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RESEARCH ARTICLE

Secondary Teachers' Views about Proof and Judgements on Mathematical Arguments

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Abstract

Despite its recognition in the field of mathematics education and mathematics, students' understanding about proof and performance on proof tasks have been far from promising. Research has documented that teachers tend to accept empirical arguments as proofs. In this study, an online survey was administered to examine how Korean secondary mathematic teachers make judgements on mathematical arguments varied along representations. The results indicate that, when asked to judge how convincing to their students the given arguments would be, the teachers tended to consider how likely students understand the given arguments and this surfaces as a controversial matter with the algebraic argument being both most and least convincing for their students. The teachers' judgements on the algebraic argument were shown to have statistically significant difference with respect to convincingness to them, convincingness to their students, and validity as mathematical proof.

Keywords Proof, Reasoning, Secondary Teacher, Teacher Knowledge.

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I. INTRODUCTION

Proof and the roles of it have been acknowledged by researchers (Alcock & Inglis, 2008; Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; De Villiers, 1990; Ellis et al., 2019; Epstein & Levy, 1995; Harel & Sowder, 1998; Knuth, 2002a, 2002b; Lakatos, 1976; Schoenfeld, 1994) and the importance placed on it in the context of school mathematics is echoed across curricula (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2015; Council of Chief State School Officers [CCSSO], 2010; Department of Education, 2014; Ministry of Education [MoE], 2011, 2015; National Council of Teachers of Mathematics [NCTM], 2000). For example, Schoenfeld (1994) argues that proof is a fundamental aspect of doing and recording mathematics. NCTM (2000) considers proof to be powerful ways of understanding mathematical phenomena: "Mathematical reasoning and proof offer a powerful way of developing and expressing insights about a wide range of phenomena" (p. 56). CCSSO (2010) considers proof to be essential for students' learning and doing mathematics. To heed the call for centrality of proof in school mathematics, there have been efforts made to unearth issues that are concerned with instruction of proof.

One of such effort is investigating students' or teachers' understandings about proof. By *proof*, I refer to a mathematical argument that is a logical sequence (or chain) of showing how one reasons from an assumption to a result (Stylianides, 2009). Using this definition of proof, in this study, faulty (or invalid) proofs that are not acceptable by mathematicians are referred to as *arguments*. I take *proving* to be one's act of developing a proof: In other words, proving is one's undertaking to develop a proof. Researchers (e.g., Inglis & Weber, 2012; Knuth, Choppin, & Bieda, 2009; Schoenfeld, 1988; Senk, 1985) reported that students' understanding about or producing proofs does not reach the level of mastery. Secondary students are not the only case for lack of understanding about proof such as overreliance on superficial features of mathematical arguments when asked to evaluate validity of them (Harel & Sowder, 1998; Knuth, 2002b), validating mathematical arguments with resort on external mathematical authority (Harel & Sowder, 1998), or even a few of confirming examples suffice to prove a mathematical conjecture (Almeida, 2000; Coe & Ruthven, 1994; Harel & Sowder, 1998; Knuth et al., 2009). Here and throughout the paper, I use examples for a proposition or conjecture to refer to all possible members within the domain of the proposition or conjecture. For example, 2 is an example of a proposition of which domain is all even numbers. In light of Simon and Blume's (1996) point that student's understanding of justification in mathematics is "likely proceed from inductive toward deductive and toward greater generality" (p. 9), of principal interest and close relevance to this study is how examples enable transition to proof by coming into recognition that examples inform the operation(s) or proof method(s) which can be applied to the general case (i.e. representative case) of a class of examples (Alcock & Inglis, 2008; Ellis et al., 2013; Lockwood et al., 2013; Ozgur et al., 2019) or that they lead to the *conceptual insight* (Sandefur, Mason, Stylianides, & Watson, 2013). In the literature, however, not much is known about what constitutes a generic example to teacher's viewpoint, and how a teacher can facilitate transition of student's thinking from example(s) to a proof. Also, teachers seem to conceptualize mathematical proof somewhat idiosyncratically due to multiplicity of what a proof serves to do in mathematics (Hanna, 2000; Hersh, 1993; Knuth, 2002a, 2002b). Furthermore, different conceptions about what a mathematical proof allow for various perspectives toward the same data and distinct interpretations (Knuth, Zaslavsky, & Kim, 2022). As a result, students have difficulty producing mathematical arguments that they find convincing and are asked to produce (Healy & Hoyles, 2000). The state of affairs calls for an action to address the aforementioned issues regarding proof, example, and their relationship. In this regard, the research question of this study are as follows:

a) How do teachers make judgements of a given mathematical argument in terms of convincingness, validity as a proof, and genericity?
b) Are the judgements different across teachers?

II. RELATED LITERATURE

Conceptualizations of Proof and Roles that Proof Plays in Mathematics

Proof plays a crucial role in mathematics. Several researchers consider proof as a tool to establish the truth of a mathematical claim (de Villiers, 1990; Harel & Sowder, 1998; Knuth, 2002b; Lakatos, 1976), provides *insight* into similar problems (Alcock & Inglis, 2008; Ellis et al., 2019; Epstein & Levy, 1995; Lynch & Lockwood, 2019), and, finally, is *a fundamental aspect* of recording and doing mathematics (Ball et al., 2002; Schoenfeld, 1994).

