}$$

and

$$\begin{aligned}
B_0 &= G_2(u_0);\\
B_1 &= u_1 G_2^{'}(u_0);\\
B_2 &= u_2 G_2^{'}(u_0) + \frac{1}{2!}u_1^2 G_2^{''}(u_0);\\
B_3 &= u_3 G_2^{'}(u_0) + u_1 u_2 G_2^{''}(u_0) + \frac{1}{3!}u_1^3 G_2^{'''}(u_0), ...
\end{aligned}$$

The components $u_0, u_1, u_2, \cdots$ are usually determined recursively by

$$\begin{aligned}
u_0 &= L^{-1}\left(\frac{f_1(x)}{\xi_k(x)}\right) + \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r,\\
u_1 &= L^{-1}\left(\frac{f_2(x)}{\xi_k(x)}\right) + \lambda_1 L^{-1}\left(\int_a^x \frac{K_1(x,t)}{\xi_k(x)}A_0(t)dt\right)\\
&\quad + \lambda_2 L^{-1}\left(\int_a^b \frac{K_2(x,t)}{\xi_k(x)}B_0(t)dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{\xi_j(x)}{\xi_k(x)}u_0^{(j)}(x)\right),\\
u_{n+1} &= \lambda_1 L^{-1}\left(\int_a^x \frac{K_1(x,t)}{\xi_k(x)}A_n(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \frac{K_2(x,t)}{\xi_k(x)}B_n(t)dt\right)\\
&\quad - \sum_{j=0}^{k-1} L^{-1}\left(\frac{\xi_j(x)}{\xi_k(x)}u_n^{(j)}(x)\right),\ n \geq 1.
\end{aligned}$$

Then, $u(x) = \sum_{i=0}^{n} u_i(x)$ as the approximate solution.

2.2. **Variational Iteration Method (VIM).** We consider the following general differential equation:

$$Lu(t) + Nu(t) = f(t),$$

where L is a linear operator, N is a nonlinear operator and $f(t)$ is inhomogeneous term. According to variational iteration method [3, 10, 27], the terms of a sequence $\{u_n\}$ are constructed such that this sequence converges to the exact solution. The terms u_n are calculated by a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)(Lu_n(\tau) + N\tilde{u}(\tau) - f(\tau))d\tau. \tag{2.4}$$

To obtain the approximation solution of IVP (1.1) − (1.2), according to the VIM, the iteration formula (2.4) can be written as follows:

$$\begin{aligned} u_{n+1}(x) &= u_n(x) + L^{-1}\Big[\lambda(x)\Big[\sum_{j=0}^{k}\xi_j(x)u_n^{(j)}(x) - f(x) \\ &\quad -\lambda_1\int_a^x K_1(x,t)G_1(u_n(t))dt - \lambda_2\int_a^b K_2(x,t)G_2(u_n(t))dt\Big]\Big]. \end{aligned}$$

To find the optimal $\lambda(x)$, we proceed as follows:

$$\begin{aligned} \delta u_{n+1}(x) &= \delta u_n(x) + \delta L^{-1}\Big[\lambda(x)\Big[\sum_{j=0}^{k}\xi_j(x)u_n^{(j)}(x) - f(x) \\ &\quad -\lambda_1\int_a^x K_1(x,t)G_1(u_n(t))dt - \lambda_2\int_a^b K_2(x,t)G_2(u_n(t))dt\Big]\Big] \\ &= \delta u_n(x) + \lambda(x)\delta u_n(x) - L^{-1}\Big[\delta u_n(x)\lambda'(x)\Big]. \qquad (2.5) \end{aligned}$$

From Eq.(2.5), the stationary conditions can be obtained as follows:

$$\lambda'(x) = 0, \text{ and } 1 + \lambda(x)|_{x=t} = 0.$$

As a result, the Lagrange multipliers can be identified as $\lambda(x) = -1$, and by substituting in Eq.(2.5), the following iteration formula is obtained:

$$\begin{aligned} u_0(x) &= L^{-1}\Big[\frac{f(x)}{\xi_k(x)}\Big] + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r, \\ u_{n+1}(x) &= u_n(x) - L^{-1}\Big[\sum_{j=0}^{k}\xi_j(x)u_n^{(j)}(x) - f(x) \\ &\quad -\lambda_1\int_a^x K_1(x,t)G_1(u_n(t))dt \\ &\quad -\lambda_2\int_a^b K_2(x,t)G_2(u_n(t))dt\Big], \quad n \geq 0. \qquad (2.6) \end{aligned}$$

The term $\sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r$ is obtained from the initial conditions, $\xi_k(x) \neq 0$. Relation (2.6) will enable us to determine the components $u_n(x)$ recursively for $n \geq 0$. Consequently, the approximation solution may be obtained by using $u(x) = \lim_{n\to\infty} u_n(x)$.

