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SETVALUED MIXED QUASI-EQUILIBRIUM PROBLEMS WITH OPERATOR SOLUTIONS

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Abstract. In this paper, we introduce and study generalized mixed operator quasi-equilibrium problems(GMQOEP) in Hausdorff topological vector spaces and prove the existence results for the solution of (GMQOEP) in compact and noncompact settings by employing 1-person game theorems. Moreover, using coercive condition, hemicontinuity of the functions and KKM theorem, we prove new results on the existence of solution for the particular case of (GMQOEP), that is, generalized mixed operator equilibrium problem (GMOEP).

1. Introduction

In 2002, Domokos and Kolumban [6] gave a fascinating analysis of variational inequalities(VI) and vector variational inequalities(VVI) in Banach space settings in terms of variational inequalities with operator solutions(OVVI). They considered (OVVI) to provide unified approach to numerous kinds of (VI) and (VVI) problems in Banach spaces and effectively depicted those problems

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in wider framework of (OVVI). This (OVVI) is also appropriate for the general development in the study of (VI) and (VVI) for abstract spaces. Therefore, the effectiveness of (OVVI) can be served as a source of inspiration for further developments. In a series of papers [15–17], the authors developed the scheme of (OVVI) from single valued case into the setvalued one. For this purpose, the generalized variational inequality with operator solutions (GOVVI) to generalized vector variational inequality (GVVI), and generalized vector quasi-variational inequalities (GVQVI) in a normed space were given.

On the other hand, the equilibrium problem (EP) is extensively studied with the work of Blum and Oettli [3], as a generalization of optimization and variational inequality problems. It turns out equilibrium problems contain, as special cases, other problems such as the fixed point and coincidence point problems, the complementarity problem, Nash equilibrium problem, etc. Kazmi and Raouf [10] introduced the operator equilibrium problems (OEP) with operator solutions and derived a Minty type lemma for this class of (OEP). They used this lemma and KKM theorem to establish the existence results for the solution of (OEP) for single valued map. Salmon [24] proved that the statement of Theorem 2.1 in [10] holds even if we omit the hemicontinuity and $c(f)-$pseudo-monotonicity conditions. Kum and Kim [15] generalized operator equilibrium problem due to Kazmi and Raouf to setvalued quasi-equilibrium problems and obtained an existence results by employing a Park fixed point theorem. Ahmad et al. [2] introduced and studied an operator mixed vector equilibrium problems(OMVEP) which is a combination of an operator vector variational inequality and an operator vector equilibrium problems. They also proved existence result for (OMVEP) by using KKM theorem and vector 0-diagonally quasi-convexity, and another existence result without KKM theorem.

Motivated and inspired by the above research work going on in this direction. In this paper, we prove existence results for the solutions of generalized mixed operator quasi-equilibrium problems(GMQOEP) in compact and noncompact settings by employing 1-person game theorems in Hausdorff topological vector spaces. Moreover, we also prove new result on the existence of solution of particular case of (GMQOEP), that is, generalized mixed operator equilibrium problem (GMOEP) by using coercive condition, hemicontinuity of the functions and KKM theorem. The results presented in this paper are generalization and unification of many known results in the literature, see for example [1, 2, 6, 9, 10, 13–17, 20–24] and the references therein.

Let X and Y be two Hausdorff topological vector spaces and let $L(X,Y)$ be the space of all continuous linear operators from X into Y equipped with the topology of point-wise convergence, and let $K \subseteq L(X,Y)$ be a nonempty convex set. Let $C : K \to 2^Y$ be a setvalued map such that $C(f)$ is a solid closed, convex cone and $0 \notin C(f)$, for all $f \in K$.

We define the ordering relationships on Hausdorff topological vector space Y with respect to cone P as follows: For $A, B \subseteq Y$, we have

$$B - A \subseteq P \Leftrightarrow A \leq B \Leftrightarrow a \leq b,\ \forall a \in A, b \in B,$$

$$B - A \nsubseteq P \Leftrightarrow A \nleq B \Leftrightarrow a \nleq b,\ \forall a \in A, b \in B.$$

If the $intP = \phi$, then the weak ordering in Y is defined as follows:

$$B - A \subseteq intP \Leftrightarrow A < B \Leftrightarrow a < b,\ \forall a \in A, b \in B,$$

$$B - A \nsubseteq intP \Leftrightarrow A \nless B \Leftrightarrow a \nless b,\ \forall a \in A, b \in B.$$