To heed the call for centrality of proof in school mathematics, there have been many attempts to conceptualize a mathematical proof (Stylianides, Stylianides, & Weber, 2016; Weber, 2014). Albeit such efforts, to date, it is generally accepted that there is no consensus on the definition of proof since different conceptualizations of proof lead to different research outcomes which are guided by unique research goals (Balacheff, 2002; Reid & Knipping, 2010; Weber, 2014). In light of studies (e.g., Almeida, 2000; Knuth, 2002b; Harel & Sowder, 1998), there are diverse conceptualizations and criterion about whether a mathematical argument qualifies as a proof. Diversity in teachers' conceptualizations of proof engenders different outcomes of instruction of proof (Weber, 2014). This line of reasoning led to investigate teachers' conceptualizations about proof in school mathematics. In particular, this study was to investigate teachers' conceptualizations about proof and roles of example use when student engages in proving-related activities.

Generic Examples that Bridge Examples and Proofs

Generic examples[1] (Balacheff, 1988) that offer insight into the general method

[1] The term *generic example* was coined by Balacheff (1988) and, by the term, he referred to examples that generic examples are examples that "involves making explicit the reasons for the

which holds true for general cases, are of instructional value in teaching and learning of proof (Knuth et al., 2019; Mason & Pimm, 1984; Walther, 2006). However, what is problematic about considering empirical arguments as proofs is that opportunities to progress from examples to proof are taken away from students (Simon & Blume, 1996; Blanton & Stylianou, 2014). This issue stems from the fact that teachers' understanding about proof is not sophisticated well enough.

Research has documented that teachers tend to accept empirical arguments as proofs. Harel and Sowder (1998) reported that teachers have empirical proof scheme of which valid mode of proof, to those who have the scheme, is nothing more than testing a few examples against a mathematical proposition. Similarly, other researchers (Almeida, 2000; Basturk, 2010; Coe & Ruthven, 1994, Knuth et al., 2020) confirmed similar results that there still persists such tendency among teachers to accept as proof empirical arguments that only test a subset of examples to evaluate truth of the arguments. To some extent, as is the case of generic example, this tendency is explained as, to mathematics educators, use of generic example is often seen as productive and accessible for secondary students (Leron & Zaslavsky, 2013; Zaslavsky, 2018).

Of particular interest of this study is *generic examples*, examples that offer insight into the general method(s) that can be applied to the objects within the domain of a mathematical conjecture (Balacheff, 1988), allowing for an accessible entry for students to gain insight into proofs that they strive to develop (Kim, 2020; Knuth et al., 2019; Lakatos, 1976; Lockwood et al., 2013; Pólya, 1954). This study investigates how teachers judge genericity of the given approaches to a mathematical conjecture so that, in teacher preparation programs, issues of equating genericity (of examples) and generality (of proof) can be addressed by noting what teachers need to know in regards to making judgements on arguments that students provide and what is admissible as valid proofs in mathematics. Equating genericity and generality is problematic in that students have no opportunity to advance their understanding. For a clear distinction between genericity and generality in this study, what distinguishes them is an argument with potential to be developed to a proof versus a proof at its own right.

This point, however, should not be misinterpreted as examples are sufficient to prove or formulate conjectures by their own right. As Knuth et al. (2009) reported, the majority of the students tended to provide empirical justifications of which primary source of conviction originate from examples. This calls for an action to emphasis limitations of empirical (or example-based) reasoning (Chazan, 1993) in instruction of mathematics and to improve students' use of examples in a more strategic way that are similar to ways how experts use examples when asked to evaluate or prove mathematical conjectures (Lockwood et al., 2013). A similar case holds true for teachers: teachers tend to misunderstand empirical justification for proof (Coe & Ruthven, 1994; Harel & Sowder, 1998) or to lower the bar for their students (Knuth, 2002a; Knuth et al., 2020),

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truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class. The account involves the characteristic properties and structures of a class, while doing so in terms of the names and illustration of one of its representatives" (p. 219).

acknowledging that their students are unlikely to write a proof and thus equating a few worked examples and a valid proof for students.

Teachers' Preference on Representations of Proof

Teachers tend to prefer proofs of certain form(s) over the other (e.g., Almeida, 2000; Harel & Sowder, 1998; Healy & Hoyles, 2000; Knuth, 2002b). According to the study by Knuth (2002b), teachers' judgements on given arguments were made taking into consideration some superficial aspects of proof rather than substance of proof. For instance, when evaluating (purported to be) proofs, teachers sought *sufficient level of detail*, *concrete features*, *familiarity* (Knuth, 2002b) or tended to rate best arguments that they had seen before or involved algebraic expressions (Coe & Ruthven, 1994).