3. Convergence results

In this section, we shall give the convergence results of Eq. (1.1), with the initial condition (1.2) and prove it.

From Eq. (1.3), we assume:

$$L^{-1}\Big[\int_a^x \frac{1}{\xi_k(x)}K_1(x,t)G_1(u_n(t))dt\Big] = \int_a^x \frac{(x-t)^k}{k!\xi_k(x)}K_1(x,t)G_1(u_n(t))dt$$

and

$$\sum_{j=0}^{k-1} L^{-1}\Big[\frac{\xi_j(x)}{\xi_k(x)}\Big]u^{(j)}(x) = \sum_{j=0}^{k-1}\int_a^x \frac{(x-t)^{k-1}\xi_j(t)}{(k-1)!\xi_k(t)}u^{(j)}(t)dt.$$

We set,

$$\Psi(x) = L^{-1}\Big[\frac{f(x)}{\xi_k(x)}\Big] + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r.$$

Before starting and proving the main results, we introduce the following hypotheses:

(H1) There exist constants α, β and $\gamma_j > 0, j = 0, 1, \cdots, k$ such that, for any $u_1, u_2 \in C(J, \mathbb{R})$

$$|G_1(u_1)) - G_1(u_2))| \leq \alpha\,|u_1 - u_2|\,,$$

$$|G_2(u_1) - G_2(u_2)| \leq \beta\,|u_1 - u_2|\,,$$

$$\left|D^j(u_1) - D^j(u_2)\right| \leq \gamma_j\,|u_1 - u_2|\,,$$

we suppose that the nonlinear terms $G_1(u(x))), G_2(u(x)))$ and $D^j(u) = (\frac{d^j}{dx^j})u(x) = \sum_{i=0}^{\infty}\gamma_{i_j}$, ($D^j$ is a derivative operator), $j = 0, 1, \cdots, k$, are Lipschitz continuous.

(H2) Suppose that for all $a \leq t \leq x \leq b$, and $j = 0, 1, \cdots, k$:

$$\left|\frac{\lambda_1(x-t)^k K_1(x,t)}{k!\xi_k(x)}\right| \leq \theta_1, \qquad \left|\frac{\lambda_1(x-t)^k K_1(x,t)}{k!}\right| \leq \theta_2,$$

$$\left|\frac{(x-t)^{k-1}\xi_j(t)}{(k-1)!\xi_k(t)}\right| \leq \theta_3, \qquad \left|\frac{(x-t)^{k-1}\xi_j(t)}{(k-1)!}\right| \leq \theta_4,$$

$$\left|\lambda_2 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big]\right| \leq \theta_5, \qquad \left|\lambda_2 L^{-1}\Big[K_2(x,t)\Big]\right| \leq \theta_6.$$

(H3) There exist three functions $\theta_3^*, \theta_4^*,$ and $\gamma^* \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R}\times\mathbb{R} : 0 \leq t \leq x \leq 1\}$ such that:

$$\theta_3^* = \max|\theta_3|\,,\ \theta_4^* = \max|\theta_4|\,,\ \text{and } \gamma^* = \max|\gamma_j|\,.$$

(H4) $\Psi(x)$ is bounded function for all x in $J = [a, b]$.

Theorem 3.1. *Suppose that* $(H1)-(H4)$*, and if* $0<\psi<1$*, hold, the series solution* $u(x)=\sum_{m=0}^{\infty}u_m(x)$ *and* $\|u_1\|_\infty<\infty$ *obtained by the m-order deformation is convergent, then it converges to the exact solution of the problem* $(1.1)-(1.2)$.

Proof. Denote as $(C[0,1],\|.\|)$ the Banach space of all continuous functions on J, with $|u_1(x)|<\infty$ for all x in J.