Suppose $T : K \to X$ is mapping and $S : K \to 2^K$ is a setvalued map with nonempty values. For a setvalued map $F : K \times K \to 2^Y$ with $F(f, f) = \{0\}$ for all $f \in K$, the *generalized mixed operator quasi-equilibrium problem(for short, GMOQEP)* is to find $f \in K$ such that

$$f \in cl_K S(f) \text{ and } \langle f - g, T(f)\rangle + F(f, g) \nsubseteq -intC(f),\ \ \forall\, g \in S(f). \tag{1.1}$$

Special cases

(i) If $S(f) = K$, for all $f \in K$, then GMOQEP (1.1) reduces to find $f \in K$ such that

$$\langle f - g, T(f)\rangle + F(f, g) \nsubseteq -intC(f),\ \forall g \in K \tag{1.2}$$

called the *generalized mixed operator equilibrium problem (GMOEP)* which appears to be new.

(ii) If $S(f) = K,\ T \equiv 0$, for all $f \in K$ and $F : K \times K \to Y$ is a single valued map, then GMOQEP (1.1) reduces to find $f \in K$ such that

$$F(f, g) \notin -intC(f),\ \forall g \in K, \tag{1.3}$$

which is called the *weak operator equilibrium problem* studied by Salamon [24].

(iii) If $S(f) = K,$ and $F \equiv 0$, then the GMOQEP (1.1) reduces to *operator variational inequality* considered by Domokos and Kolumban [6].

(iv) If the ordering is not weak, then the problem (1.3) reduces to find $f \in K$ such that

$$F(f, g) \notin -C(f),\ \forall g \in K, \tag{1.4}$$

which is called *operator equilibrium problem* introduced and studied by Kazmi and Raouf [10].

(v) If $F(f,g) = \langle \eta(f,g), T(f) \rangle$, where $T : K \to X$ and $\eta : K \times K \to K$, then OEP (1.4) reduces to find $f \in K$ such that

$$\langle \eta(f,g), T(f) \rangle \notin -C(f), \forall g \in K,$$

which appears to be new. We call it the *operator variational-like inequality problem.*

(vi) If $F(f,g) = \phi(f) - \phi(g)$, where $\phi : K \to Y$, then OEP (1.4) reduces to a problem of finding $f \in K$ such that

$$\phi(f) - \phi(g) \notin -C(f), \ \forall g \in K,$$

which appears to be new and we call it as *operator minimization problem.*

(vii) If $K \subseteq X$, a topological vector space, then the operator equilibrium problem OEP (1.4) reduces to *vector equilibrium problem* studied by Kazmi [11, 12], and Lee et al. [18].

2. Preliminaries

Now we give some definitions and preliminary results which are needed in the latter sections.

Definition 2.1. Let $T : D \subseteq L(X,Y) \to 2^Y$ be a setvalued mapping. Then

(i) T is said to be *upper semicontinuous* on D, if for each $f \in D$ and any open set V in Y containing $T(f)$, there exists an open neighborhood U of $f \in D$ such that $T(f) \subseteq V$ for all $g \in U$.

(ii) The inverse T^{-1} of T is the setvalued mapping from $\mathcal{R}(T)$, range of T to D defined by $f \in T^{-1}(y)$ if and only if $y \in T(f)$.

(iii) The graph of T, denoted by $\mathcal{G}(T)$, and is defined as

$$G(T) = \{(f,y) \in D \times Y : f \in D, \ y \in T(f)\}.$$

Definition 2.2. Let K be a nonempty convex subset of a vector space X. A function $f : K \to \mathbb{R}$ is said to be *quasiconvex*, if for all $x, y \in K$ and $\lambda \in (0,1)$,

$$f(x + \lambda(y - x)) \le \max\{f(x), f(y)\}.$$

Definition 2.3. Let $C : K \to 2^Y$ be a setvalued map such that for each $f \in K$, $C(f)$ is a setvalued map with convex cone values in Y. A setvalued bifunction $F : K \times K \to 2^Y$ is called weakly $C(f)$-*quasiconvex*, if for all $f, g_1, g_2 \in K$ and $\lambda \in [0,1]$, $g_\lambda = \lambda g_1 + (1-\lambda) g_2$, we have

$$F(f, g_\lambda) \subseteq F(f, g_1) - intC(f)$$

or

$$F(f, g_\lambda) \subseteq F(f, g_2) - intC(f).$$

Definition 2.4. Let $C : K \to 2^Y$ be a setvalued map such that $C(f)$ is a closed convex cone with $0 \notin C(f)$ in Y for all $f \in K$. Then the setvalued map $F : K \times K \to 2^Y$ is said to be weakly $C(f)-$*pseudomonotone*, if

$$F(f,g) \nsubseteq -intC(f)$$

then

$$F(g,f) \nsubseteq intC(f).$$

Definition 2.5. Let $C : K \to 2^Y$ be a setvalued map such that $C(f)$ is a closed convex cone with $0 \notin C(f)$ in Y for all $f \in K$. Then the setvalued map $F : K \times K \to 2^Y$ is said to be *hemicontinuous*, if the function $\lambda \to F(f+\lambda g, h)$ is upper semicontinuous at 0^+ for all $f, g, h \in K$, as a mapping from $\mathbb{R}_+$ into Y.