This tendency that teachers show when asked to evaluate mathematical arguments seems to be known to students as well. Healy and Hoyles (2000) investigated students' proof conceptions in algebra and learned that students have a dual criterion on proof: Proofs that are convincing to them and those that are likely to receive best marks from their teachers. Not surprisingly, students evaluated arguments more highly that they find to be confusing due to algebraic expressions involved rather than those that they find to be convincing. To address the issues in regard to teachers' preference (or tendency to favor proofs of certain forms), there is need to examine what ground(s) teachers base their judgements of mathematical arguments when asked to do so in general, how their judgments change by taking different lens (e.g., convincingness, validity, genericity) in particular.

III. METHODS

The participants of the study were secondary mathematics teachers or prospective secondary mathematics teachers in Korea. Per the mathematics curriculum (MoE, 2011, 2015), all of the participants were exposed to explicit instruction about proof at their 8^{th} grade. In Korea, the teacher preparation programs for secondary mathematics teachers consist of required mathematics courses that account for 45% of credit hours required for graduation (Kwon, Kim, & Cho, 2012) and the courses include real analysis, abstract algebra, linear algebra, and topology. Given the undergraduate curricula, either prospective or inservice secondary mathematics teachers were eligible to participate in the online survey. The select participants were provided with a small incentive as compensation for their time and effort.

The solicitation for participation was sent primarily through venues including emails and messages to the departmental pages in a social network platform. In an attempt to reach out to the teacher preparation programs in Korea, I searched all the departmental websites to look for the email addresses and sent emails to the administrative staffs that include the overview of the study and who are eligible to participate in the survey. While a number of the programs responded that they would send the flyer to their students, the majority of them had no response. Then, alternatively, I sent the flyer to the student council websites of the mathematics education department on a social network platform. Among about twenty messages sent, a few of them returned with their willingness to partake in the survey. Finally, for recruiting inservice teachers, I took the approach so called *snowballing* to start from teachers whom I used to work with or who were in contact with me, asking them to disseminate the flyers to other teachers who are eligible. This recruitment took a month and the flyer was sent periodically during the recruitment period. As a result, the resulting number of the participants was 104 consisting of 84 prospective teachers and 20 inservice teachers. The number of the participants who completed all the questionnaire was 27.

The demographic of the participants varied in terms of their years of teaching, the highest degrees they earned, grades they were teaching at the time of their participation and whether they completed student teaching. The vast majority (about 71%) of the prospective teachers did not complete their student teaching by the time of their participation. Most of the inservice teachers reported that they had completed required coursework for their master's degrees and 40% of the teachers responded that they only completed undergraduate degrees, leaving the rest of them who completed their doctoral studies regardless of degrees earned. It should be noted that, in the section grades they are currently teaching, the aggregate percentage exceeds one hundred percent due to the fact that the participants would choose multiple grades where applicable. Table 1 provides the detailed demographic of the participants.

Table 1. Demographic of the participants

Prospective teachers	Student Teaching Completed					
	Yes			No		
	28.55%			71.45%		
Inservice teachers	Highest Degrees they earned					
	Undergraduate		Master's		Doctoral	
	40%		45%		15%	
	Years of Teaching					
	~ 5 years	~ 10 years	10~15 years	15~20 years	20 years~	
	5%	55%	25%	0%	15%	
	Grades they are currently teaching					
	Grade 7	Grade 8	Grade 9	Grade 10	Grade 11	Grade 12
	4.55%	13.64%	0.00%	31.82%	45.45%	50.00%

Data Collection

Teachers (including undergraduate students in the track of teacher preparation) were invited to respond to an online survey with a collection of questions focused on teachers' own definitions of proof, perceptions of example use in various didactical situations, and their judgement on whether the given arguments constitute valid proofs and the extent to which they thought their students (imaginary students for prospective teachers) were strategic in using examples.

The intent of the online survey was to learn about teachers' judgements on students' use of examples and how they make judgements of validity, convincingness, and genericity of the given arguments. One of the assumptions in identifying the sub-domains was that results of this study would suffice to formulate hypotheses about relationships between teachers' definitions of proof, judgements in terms of genericity, convincingness, and validity of an argument that may be tested in future research. For this study, my hypothesis is that teachers' judgements on mathematical arguments might be differed by how mathematical arguments are represented, how convincing teachers find the arguments to be, how generic examples involved in given arguments are thought to be by the teachers and how valid as mathematical proof teachers deem the arguments to be. In the literature, not much is known about how one reaches conviction about a mathematical argument of which principal evidence is a generic example. The results of this study would shed light on how one reaches judgement on genericity of an example. The following table (Table 2) provides an overview of the survey to better align the aforementioned sub-domains with the research questions and the survey questions.