First we define the sequence of partial sums $\{s_n\}$, let s_n and s_m be arbitrary partial sums with $n\geq m$. We are going to prove that $s_n=\sum_{i=0}^{n}u_i(x)$ is a Cauchy sequence in this Banach space:

$$\begin{aligned}
\|s_n-s_m\|_\infty &= \max_{\forall x\in J}|s_n-s_m|=\max_{\forall x\in J}\Big|\sum_{i=0}^{n}u_i(x)-\sum_{i=0}^{m}u_i(x)\Big|\\
&= \max_{\forall x\in J}\Big|\sum_{i=m+1}^{n}u_i(x)\Big|\\
&= \max_{\forall x\in J}\Big|\sum_{i=m+1}^{n}\int_a^x\frac{\lambda_1(x-t)^kK_1(x,t)}{k!\xi_k(x)}A_{i-1}dt\\
&\quad+\sum_{i=m+1}^{n}\int_a^b\lambda_2L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big]B_{i-1}dt\\
&\quad-\sum_{j=0}^{k-1}\int_a^x\frac{\xi_j(t)(x-t)^{k-1}}{(k-1)!\xi_k(t)}L_{(i-1)J}dt\Big|\\
&= \max_{\forall x\in J}\Big|\int_a^x\frac{\lambda_1(x-t)^kK_1(x,t)}{k!\xi_k(x)}\Big(\sum_{i=m}^{n-1}A_i\Big)dt\\
&\quad+\int_a^b\lambda_2L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big]\Big(\sum_{i=m}^{n-1}B_i\Big)dt\\
&\quad-\sum_{j=0}^{k-1}\int_a^x\frac{\xi_j(t)(x-t)^{k-1}}{(k-1)!\xi_k(t)}\Big(\sum_{i=m}^{n-1}L_{iJ}dt\Big)\Big|.
\end{aligned}$$

From (2.2), we have

$$\sum_{i=m}^{n-1}A_i=G_1(s_{n-1})-G_1(s_{m-1}),\ \sum_{i=m}^{n-1}B_i=G_2(s_{n-1})-G_2(s_{m-1}),$$
$$\sum_{i=m}^{n-1}L_i=D^j(s_{n-1})-D^j(s_{m-1}).$$

So,

$$\begin{aligned} \|s_n - s_m\|_\infty &= \max_{x\in J}\Big| \int_0^x \frac{\lambda_1 (x-t)^k K_1(x,t)}{k!\xi_k(x)}[G_1(s_{n-1}) - G_1(s_{m-1})]dt \\ &\quad + \int_a^b \lambda_2 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big][G_2(s_{n-1}) - G_2(s_{m-1})]dt \\ &\quad - \sum_{j=0}^{k-1}\int_a^x \frac{\xi_j(t)(x-t)^{k-1}}{(k-1)!\xi_k(t)}[D^j(s_{n-1}) - D^j(s_{m-1})]dt\Big| \\ &\le \max_{x\in J}\int_0^x \Big|\frac{\lambda_1 (x-t)^k K_1(x,t)}{k!\xi_k(x)}\Big|\Big|G_1(s_{n-1}) - G_1(s_{m-1})\Big|dt \\ &\quad + \int_a^b \Big|\lambda_2 L^{-1}\Big[\frac{K_2(x,t)}{\xi_k(x)}\Big]\Big|\Big|G_2(s_{n-1}) - G_2(s_{m-1})\Big|dt \\ &\quad + \sum_{j=0}^{k-1}\int_a^x \Big|\frac{\xi_j(t)(x-t)^{k-1}}{(k-1)!\xi_k(t)}\Big|\Big|D^j(s_{n-1}) - D^j(s_{m-1})\Big|dt. \end{aligned}$$

Let $n = m+1$. Then

$$\begin{aligned} \|s_n - s_m\|_\infty &\le \psi\|s_m - s_{m-1}\|_\infty \\ &\le \psi^2\|s_{m-1} - s_{m-2}\|_\infty \\ &\vdots \\ &\le \psi^m\|s_1 - s_0\|_\infty, \end{aligned}$$

so,

$$\begin{aligned} \|s_n - s_m\|_\infty &\le \|s_{m+1} - s_m\|_\infty + \|s_{m+2} - s_{m+1}\|_\infty + \cdots + \|s_n - s_{n-1}\|_\infty \\ &\le [\psi^m + \psi^{m+1} + \cdots + \psi^{n-1}]\|s_1 - s_0\|_\infty \\ &\le \psi^m[1 + \psi + \psi^2 + \cdots + \psi^{n-m-1}]\|s_1 - s_0\|_\infty \\ &\le \psi^m(\frac{1-\psi^{n-m}}{1-\psi})\|u_1\|_\infty. \end{aligned}$$

Since $0 < \psi < 1$, we have $(1 - \psi^{n-m}) < 1$, then

$$\|s_n - s_m\|_\infty \le \frac{\psi^m}{1-\psi}\|u_1\|_\infty.$$

But $|u_1(x)| < \infty$, so, as $m \longrightarrow \infty$, then $\|s_n - s_m\|_\infty \longrightarrow 0$.