Definition 2.6. Let B be a convex compact subset of K. Then the setvalued mapping $F : K \times K \to 2^Y$ is said to be *weakly coercive with respect to* B, if there exists $g_0 \in B$ such that for all $f \in K \setminus B$,

$$\langle f - g_0, T(f)\rangle + F(f, g_0) \subseteq -intC(f).$$

To prove the existence of solution of GMOQEP (1.1), we shall use the following theorems which are the special cases of Theorem 2 in Ding et al. [4] and Theorem 2.1 in Ding et al. [5].

Now, we need concept of some special setvalued maps as:

Let $S, T : K \to 2^{L(X,Y)}$ be setvalued maps. Then the setvalued maps $cl_K S, co(S),\ S \cap T : K \to 2^{L(X,Y)}$ are defined, respectively as

$$(cl_K S)(f) = cl_K S(f), (coS)(f) = coS(f)$$

and

$$(S \cap T)(f) = S(f) \cap T(f),\ \forall f \in K,$$

where $coS(f)$ denotes the convex hull of S(f).

Theorem 2.7. ([4]) *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Suppose that $S,\ cl_X S,\ P : K \to 2^K$ are setvalued maps such that for each $x \in K,\ S(x)$ is nonempty convex set, for each $y \in K,\ S^{-1}(y)$ is open set in $K,\ cl_X S$ is upper semicontinuous, for each $x \in K,\ x \notin coP(x)$ and for each $y \in K,\ P^{-1}(y)$ is open in K. Then there exists $x^* \in K$ such that $x^* \in cl_K S(x^*)$ and $S(x^*) \cap P(x^*) = \phi$.*

Theorem 2.8. ([5]) *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space X and D be a nonempty compact subset of K. Suppose that $A,\ P : K \to 2^D$ and $cl_X A : K \to 2^K$ are setvalued maps such* *that for each $x \in K$, $A(x)$ is a nonempty convex set, for each $y \in K$, $A^{-1}(y)$ is open set in K, $cl_X(A)$ is upper semicontinuous, for each $x \in K$, $x \notin coP(x)$ and for each $y \in D$, $P^{-1}(y)$ is open in K. Then there exists $x^* \in K$ such that $x^* \in cl_K A(x^*)$ and $A(x^*) \cap P(x^*) = \phi$.*

3. Result in compact setting

In this section, we prove an existence result for GMOQEP (1.1) in compact setting.

Theorem 3.1. *Let $K \subseteq L(X,Y)$ be a nonempty, compact and convex set. Let $C : K \to 2^Y$ be a setvalued map such that for each $f \in K$, $C(f)$ is a solid convex closed cone with apex at the origin of Y. Let $(Y, C(f))$ be a ordering topological vector space. Let $F : K \times K \to 2^Y$ and $cl_K S : K \to 2^K$ be setvalued maps such that for each $f \in K$, $S(f)$ is nonempty convex, for each $g \in K$, $S^{-1}(g)$ is open in K and $cl_K S$ is upper-semicontinuous. Assume that*

(i) *F is weakly $C(f)-$quasiconvex,*
(ii) *graph of $W(f) = Y \setminus (-intC(f))$ is closed, for all $f \in K$,*
(iii) *for each $g \in K$, $F(.,g)$ is upper semicontinuous with compact values on K,*
(iv) *$\langle f - g, T(f)\rangle \subseteq -intC(f)$, $\forall g \in K$,*
(v) *$F(f,f) \subseteq -intC(f)$, $\forall f \in K$.*

Then GMOQEP has a solution.