Table 2. Alignment of sub-domains with the research questions and survey questions

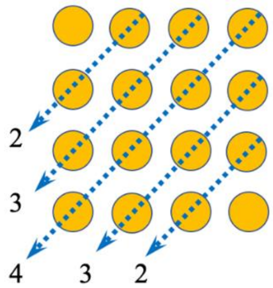
Sub-domain	Relevant Survey Question	What to be elicited by survey question
Judgement on Convincingness	3	Teachers' judgements on *convincingness* from their viewpoint and reasons
	4	Teachers' judgements on *convincingness* from the viewpoint of their students and reasons
Judgement on Validity as Proof	5	Teachers' judgements on *validity* and reasons about their judgements

Upon the development of the initial survey questions, the questions were sent to two reviewers who were teachers and their comments were reflected in the questions revised accordingly. The comments were concerned with cognitive load laid on participants regarding the length of the survey and difficulty of mathematical propositions given in the initial questionnaire. In the initial questionnaire, there were three mathematical propositions with each of them accompanied by four approaches. Also, there came four questions along with each of the twelve approaches. In the following revision, two of the propositions were removed from the questionnaire and the least demoing proposition remained. The revised questionnaire was sent out to a few teachers for a pilot-testing, finally, the resulting questionnaire (see Appendix for detail) was responded by the participants and the results will be reported.

Argument 1) I substitute n with 3 and it worked out:
$1+2+3+2+1=9=3^2$

Argument 2) I substitute n with 4 and it worked out:
$1+2+3+4+3+2+1=16=4^2$ and the same will hold true for all whole numbers.

Argument 3) I substitute n with 4 and it worked out: $1+2+3+4+3+2+1=16=4^2$
As shown below, for other cases, squares with n circles in a diagonal line will be constructed and the squares should have n times n circles. Therefore, the given conjecture holds true for all whole numbers.



Argument 4)
$1+2+\cdots(n-1)+n+(n-1)+(n-2)+\cdots+2+1$
$=2\{1+2+\cdots+(n-1)\}+n$
$=[\{1+(n-1)\}+\{2+(n-2)\}+\cdots+\{k+(n-k)+\cdots+\{(n-1)+1\}+n$
$=(n+n+\cdots+n)+n$
$=(n-1)n+n$
$=n^2-n+n$
$=n^2$

Figure 1. Arguments 1, 2, 3, and 4.

For the survey questions 3 to 5, four arguments to the conjecture (for a whole number n, $1+2+\cdots(n-1)+n+(n-1)+(n-2)+\cdots+2+1=n^2$) were provided to investigate on what ground(s) teachers' judgements on mathematical arguments are made in terms of convincingness, validity, and genericity. The arguments were varied with respect to representations (e.g., linguistic, diagrammatic/pictorial, symbolic/algebraic) and particularity of example involved in each argument. The first argument (argument 1) was purely an empirical argument involving a particular example ($n = 3$) and written in an algebraic form. The second argument was similar to the first one in that it involved a particular example ($n = 4$) and written in a similar way to the first argument, however, it was explicit that the second argument showed the argument provider's consideration about generalizability of the operation that would be replicated to all other examples within the domain of the conjecture. The third argument contained the same component of using the particular example ($n = 4$) and was written in an algebraic form. On top of that, the third argument included justification on the generality of the operation that was absent in the second argument. The last argument involved a general case that is usually seen in the case of mathematical proof and was written in algebraic and diagrammatic representation supporting the proof method of counting the number of circles in a square with n circles in each side.

Data Analysis

The analysis of the responses was conducted in a complementary way. For all of the survey questions, there were forced choices given to participants. The analysis of the forced choices resulted in a quantitative summary. Along with the forced choice questions, there were written prompts for participants to provide their reasons or explanations about their choices.

The written responses were first analyzed through inductive coding (Creswell & Poth, 2016). In the subsequent coding, when the emergent codes seemed to be pertaining to the literature, the codes were modified based on the relevant literature and other codes underwent further revisions to form a set of codes that allows for manageable coding.

Finally, to explore relationships between their judgements on the given arguments, I conducted several dependent t-tests on each argument. The t-tests were conducted to test whether there was statistically significant difference between any pair of judgements (e.g., convincingness to them versus validity as mathematical proof) by each participant. Given that all of the judgements were given as 4-point Likert scale and clustered around each participant and each of the given argument, dependent t-test was the appropriate model to address the last research question. For the degrees of freedom for the t-test, I considered the number of the participants who responded to all the questionnaire which is 28 (equivalently, 27 of the degrees of freedom).

Ⅳ. DISCUSSION

Teachers' judgments on mathematical arguments vary by the viewpoint they take at a moment. As reported earlier, their judgements on the given arguments were differed when they viewed the arguments from the viewpoint of them or their students. What seems most striking to me is that the judgements to the argument 4 were controversial as being both most and least convincing when viewing it from the viewpoint of students. Given the reasons the teachers provided for justifying their ratings to the argument, by and large, their choices were made based on how likely a mathematical argument is to be understood by a particular group of reader. This suggests a hypothesis that may be tested in future research: to what extent teachers' selections of mathematical arguments for their instruction are associated with their expectations about how likely a particular group of students are to understand the arguments.