We conclude that $\{s_n\}$ is a Cauchy sequence in $C[0,1]$, therefore $u = \lim_{n\to\infty} u_n$. Then, the series is convergence and the proof is complete. □

Theorem 3.2. *If problem* (1.1)−(1.2) *has a unique solution, then the solution* $u_n(x)$ *obtained from the recursive relation* (2.6) *using VIM converges when* $0 < \phi = (\alpha\theta_5 + \beta\theta_6 + k\gamma^*\theta_4^*)(b-a) < 1$.

Proof. We have from equation (2.6):

$$\begin{aligned} u_{n+1}(x) - u(x) &= u_n(x) - u(x) - \Big(L^{-1}\Big[\sum_{j=0}^{k} \xi_j(x)[u_n^{(j)}(x) - u^{(j)}(x)]\Big] \\ &\quad - L^{-1}\Big[\lambda_1 \int_a^x K_1(x,t)[G_1(u_n(t)) - G_1(u(t))]dt \\ &\quad - L^{-1}\Big[\lambda_2 \int_a^b K_2(x,t)[G_2(u_n(t)) - G_2(u(t))dt]\Big]\Big). \end{aligned}$$

If we set, $\xi_k(x) = 1$, and $W_{n+1}(x) = u_{n+1}(x) - u(x)$, $W_n(x) = u_n(x) - u(x)$ since $W_n(a) = 0$, then

$$\begin{aligned} W_{n+1}(x) &= W_n(x) + \int_a^x \frac{\lambda_1 K_1(x,t)(x-t)^k}{k!}[G_1(u_n(t)) - G_1(u(t))]dt \\ &\quad + \int_a^b \lambda_2 L^{-1}\Big[K_2(x,t)[G_2(u_n(t)) - G_2(u(t))dt]\Big] - \sum_{j=0}^{k-1}\int_a^x \\ &\quad \times \frac{\xi_j(t)(x-t)^{k-1}}{(k-1)!}[D^j(u_n(t)) - D^j(u(t))]dt - (W_n(x) - W_n(a)). \end{aligned}$$

Therefore,

$$\begin{aligned} \Big|W_{n+1}(x)\Big| &\le \int_a^x \Big|\frac{\lambda_1 K_1(x,t)(x-t)^k}{k!}\Big|\Big|W_n\Big|\alpha dt \\ &\quad + \int_a^b \Big|\lambda_2 L^{-1}\Big[\Big|K_2(x,t)\Big|\Big|W_n\Big|\beta dt\Big] \\ &\quad + \sum_{j=0}^{k-1}\int_a^x \Big|\frac{\xi_j(t)(x-t)^{k-1}}{(k-1)!}\Big| \max|\gamma_j|\Big|W_n\Big|dt \\ &\le \Big|W_n\Big|\Big[\int_a^x \alpha\theta_5 dt + \int_a^b \beta\theta_6 dt + \sum_{j=0}^{k-1}\int_a^x \theta_4^* \max|\gamma_j|\Big] \\ &\le |W_n|(\alpha\theta_5 + \beta\theta_6 + k\gamma^*\theta_4^*)(b-a) = |W_n|\phi. \end{aligned}$$

Hence,

$$\|W_{n+1}\| = \max_{\forall x\in J}|W_{n+1}(x)| \le \phi \max_{\forall x\in J}|W_n(x)| = \phi\|W_n\|.$$

Since $0 < \phi < 1$, $\|W_n\| \longrightarrow 0$. So, the series converges and the proof is complete. □

4. Applications

Example 4.1. Consider the following Volterra-Fredholm integro-differential equation.

$$u'(x) + xu(x) = 2x + x^3 - \frac{x^5}{5} - \frac{0.9^7}{7}x + \int_0^x u^2(t)dt + \int_0^{0.9} xu^3(t)dt,$$

with the initial condition $u(0) = u'(0) = 0$ and the the exact solution is $u(x) = x^2$.