Proof. For each $f \in K$, we define a setvalued map $P : K \to 2^K$ by

$$P(f) = \{g \in K : \langle f - g, T(f)\rangle + F(f,g) \subseteq -intC(f)\}.$$

We show that $f \notin coP(f)$ for all $f \in K$. Suppose that $f \in coP(f)$, for some $f \in K$. Then there exist $f_0 \in K$ such that $f_0 \in coP(f_0)$. This implies that f_0 can be expressed as

$$f_0 = \sum_{i \in I} \lambda_i g_i \text{ with } \lambda_i \geq 0, \ \sum_{i \in I} \lambda_i = 1, \ i = 1, 2, \cdots, n,$$

where $\{g_i : i \in \mathbb{N}\}$ is a finite subset of K, $I \subseteq \mathbb{N}$ is an arbitrary nonempty subset. Therefore

$$\langle f_0 - g_i, T(f_0)\rangle + F(f_0, g_i) \subseteq -intC(f_0), \ \forall\, i = 1, 2, \cdots, n.$$

Since F is $C(f)$-quasiconvex, we have

$$\begin{aligned} \langle f_0 - g_i, T(f_0)\rangle + F(f_0, g_i) &\subseteq \langle f_0 - g_i, T(f_0)\rangle + F(f_0, f_0) + intC(f_0) \\ &\subseteq -intC(f_0) - intC(f_0) + intC(f_0) \\ &\nsubseteq -intC(f_0), \ \forall\, i = 1, 2, \cdots, n. \end{aligned}$$

This implies that

$$\langle f_0 - g_i, T(f_0)\rangle + F(f_0, g_i) \nsubseteq -intC(f_0),\ \forall\, i = 1, 2, \cdots, n,$$

which is a contradiction. Hence $f \notin coP(f)$, for all $f \in K$.

Now we show that $P^{-1}(g)$ is open in K. That is, $K \setminus P^{-1}(g) = [P^{-1}(g)]^c$ is closed in K. So, we have

$$\begin{aligned} P^{-1}(g) &= \{f \in K : g \in P(f)\} \\ &= \{f \in K : \langle f - g, T(f)\rangle + F(f, g) \subseteq -intC(f)\} \end{aligned}$$

and

$$[P^{-1}(g)]^c = \{f \in K : \langle f - g, T(f)\rangle + F(f, g) \nsubseteq -intC(f)\}\,.$$

By assumption (ii) and (iii), we claim that $[P^{-1}(g)]^c$ is closed in K for all $g \in K$. Now let $\{f_\alpha\}_{\alpha\in\Delta}$ be an arbitrary net in $[P^{-1}(g)]^c$ such that $\{f_\alpha\}$ converges to f (w.r.t.p.c). Then, we have

$$\langle f_\alpha - g, T(f_\alpha)\rangle + F(f_\alpha, g) \nsubseteq -intC(f_\alpha),\ \forall\, g \in K.$$

That is, there exist $h_\alpha \in \langle f_\alpha - g, T(f_\alpha)\rangle + F(f_\alpha, g)$ such that $h_\alpha \notin -intC(f_\alpha)$ or $h_\alpha \in W(f_\alpha)$ for all $\alpha \in \Delta$. Let

$$A = \{f_\alpha\} \cup \{f\}.$$

Then A is compact and $h_\alpha \in \langle A - g, T(A)\rangle + F(A, g)$, which is compact. Therefore h_α converges to h (w.r.t.p.c). Hence by the upper-semicontinuity of F(.,g), we have

$$h \in \langle f - g, T(f)\rangle + F(f, g).$$

Also, since $W(.)$ has a closed graph in $K \times Y$, we have $h \in W(f)$. Consequently, $h \in \langle f - g, T(f)\rangle + F(f, g)$ and $h \notin -intC(f)$, that is,

$$\langle f - g, T(f)\rangle + F(f, g) \nsubseteq -intC(f).$$

Hence $f \in [P^{-1}(g)]^c$ and so $[P^{-1}(g)]^c$ is closed in K, for all $g \in K$. Thus all the hypotheses of Theorem 2.7 are satisfied. Hence there exist $f_0 \in K$ such that

$$f_0 \in cl_K S(f_0) \ \text{ and } S(f_0) \cap P(f_0) = \phi.$$

Thus, there exist $f_0 \in K$ such that

$$f_0 \in clS(f_0) \ \text{ and } \langle f_0 - g, T(f_0)\rangle + F(f_0, g) \nsubseteq -intC(f_0),\ \ \forall\, g \in S(f_0).$$

That is, GMOQEP is solvable. □

4. Results in noncompact setting

For the solution of GMOQEP (1.1) in noncompact setting, we need the following concept of an escaping sequence.