The results of the study need further examinations on several grounds. Taken into consideration prior education of the participants, years of teaching, and grade levels that they were teaching at the time of their participation, there may be formulated a variety of research questions involving different groups of participants, particular content area, or years of teaching. As the participants of this study were consisting of prospective teachers and inservice teachers (working at either middle school or high school) without particular attention to a specific content area or grade level, such variations in characteristics of participants would nourish the literature, providing insight into what such variations lead to. Since the study is based on the data collected from an online survey, other forms (e.g., interview, classroom observation) of data would enable more insightful results that have not been documented elsewhere. However, for a more robust finding that may yield a more general claim to be made, the recruitment method taken in this study should be reconsidered and more participants must be recruited for future study.

The results of the study confirm results reported previously in the literature. For one, when asked to evaluate mathematical arguments, teachers show overreliance on some superficial features (e.g., appearance, variable use) of the arguments and that is in resonance with the results reported previously in the literature (Coe & Ruthven, 1994; Harel & Sowder, 1998; Knuth, 2002b). Though there were different reasons why the teachers rated the argument with symbolic representations and a general case as most valid as mathematical proof, it may be argued that they prefer arguments with algebraic representations over other arguments. In light of the results (Harel & Sowder, 1998; Healy & Hoyles, 2000; Knuth, 2002b), this argument seems to be viable. Future research needs to investigate this phenomenon with a variety of mathematical arguments involved. For another, this study reiterates that teachers' views about centrality of proof in school mathematics still remain a matter of debate among teachers as reported by Knuth (2002b). As mentioned earlier, I have reservation to argue that the results of this study are sufficient enough to make generalized claims for the population of interest, but it can be said that researchers see the merit of this study to formulate hypotheses to be tested in future study.

Lastly, noteworthy of the results is that teachers' judgements on the algebraic argument were differed by viewing it from different perspectives. The algebraic argument (argument 4) was a controversial one given that teachers considered it as being most and least convincing to their students. Their judgements showed statistically significant difference in the survey questions 3 to 5. Particularly, with respect to convincingness to them and validity as mathematical proof, difference in their judgements comes as call for research that investigate teacher's own meaning for convincing arguments versus valid proofs.

V. CONCLUSION

Teachers' judgements on convincingness, validity, and genericity of the given arguments to the conjecture

The participants were given the four arguments that varied along several dimensions including representation, particularity/generality of a case involved, and genericity of operation. convincingness, validity, and genericity. The relevant survey questions include the questions 3 to 5. The survey questions 3 to 5 were given marks using 4-point Likert scale: 1 as being the most convincing/valid, 2 as being the second most convincing/valid followed by 3 and 4 as being the least convincing/valid. With respect to convincingness, they were asked to make judgements on each argument taking the viewpoint from teacher (the survey question 3) or from student (the survey question 4). With regard to validity, they considered validity of each of the given arguments as a mathematical proof and left marks of them based on their judgements. Similarly, they were asked to evaluate genericity of each argument taking into consideration of how likely the proof method/operation involved in the argument could be applicable to all other elements (i.e., whole numbers) within the domain of the conjecture.

The vast majority of (72.4%) the teachers seemed to be most convinced by the argument 4. The same proportion of the teachers found second most convincing the argument 3 followed by the arguments 2 and 1. Though not apparent in Table 3, the half of the teachers ranked order the arguments 4, 3, 2, and 1 in descending order with respect to convincingness: argument 4 as being most convincing and argument 1 as being least convincing. The reasons they provided included:

> As a manipulation of algebraic expressions, the argument 4 is explicit in explaining that the conjecture holds true, thus being most convincing. Intuitively, the argument 3 shows a generalizable proof method that is valid. Though the argument 2 is quite a common type of reasoning among students, it is not mathematically valid. The arguments 2 and 1 are no way convincing since both of them involve inductive reasoning that cannot be justified in mathematics
>
> I ranked order the arguments in terms of potential existence of counterexamples. The argument 4 seems not to have any counterexamples while the argument 1 seems most likely to be encountered with counterexamples

It seems apparent in their responses that the teachers tended to find most convincing algebraic representation in argument 4 rather than those with particular numbers. In comparison to a written description contained in the argument 2, the teachers seemed to be more convinced by the diagrammatic representation along with the written description that appears in argument 3. Table 3 provides the detailed overview of the result.

Table 3. Convincingness to teachers

	Most convincing	Somewhat Convincing	Less Convincing	Least Convincing
Argument 1	10.3%	3.4%	20.7%	65.5%
Argument 2	13.8%	20.7%	55.2%	10.3%
Argument 3	3.4%	72.4%	17.2%	6.9%
Argument 4	72.4%	3.4%	6.9%	17.2%

Taking the viewpoint from students, the teachers' judgements on convincingness of the given arguments varied greatly from when taking the viewpoint from them. Each argument was not received the same rating from the majority of the teachers except argument 1 which was rated as third most convincing by the slight majority (51.7%) of the participants. Arguments 2 and 4 seemed to be controversial in that the ratings bifurcated. Argument 2 was rated as second most convincing by 37.9% of the participants and, at the same time, 31% of the participants rated it as least convincing. The exact same number of participants deemed argument 4 as being both most convincing and least convincing. This stark contrast in the results between what was reported in the preceding paragraph and this paragraph made sense in the written responses the teachers provided:

> Easy to be understood by students (in the preceding prompt, "I ordered the approaches in logical order")
> For those who might have difficulty following the algebraic manipulation, it seems intuitively easier for them to understand what the arguments are explaining(previously, "derivation involving an algebraic representation seems more viable than the intuitive one")
> Involving algebraic representations, the argument 4 might be difficult for students to be grappled with grasping meaning. […](previously, "I ordered them in ascending order of doubt that there are likely to exist exceptions")

Though their responses showed a variety of reasons why they ranked order the arguments differently in the prompts (prompts asking their judgements on convincingness taking their views as teachers and the viewpoint from their students), it became explicit that their judgements were based on different criterion when changing the viewpoints. This change comes as no surprise given that convincingness of a mathematical argument seems to be associated with the reader's understanding of the argument (Sowder & Harel, 2003).

Table 4. Convincingness to students

	Most convincing	Somewhat Convincing	Less convincing	Least convincing
Argument 1	24.1%	6.9%	51.7%	17.2%
Argument 2	20.7%	31.0%	10.3%	37.9%
Argument 3	17.2%	48.3%	27.6%	6.9%
Argument 4	37.9%	13.8%	10.3%	37.9%

The teachers made judgements on validity of the given arguments as mathematical proof. Over three quarters of the teachers showed a tendency of ranking order the arguments in ascending order with respect to the numbers assigned to the arguments. In other words, about 86 percent of the teachers found the argument 4 most valid as a mathematical proof; about 83% of them deemed the argument 3 to be second most valid; about 76% of them considered the argument 2 as less valid; finally, about 86% of them saw the argument 1 as least valid. The teachers' written responses included:

> A mathematical proof for the conjecture should show that the conjecture holds true for all whole numbers
> It (the argument 4) is a deductive proof that can be applicable more generally
> To qualify as a valid proof, an argument must begin with a general case n rather than a particular number
> As it (the argument 4) involves an algebraic equation, it seems most valid

Table 5. Judgements on validity as a mathematical proof

	Most Valid	Somewhat Valid	Less Valid	Least Valid
Argument 1	10.3%	0.0%	3.4%	86.2%
Argument 2	3.4%	17.2%	75.9%	3.4%
Argument 3	0.0%	82.8%	17.2%	0.0%
Argument 4	86.2%	0.0%	3.4%	10.3%

The responses were concerned with generality of the proof method or generality of the case involved in the argument. One of the responses seems that the teacher based her or his judgement on the appearance of the argument: that is, algebraic representation. This is consistent with the views appeared in Knuth's (2002b) that teachers often rely on superficial features of mathematical arguments when asked to judge the arguments qualify as mathematical proof. Table 5 provides an overview of the results about teachers' judgments in terms of validity as mathematical proof.

Finally, the vast majority (about 70.4%) of the teachers tended to favor the proof method used in the argument 4 over other approaches. Following argument 4, argument 3 was placed in the second place by the majority vote with the proportion of about 63% of the teachers while the arguments 2 and 1 were chosen by 37% and 11.1% of the teachers, respectively. The sample responses that supported their preference on the proof method used in argument 4 were as follows:

> The arguments other than the argument 4 were merely confirming examples for which the conjecture holds true, so that the proof methods cannot be applicable to other examples
>
> The algebraic derivation used in the argument 4 can be applicable to other similar algebraic proofs so that the method is mathematically valid to be applied to other mathematical objects

In the first response shown above, the teacher articulated distinction between examples and a proof that is pertaining to the limitation of examples: examples do not suffice to prove any conjectures. The following response attended to the generality of the proof method: the method can be applicable to other objects within the domain of the conjecture (i.e. all whole numbers). However, there were some responses that seemed to base their judgements on *superficial features* (e.g., appearance, variable use, diagram) of an argument:

> It (the argument 4) uses variables
>
> It (the argument 4) involves algebraic manipulation […]
>
> It (the argument 4) does not contain any specific number, thus being applicable to other proofs
>
> It (the approach 4) has more detail than other arguments

These responses do not seem surprising at all given the results (see e.g., Coe & Ruthven, 2002; Harel & Sowder, 1998; Knuth, 2002b): teachers' tendency to base their judgements about mathematical arguments on appearance of the arguments. Though the teachers might have more to say what was not explicit in the aforementioned responses, such responses sufficed to conclude that the teachers' judgements were made based on some of aspects (e.g., representation, proof method, appearance) of mathematical argument.

In the survey question 6, teachers were asked to which argument(s) they found the proof method to be applicable to all other examples within the domain conjecture (i.e. the set of whole numbers) of the conjecture. The vast majority (70.4%) of the teachers generally accepted the argument 4 as being generic and the argument 3 was accepted as being general in the proof method of the argument by 63% of the participants. Following the arguments 3 and 4, 37% of the participants deemed the proof method of the argument 2 to be generic while only 11.1% of them considered the argument 1 as generic in method.