Table 1. Numerical Results of the Example 4.1.

x	Exact solution	MDM	VIM
0.1	0.010000	0.016377	0.010024
0.2	0.040000	0.046990	0.040394
0.3	0.090000	0.094713	0.091969
0.4	0.160000	0.148751	0.151274
0.5	0.250000	0.236624	0.243752
0.6	0.360000	0.342563	0.350874
0.7	0.490000	0.478846	0.483681
0.8	0.640000	0.635372	0.630257
0.9	0.810000	0.790145	0.801487

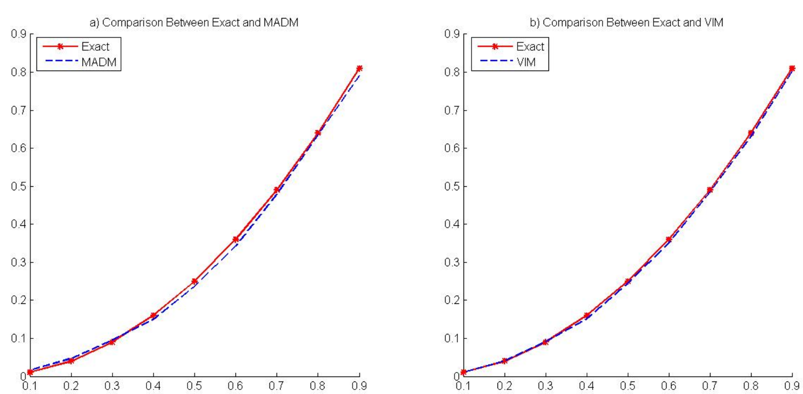

Figure 1. Comparison Between MADM and VIM, for Example 4.1.

Example 4.2. Consider the nonlinear Volterra-Fredholm integro-differential equation:

$$\begin{aligned}(x^3-1)u^{(4)}(x)+(x^2+1)u''(x) &= e^{-x}(x^2+x^3)-\frac{x^2}{2e^2}-(0.130639)x \\ &\quad +\int_0^x [u(t)]^2dt+\int_0^{0.5} xt(1+u^2(t))dt,\end{aligned}$$

with the initial conditions: $u(0)=u''(0)=1,\ u'(0)=u'''(0)=-1$, The exact solution is $u(x)=e^{-x}$.

TABLE 2. Numerical Results of the Example 4.2.

x	Exact solution	MADM	VIM
0.05	0.951229	0.908237	0.948796
0.1	0.904837	0.921381	0.908269
0.2	0.818731	0.796469	0.815826
0.3	0.740818	0.708649	0.739765
0.4	0.670320	0.649382	0.663792
0.5	0.606531	0.579846	0.607912

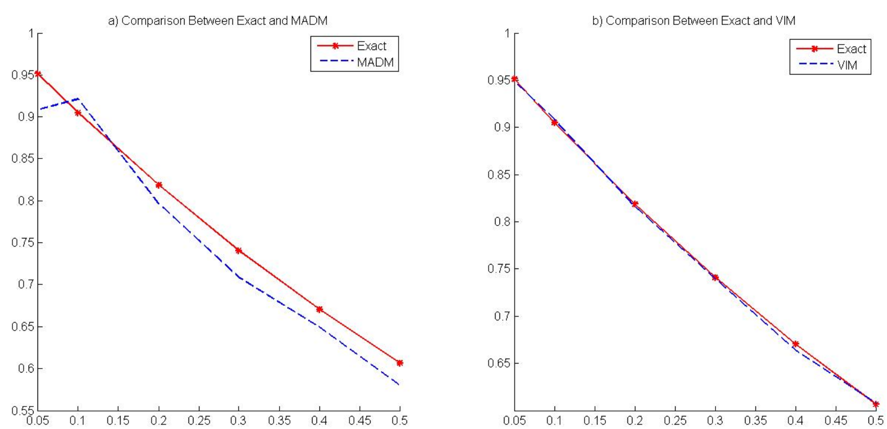

FIGURE 2. Comparison Between MADM and VIM, for Example 4.2.

5. Concluding remarks

In this work, we have examined a few recent familiar numerical methods for solving integro-differential equations. The numerical studies showed that all the methods give highly accurate solutions for given equations. The MADM and the VIM are uncomplicated and comfortable. Despite this, they are not converging to a closed form. One advantage of VIM is that the initial solution can be freely chosen with some unknown parameters. An interesting point about this method is that with few number of iterations. The comparison reveals that although the numerical results of these methods are similar approximately, VIM is the easiest, the most efficient and convenient.

The problem considered in this paper can be generalized to a higher dimension involving a general formulation of fractional derivative with respect to another function. Also, study nonlinear fractional systems of Volterra-Fredholm integro-differential equations with nonlocal conditions is a direction which we are working on.

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