Definition 4.1. Let K be a subset of $L(X,Y)$ such that $K = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact sets in the sense that $K_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}$. A sequence $\{f_n\}_{n=1}^{\infty}$ in K is said to be *escaping from K (relative to $K = \bigcup_{n=1}^{\infty} K_n$) if for each $n \in \mathbb{N}$ there exists $M > 0$ such that $k \geq M$ implies $f_k \notin K_n$.*

Theorem 4.2. *Let $K \subseteq L(X,Y)$ be a nonempty subset such that $K = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty, compact and convex subset of K. Let $C : K \to 2^Y$ be a setvalued map such that for each $f \in K$, $C(f)$ is a solid convex and closed cone, $0 \notin C(f)$. Let $(Y, C(f))$ be an ordered topological vector space. Let $F : K \times K \to 2^Y$ be a setvalued map such that for each $f \in K$, $S(f)$ is nonempty convex, for each $g \in K$, $S^{-1}(g)$ is open in K and $cl_K S$ is upper-semicontinuous. Assume that*

(i) *F is weakly $C(f)$-quasiconvex,*
(ii) *graph of $W(f) = Y \setminus (-intC(f))$ is closed for all $f \in K$,*
(iii) *for each $g \in K$, $F(.,g)$ is upper-semicontinuous with compact values on K,*
(iv) *$F(f,f) \subseteq -intC(f)$, $\forall f \in K$,*
(v) *$\langle f-g, T(f)\rangle \subseteq -intC(f)$, $\forall g \in K$,*
(vi) *for each sequence $\{f_n\}$ in K with $f_n \in K_n$, $n \in \mathbb{N}$, which is escaping from K relative to $\{K_n\}_{n=1}^{\infty}$, there exists $m \in \mathbb{N}$ and $g_m \in K_m \cap S(f_m)$ such that*

$$\langle f-g, T(f)\rangle + F(f_m, g_m) \subseteq -intC(f_m),\ \forall f_m \in cl_K S(f_m).$$

Then there exist $f_0 \in K$ such that

$$f_0 \in cl_K S(f_0) \text{ and } \langle f-g, T(f)\rangle + F(f,g) \nsubseteq -intC(f_0),\ \forall g \in S(f_0).$$

Proof. Since for each $n \in N$, K_n is compact and convex subset of $L(X,Y)$, applying Theorem 3.1, we have for all $n \in \mathbb{N}$, there exists $f_n \in K_n$ such that

$$f_n \in cl_K S(f_n) \text{ and } \langle f_n - h, T(f)\rangle + F(f_n, h) \nsubseteq -intC(f_n), \forall h \in S(f_n). \quad (4.1)$$

Suppose that the sequence $\{f_n\}_{n=1}^{\infty}$ in K is escaping from $K = \bigcup_{n=1}^{\infty} K_n$. By assumption (vi), there exists $m \in N$ and $h_m \in K_m \cap S(f_m)$ such that for each $f_m \in cl_K S(f_m)$,

$$\langle f_m - h_m, T(f_m)\rangle + F(f_m, h_m) \subseteq -intC(f_m)),$$

which contradicts (4.1). Hence $\{f_n\}_{n=1}^{\infty}$ is not an escaping sequence from K relative to $\{K_n\}_{n=1}^{\infty}$. Thus using the similar argument, which have been used by Qun [19] in proving Theorem 2, there exists $r \in N$ and $f_0 \in K_r$ such that $f_n \to f_0$ (w.r.t.p.c) and $\langle f_0 - g, T(f_0)\rangle + F(f_0, g) \subseteq W(f_0)$.

Since $cl_K S : K \to 2^K$ is upper-semicontinuous with compact values, hence there exists $f_0 \in K$ such that

$$f_0 \in cl_K S(f_0) \ \ and\ \langle f_0 - g, T(f)\rangle + F(f_0, g) \nsubseteq -intC(f_0),\ \forall g \in S(f_0).$$

□

Theorem 4.3. *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space $L(X, Y)$, and D be a nonempty compact subset of K. Let $C : K \to 2^Y$ be a setvalued map such that for each $f \in K$, $C(f)$ is a solid, convex and closed cone, $0 \notin C(f)$. Let $(Y, C(f))$ be an ordered Hausdorff topological vector space. Let $F : K \times K \to 2^Y$ and S, $cl_K S : K \to 2^K$ be setvalued maps such that for each $f \in K$, $S(f)$ is nonempty convex, for each $g \in K$, $S^{-1}(g)$ is open in K and $cl_K S$ is upper-semicontinuous. Assume that*