Generic proofs (or proofs involving generic examples) are of both instructional value and with potential risk. As researchers argued (Basturk, 2010; Knuth et al., 2020), generic examples are certainly of instructional value in that they possess potential to guide students to gain insight into solution or proof method. However, Leron and Zaslavsky (2013) argue: "The main weakness of a *generic proof* is, obviously, that it does not really prove the theorem. The "fussiness" of the full, formal, deductive proof is necessary to ensure that the theorem's conclusion infallibly follows from its premises" (p. 27, italics added). With the conflicting arguments in mind, I argue that generic proofs should be deemed to be *seeds of proofs* or *seeds of generality* that possess potential to be developed into proofs with some modifications to gradually increase the level of generality and the level of formalism from those of a student's initial proof with the assumption that students' proofs are often undervalued on surface due to use of everyday language (Zack, 1999).

To that end, what we as mathematics educators need to address is to map the terms *genericity*, convincingness, generality, and formalism in the terrain of proof. To the teachers' views about genericity, what seemed apparent in their responses was lack of distinction between genericity, convincingness, generality, and formalism. In other words, their judgements on genericity seemed to be based on one over the other without paying attention to generality and formalism. I believe that convincingness should be seen as a community (or reader)-specific entity of which judges are members of a community: In other words, this might not be consistent across communities, thus being considered to be not so significant when determining whether an argument is a valid proof. However, levels of generality and formalism are to be determined by professional mathematicians and the judgements can be uniform across communities. Therefore, what must be addressed in teacher preparation programs and professional development courses is to have teachers to develop robust understanding about generality and formalism in regards to proof rather than convincingness and the mapping and trajectories of progression from empirical arguments to formal proofs must be part of students' learning about proof. As Porteous (1990) argued,

> It must be emphasised that this view does not entail a rejection of empirical methods in learning about mathematics, or an undervaluing of concrete experiences. [F]irst-hand experience is vital to learning [...] The proof types used by children are naturally informal, but it would be a mistake to devalue them because of this. Proving, in the sense of explaining generalities, should be part of the normal activity of all learners of mathematics. (p. 5)

Empirical methods or empirical reasoning should have its place somewhere in K-12 education not as the end but as the beginning or transitional phase of learning. If we imagine a continuum with two extremes of students' informal ways of doing mathematics in an end and formal ways in the other, empirical methods must be placed in anywhere but the extreme of formal. What should be noted in instruction of proof is distinction between valid types of proof and types of proof that are admissible as part of learning mathematics. Rather than placing empirical arguments nowhere in learning mathematics, they must be seen as *seeds* or *sprouts* to be grown up later, thus having reservation to call it mature plants as people differentiate ducklings from ducks: "empiricism is an essential component of the machinery of deduction, conversely, however deduction makes possible a discovery that is inaccessible to insight or empiricism" (Schoenfeld, 1986, p. 249). Therefore, potential to be developed into a valid proof must not be considered equally as what has been accomplished in the proof and teachers need not reject such potential but have an eye for it so as to support students to weather through struggles they may encounter since it takes huge cognitive struggle for a student to make transition from examples to a proof (Tall, 1999).

The relationship between Teachers' judgements

Several dependent t-tests were conducted to explore relationships between the judgements that the teachers made with respect to convincingness to them, convincingness to their students, and validity as mathematical proof. Given that the judgements were measured as 4-point Likert scale and clustered around each participant and each of the given arguments, dependent t-tests by each argument were conducted to address the last research question. The degree of freedom for each of the t-tests was 25 since there were 26 participants who responded to all the survey questions 3 to 5. To denote what judgements are being compared, I use T for convincingness to you (the survey question 3), S for convincingness to your students (the survey question 4), and V for validity as mathematical proof (the survey question 5). For example, in the table 6, the second row from top in the second column from left is denoted as T-S, thus implying that the t-test was conducted to test whether there is statistically significant difference in the judgements for the argument 1 between the survey question 3 and 4. The full test results are provided in Table 6.

Table 6. Descriptive statistics for judgements made for each argument

	Judgements	Mean	Standard Deviation
Argument 1	T(Survey question 3)	3.42	1.01
	S(Survey question 4)	2.65	1.38
	V(Survey question 5)	3.69	0.75
Argument 2	T(Survey question 3)	2.77	0.33
	S(Survey question 4)	2.62	0.62
	V(Survey question 5)	2.85	0.13
Argument 3	T(Survey question 3)	2.08	0.69
	S(Survey question 4)	2.00	1.23
	V(Survey question 5)	2.19	0.31
Argument 4	T(Survey question 3)	1.73	1.27
	S(Survey question 4)	2.73	1.43
	V(Survey question 5)	1.27	0.66