(i) *F is weakly $C(f)-$ quasiconvex,*
(ii) *the graph of $W(f) = Y \setminus (-intC(f))$ is closed, $\forall f \in K$,*
(iii) *for each $g \in K$, $F(., g)$ is upper-semicontinuous with compact values on K,*
(iv) *$F(f, f) \subseteq -intC(f)$, $\forall f \in K$,*
(v) *$\langle f - g, T(f)\rangle \subseteq -intC(f)$, $\forall g \in K$.*

Then there exists $f_0 \in K$ such that

$$f_0 \in cl_K S(f_0)\ and\ \langle f_0 - g, T(f)\rangle + F(f_0, g) \nsubseteq -intC(f_0),\ \forall g \in S(f_0).$$

Proof. Let $P : K \to 2^D$ be a setvalued map defined by

$$P(f) = \{g \in D : \langle f - g, T(f)\rangle + F(f, g) \subseteq -intC(f)\},\ \forall f \in K.$$

Then by using the similar argument, which we have used in proving Theorem 3.1, we have $f \notin coP(f)$, for all $f \in K$ and $P^{-1}(g)$ is open for each $g \in D$. Thus all the conditions of Theorem 2.8 are satisfied. Hence there exists $f_0 \in K$ such that

$$f_0 \in cl_K S(f_0) \text{ and } S(f_0) \cap P(f_0) = \phi.$$

Therefore, there exists $f_0 \in K$ such that

$$f_0 \in cl_K S(f_0) \ \ and \ \langle f_0 - g, T(f_0\rangle + F(f_0, g) \nsubseteq -intC(f_0), \ \forall g \in S(f_0).$$

□

5. Particular Case

In this section, we prove the existence result for the solution of particular case of GMQOEP (1.2) by using the coercive conditions, hemicontinuity and KKM theorem. For this, we first need the following lemmas:

Lemma 5.1. *Let $K \subseteq L(X,Y)$ be a nonempty convex set. Let $(Y, C(f)$ be an ordered topological vector space with solid convex closed cone $C(f)$, $0 \notin C(f)$, for each $f \in K$. Then we have that*

(i) *if $g - f \subseteq intC(f)$ and $g \nsubseteq intC(f)$, then $f \nsubseteq intC(f)$.*
(ii) *if $g - f \subseteq -intC(f)$ and $g \nsubseteq -intC(f)$, then $f \nsubseteq -intC(f)$.*

Proof. (i) Let $g - f \subseteq intC(f)$ and $g \nsubseteq intC(f)$. We have to show that $f \nsubseteq intC(f)$. Since $g - f \subseteq intC(f)$, $f \subseteq -intC(f) + g \subseteq -intC(f) + Y \setminus intC(f) \subseteq Y \setminus intC(f)$. Hence we have $f \nsubseteq intC(f)$.

(ii) Let $g - f \subseteq -intC(f)$ and $g \nsubseteq -intC(f)$. We have to show that $f \nsubseteq -intC(f)$. Since $g - f \subseteq -intC(f)$, $f \subseteq intC(f) + g \subseteq intC(f) + Y \setminus \{-intC(f)\}$. Hence we have $f \subseteq Y \setminus \{-intC(f)\}$, that is, $f \nsubseteq -intC(f)$. □

Lemma 5.2. *Let X, Y be Hausdorff topological vector spaces and $K \subseteq L(X,Y)$ be a nonempty convex set. Let the setvalued mapping $F : K \times K \to 2^Y$ be weakly $C(f)-$pseudomonotone and hemicontinuous in the first argument and weakly $C(f)-$quasiconvex in the second argument and let $B \subseteq K$. Then the following are equivalent:*

(a) *There exists $f \in B$ such that $\langle f - g, T(f)\rangle + F(f,g) \nsubseteq -intC(f)$, for all $g \in K$.*
(b) *There exists $f \in B$ such that $\langle g - f, T(g)\rangle + F(g,f) \nsubseteq intC(f)$, for all $g \in K$.*

Proof. By the definition of weak $C(f)-$pseudomonotonicity of T and F, we have (a) implies (b).

Conversely, suppose that there exists $f \in B$ such that $\langle g - f, T(g)\rangle + F(g,f) \nsubseteq intC(f)$, for all $g \in K$. Then, since K is convex, for all $f, g \in K, \lambda \in [0,1]$,

$$h_\lambda = \lambda g + (1 - \lambda) f \in K.$$

Hence, we have

$$\langle h_\lambda - f, T(h_\lambda)\rangle + F(h_\lambda, f) \nsubseteq intC(f).$$

Since F is weakly $C(f)-$quasiconvex, we have

$$\begin{aligned}\{0\} = \langle h_\lambda - h_\lambda, T(h_\lambda)\rangle + F(h_\lambda, h_\lambda) \subseteq\ & (1-\lambda)\langle h_\lambda - h_\lambda + f - f, T(h_\lambda)\rangle \\ & + F(h_\lambda, \lambda g + (1-\lambda)f) + \lambda F(h_\lambda, g) + (1-\lambda)F(h_\lambda, f) - intC(f) \\ \subseteq\ & (1-\lambda)\{\langle h_\lambda - f, T(h_\lambda)\rangle + F(h_\lambda, f)\} \\ & + (1-\lambda)\langle f - h_\lambda, T(h_\lambda)\rangle + \lambda F(h_\lambda, g) - intC(f).\end{aligned}$$

This implies

$$(1-\lambda)\{\langle h_\lambda - f, T(h_\lambda)\rangle + F(h_\lambda, f)\} + (1-\lambda)\langle f - h_\lambda, T(h_\lambda)\rangle + \lambda F(h_\lambda, g) \subseteq intC(f).$$

Therefore, by Lemma 5.1, we have

$$\lambda F(h_\lambda, g) + (1-\lambda)\langle f - h_\lambda, T(h_\lambda)\rangle \nsubseteq -intC(f)$$

or

$$\lambda F(h_\lambda, g) + \lambda(1-\lambda)\langle f - h_\lambda, T(h_\lambda)\rangle \nsubseteq -intC(f),$$

that is,

$$F(h_\lambda, g) + (1-\lambda)\langle f - h_\lambda, T(h_\lambda)\rangle \nsubseteq -intC(f).$$

So, by using the hemicontinuity of F, T and closedness of $Y \setminus -intC(f)$, we have

$$F(f, g) + \langle f - g, T(f)\rangle \ \nsubseteq -intC(f).$$

□

Next, we define KKM map, and KKM theorem as follows:

Definition 5.3. ([7]) Let K be a nonempty subset of a topological vector space X. Then a setvalued map $F : K \to 2^X$ is said to be KKM, if for every finite subset $\{x_1, x_2, \cdots, x_n\}$ of K,

$$co\{x_1, x_2, \cdots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i).$$

Lemma 5.4. (KKM-lemma [7]) *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $T : K \to 2^X$ be a KKM-map such that for any $y \in K$, $T(y)$ is closed and $T(y^*)$ is contained in a compact set $B \subseteq X$, for some $y^* \in K$. Then there exist $x^* \in B$ such that*

$$x^* \in T(y),\ \forall y \in K.$$

That is, $\bigcap_{y\in K} T(y) \neq \phi$.

Theorem 5.5. *Let $K \subseteq L(X,Y)$ be a nonempty convex set. Let $C : K \to 2^Y$ be a setvalued map such that for each $f \in K$, $C(f)$ is solid convex closed cone with apex at the origin of Y. Let $S : K \to 2^K$ be a setvalued map such that for each $f \in K$, $S(f)$ is nonempty convex. Let $(Y, C(f))$ be a ordering topological vector space and $F : K \times K \to 2^Y$ be a setvalued mapping. Assume that*

(i) *F is weakly $C(f)-$quasiconvex in the second argument,*
(ii) *F is weakly $C(f)-$pseudomonotone and hemicontinuous in the first argument,*
(iii) *F is weakly coercive with respect to the compact convex set $B \subseteq K$,*
(iv) *$F(g,.)$ is upper semicontinuous with compact values on B, for each $g \in K$,*
(v) *the graph of $Y \setminus \{-intC(f)\}$ is closed with respect to B for all $f \in K$,*
(vi) *$F(f,f) \subseteq -intC(f),\ \forall f \in K$,*
(vii) *$\langle f-g, T(f)\rangle \subseteq -intC(f),\ \forall g \in K$.*

Then there exists $f_0 \in K$ such that

$$\langle f_0 - g, T(f)\rangle + F(f_0, g) \nsubseteq -intC(f_0),\ \forall g \in S(f_0).$$

That is, GMOQEP has a solution.