Note. The closer to 1 ratings are, the more convincing/valid the arguments are

There were several tests that show statistically significant difference. Such difference exists for arguments 1 and 4. For argument 1, the results from judgments between the survey questions 3 (M=3.42, SD=1.01) and 4 (M=2.65, SD=1.38) indicate that shifting their viewpoint from teachers to students results in difference in their judgements on the argument, $t(25)=-3.24$ and $p=0.0034$ (<0.05). Also, the results from judgments between the survey questions 4 (M=2.65, SD=1.38) and 5 (M=3.69, SD=0.75) indicate that the judgements on convincingness to students and validity as mathematical proof result in difference in their judgements on the argument 1, $t(25)=-3.95$ and $p=0.0006$ (<0.05). For argument 4, judgements from every lens (e.g., convincingness to teachers or students, validity as mathematical proof) show statistically significant difference in responses between any pair of the survey questions 3 to 5. The detailed analysis is provided in Table 7 shown below.

Table 7. Results of dependent t-tests between judgements (Note: p-values with asteroids mean that there is statistically significant difference)

	Comparison	Mean	Standard error	t-statistic	p-value
Argument 1	T-S	-0.77	0.06	-3.24	0.0034*
	S-V	-1.04	0.07	-3.95	0.0006*
	T-V	0.27	0.04	1.27	0.2154
Argument 2	T-S	-0.15	0.02	-1.07	0.2939
	S-V	0.23	0.03	1.36	0.1848
	T-V	0.08	0.01	0.7	0.4903
Argument 3	T-S	-0.08	0.03	-0.4402	0.6636
	S-V	0.19	0.03	1.0444	0.3063
	T-V	0.12	0.02	0.9013	0.3761
Argument 4	T-S	1	0.08	3.6056	0.0014*
	S-V	-1.46	0.09	-4.9590	0.0000*
	T-V	-0.46	0.04	-2.29	0.0309*

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Appendix

The survey questionnaire

1. Are you a pre-service teacher?
a) Yes, I am a pre-service teacher.
b) No, I am an in-service teacher.
2. Please provide your definition of a mathematical proof.

For the following questions 3 to 5, please consider the four arguments that ensue and rank order them in terms of the way presented in each question.

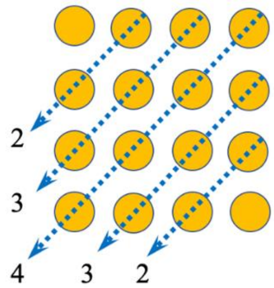
For a whole number n,

$$1+2+\cdots+(n-1)+n+(n-1)+(n-2)+\cdots 2+1=n^2$$

Argument 1) I substitute n with 3 and it worked out:
$1+2+3+2+1=9=3^2$

Argument 2) I substitute n with 4 and it worked out:
$1+2+3+4+3+2+1=16=4^2$ and the same will hold true for all whole numbers.

Argument 3) I substitute n with 4 and it worked out: $1+2+3+4+3+2+1=16=4^2$ As shown below, for other cases, squares with n circles in a diagonal line will be constructed and the squares should have n times n circles. Therefore, the given conjecture holds true for all whole numbers.



Argument 4)
$1+2+\cdots(n-1)+n+(n-1)+(n-2)+\cdots+2+1$
$=2\{1+2+\cdots+(n-1)\}+n$
$=[\{1+(n-1)\}+\{2+(n-2)\}+\cdots+\{k+(n-k)+\cdots+\{(n-1)+1\}+n$
$=(n+n+\cdots+n)+n$
$=(n-1)n+n$
$=n^2-n+n$
$=n^2$

3. Which argument do you think is *most convincing to you*? The lesser the number is, the more convincing the approach is: 1 as being most convincing and 4 as being least convincing. Please provide your explanation on ranking order the given approaches.

	1	2	3	4
Argument 1	○	○	○	○
Argument 2	○	○	○	○
Argument 3	○	○	○	○
Argument 4	○	○	○	○

4. Which argument do you think is *most convincing to your students*? The lesser the number is, the more convincing the approach is: 1 as being most convincing and 4 as being least convincing. Please provide your explanation on ranking order the given approaches.

	1	2	3	4
Argument 1	○	○	○	○
Argument 2	○	○	○	○
Argument 3	○	○	○	○
Argument 4	○	○	○	○

5. Which argument do you think is *most valid as proof*? The lesser the number is, the more valid as proof the approach is: 1 as being most valid as proof and 4 as being least valid as proof. Please explain why you think so.

	1	2	3	4
Argument 1	○	○	○	○
Argument 2	○	○	○	○
Argument 3	○	○	○	○
Argument 4	○	○	○	○

6. For prospective teachers, have you completed student teaching?
a) Yes
b) No
7. For inservice teachers, what is your highest level of degree earned?
a) Undergraduate
b) Master’s
c) Doctoral
8. For inservice teachers, how long have you been teaching mathematics?
a) Less than 6 years
b) 6 to 10 years
c) 11 to 20 years

d) More than 20 years

9. For inservice teachers, what grade level(s) are you currently teaching?

a) Grade 6

b) Grade 7

c) Grade 8

d) Grade 9

e) Grade 10

f) Grade 11

g) Grade 12