Proof. For each $g \in K$, define the setvalued mappings $P, T : K \to 2^K$ by

$$P(g) = \{f \in K : \langle f-g, T(f)\rangle + F(f,g) \nsubseteq -intC(f)\}$$

and

$$T(g) = \{f \in B : \langle f-g, T(f)\rangle + F(g,f) \nsubseteq intC(f)\}.$$

First, we claim that P is a KKM-mapping. Indeed, let $\{g_1, g_2, \cdots, g_n\}$ be a finite subset of K and $g \in co\{g_1, g_2, \cdots g_n\}$ be arbitrary. Then

$$g = \sum_{i=1}^{n} \lambda_i g_i,\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1.$$

Suppose, if possible $g \notin \cup_{i=1}^{n} P(g_i)$, then

$$\langle g - g_i, T(g)\rangle + F(g, g_i) \subseteq -intC(g),\ \ \forall\, i = 1, 2, \cdots n. \tag{5.1}$$

Since F is weakly $C(f)-$quasiconvex in the second argument, we have

$$\begin{aligned}\langle g - g_i, T(g)\rangle + F(g, g_i) &= \langle g - g_i, T(g)\rangle + F(g,g) + intC(g)\\ &\subseteq -intC(g) - intC(g) + intC(g)\\ &\nsubseteq -intC(g),\ \ \forall\, i = 1, 2, \cdots, n,\end{aligned}$$

which is not true by (5.1).

$$\text{Thus } g = \sum_{i=1}^{n} \lambda_i g_i \in \bigcup_{i=1}^{n} P(g_i),$$

which means that $co\{g_1, g_2, \cdots, g_n\} \subseteq \sum_{i=1}^{n} P(g_i)$. Hence a mapping $\overline{P} : K \to 2^K$ defined by $\overline{P}(g) = \overline{P(g)}$, the closure of $P(g)$, is also a KKM-mapping. The coercivity of F with respect to B implies that $\overline{P(g_0)} \subseteq B$. Hence $\overline{P(g_0)}$ is compact. Thus by Lemma 5.4, it follows that $\cap_{g\in K}\overline{P(g)} \neq \phi$.

Next, we claim that

$$\cap_{g\in K}\overline{P(g)} \subseteq T(h), \;\; \forall\, h \in K.$$

Now, let $f \in \bigcap_{g\in K} \overline{P(g)}$. Since

$$\bigcap_{g\in K} \overline{P(g)} = \bigcap_{g\in K} (\overline{P(g)} \cap \overline{P(g_0)}) \subseteq \overline{P(g)} \cap B \subseteq B,$$

we have $f \in \bigcap_{g\in k} \overline{P(g)} \cap B$, for all $g \in K$.

Let $h \in K$ be an arbitrary. Then, there exists a net $\{f_\alpha\}_{\alpha\in\Delta}$ in $P(h)$ such that $\{f_\alpha\}$ converges to $f \in B$. Hence, we have

$$\langle f_\alpha - h, T(f_\alpha)\rangle + F(f_\alpha, h) \nsubseteq -intC(f_\alpha), \; \forall\, h \in K.$$

Since F is weakly C(f)-pseudomonotone, it follows that

$$\langle f_\alpha - h, T(f_\alpha)\rangle + F(h, f_\alpha) \nsubseteq intC(f_\alpha), \; \forall\, h \in K.$$

That is, there exists $p_\alpha \in \langle f_\alpha - h, T(f_\alpha)\rangle + F(h, f_\alpha)$ such that

$$p_\alpha \notin intC(f_\alpha), \;\; \forall\, \alpha \in \Delta.$$

Since the set $A = \{f_\alpha\} \cup \{f\}$ is compact,

$$p_\alpha \in \langle A - h, T(A)\rangle + F(h, A), \; \forall\, \alpha \in \Delta.$$

Since $\langle A - h, T(A)\rangle + F(h, A)$ is compact, $\{p_\alpha\}$ has a convergent subnet with limit, say p. We can assume that $\{p_\alpha\}$ converges to p. Further, since the graph of $Y \setminus \{-intC(f)\}$ is closed. Clearly, the graph of $Y \setminus intC(f)$ is also closed. Hence the upper semicontinuity of $F(h, .)$ implies $p \in \langle f - h, T(f)\rangle + F(h, f)$. Hence

$$\langle f - h, T(f)\rangle + F(h, f) \nsubseteq intC(f),$$

that is, $f \in T(h)$ for all $h \in K$. Therefore, we have

$$\phi \neq \bigcap_{g\in K} \overline{P(g)} \subseteq \bigcap_{g\in K} \overline{T(g)} \subseteq B.$$

Finally by Lemma 5.4, $\bigcap_{g\in K} P(g) = \bigcap_{g\in K} T(g)$. Thus $\bigcap_{g\in K} P(g) \neq \phi$. Therefore, there exists $f_0 \in K$ such that

$$\langle f_0 - g, T(f)\rangle + F(f_0, g) \nsubseteq -intC(f_0),\ \forall\, g \in K.$$

Hence GMOEP(1.2) is solvable. □

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