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POSITIVE EXPANSIVITY, CHAIN TRANSITIVITY, RIGIDITY, AND SPECIFICATION ON GENERAL TOPOLOGICAL SPACES

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Abstract. We discuss the notions of positive expansivity, chain transitivity, uniform rigidity, chain mixing, weak specification, and pseudo orbital specification in terms of finite open covers for Hausdorff topological spaces and entourages for uniform spaces. We show that the two definitions for each notion are equivalent in compact Hausdorff spaces and further they are equivalent to their standard definitions in compact metric spaces. We show that a homeomorphism on a Hausdorff uniform space has uniform h-shadowing if and only if it has uniform shadowing and its inverse is uniformly equicontinuous. We also show that a Hausdorff positively expansive system with a Hausdorff shadowing property has Hausdorff h-shadowing.

1. Introduction

Generally, a topological dynamical system consists of a pair (X, f), where X is a compact metric space and f is a self continuous map on X. It is well known that the notions such as sensitivity and expansiveness are defined in terms of metric. In case of compact metric space, when one take an equivalent metric the expansiveness and sensitivity of the system (X, f) remain unchanged although their constants are altered. In case when we take a non compact metric space X, these properties may hold on a particular equivalent metric of X but may not hold for another equivalent metric (see [13, Example 6]). Several notions of a dynamical system are defined in purely topological terms such as transitivity, minimality, mixing, etc. while many others are defined in terms of the metric such as equicontinuity, sensitivity, chain transitivity, shadowing, specification, etc. When a notion is defined in topological term, it is independent of the equivalent metric of the compact or non compact metric space X. Interestingly, transitivity and periodicity which are defined purely in topological terms are related to sensitivity which is defined in terms of metric. Banks et al. [7] and

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other authors (see [16, 25]) show that sensitivity follows from transitivity and dense periodic points.

Recently, dynamical systems are being studied in spaces that are not necessarily compact or metric spaces. One of the natural candidates to replace the metric oriented definitions is uniformity because it mimics the existing metric proofs. In doing so, we are studying dynamical systems on the class of completely regular topological spaces or Tychonof spaces. In uniform space also, the inconsistency of the definitions of various notions arises with respect to the compatible uniformities of the uniform topology [12]. These inconsistencies cease once we take compact uniform space where the uniformity is unique.

The uniform approach has been considered in many cases. In [4], D. Alcaraz and M. Sanchis first extended the study of dynamical systems on the completion of totally bounded uniform spaces. In [24], Sejal et al. have provided sufficient conditions for a map to have the specification property. Further, they have proved that if a self homeomorphism on a totally bounded uniform space is mixing, expansive, and has a shadowing property, then it has the specification property. In [12], Das et al. have defined the topological pseudo orbital specification, topological weak specification, topological ergodic shadowing, topological $\underline{d}$ -shadowing for a continuous map on a uniform space and it has been shown that they are equivalent for a uniformly continuous map with a topological shadowing property on a totally bounded uniform space. Xinxing et al. [28] introduced the concepts of weak uniformity, uniform rigidity, and multi-sensitivity for uniform (not necessarily compact or metric) spaces and obtain some equivalent characterizations of uniform rigidity. For more results on dynamical systems in uniform spaces, see [2, 3, 11, 28, 29].

Many dynamical notions could be defined naturally on Hausdorff (but not necessarily compact or metric) spaces. When we study dynamical systems on the class of Hausdorff topological spaces, indeed we are extending the dynamical notions of compact metric spaces to a bigger class of Hausdorff topological spaces. The Hausdorff definition or open cover definition of the various notions has been introduced in [10, 17]. In [10], Brian explicitly defined chain transitivity in terms of open covers. Good and Macías [17] have considered sensitivity and shadowing in terms of open covers. In [26], Wang has studied the notion of equicontinuity and sensitive point in terms of open covers. In both [17] and [26], the authors have shown that the Hausdorff and uniform versions coincide in compact Hausdorff spaces and are equivalent to their metric definitions in compact metric spaces. Inspired by the results mentioned above, we discuss the notions of chain transitivity, uniform rigidity, chain mixing, weak specification, pseudo orbital specification and positive expansivity in terms of finite open covers for Hausdorff topological spaces, and in terms of entourages for uniform spaces. We show that the Hausdorff and the uniform definitions of the respective dynamical notions are equivalent in compact Hausdorff spaces and further they are equivalent to their standard definitions in compact metric spaces.

In Section 2, we give some preliminaries of a dynamical system and uniform space which are used in this paper. In Section 3, we give definitions of various dynamical notions, in terms of open cover for Hausdorff topological spaces and in terms of entourages for uniform spaces. We find results establishing the equivalence of the Hausdorff definitions and uniform definitions on compact Hausdorff spaces. We give examples to support our claims. In Theorem 3.1, we prove that Hausdorff positive expansivity and uniform positive expansivity are equivalent on compact Hausdorff spaces and further they are equivalent to standard positive expansivity on compact metric spaces. In Example 3.2, we give an example where the dynamical system is positively expansive but not Hausdorff positively expansive on a non-compact space. In Example 3.3, we verify that positive expansivity, uniform positive expansivity, and Hausdorff positive expansivity are equivalent on a compact metric space. It has been proved in Theorem 3.4 that a Hausdorff positively expansive dynamical system on a Hausdorff space with Hausdorff shadowing property has Hausdorff h-shadowing property. We show in Theorem 3.5 that a homeomorphism f on a compact Hausdorff uniform space has uniform h-shadowing if and only if f has uniform shadowing and f^{-1} is uniformly equicontinuous. In Theorem 3.6 we show that Hausdorff shadowing is preserved by a topological conjugacy. Similar results of the invariance of topological conjugacy for various other dynamical notions are given in Appendix (Section 4). In Examples 3.7 and 3.9, we verify that Hausdorff shadowing is preserved by a topological conjugacy whereas uniform shadowing and standard shadowing are not. In Theorem 3.10, we prove that Hausdorff chain transitivity and uniform chain transitivity are equivalent on compact Hausdorff spaces and further they are equivalent to standard chain transitivity on compact metric spaces. Following in Theorem 3.12, we prove that uniform chain transitivity is invariant under a topological conjugacy. In Theorem 3.14, we show that Hausdorff uniformly rigid is equivalent to Uniform uniformly rigid on compact Hausdorff spaces and further they are equivalent to uniformly rigid on compact spaces. In Theorem 3.15, we show that a point transitive, uniformly equicontinuous system on a Hausdorff uniform space is Uniform uniformly rigid. We know that uniform chain transitivity depends on the compatible uniformity, however it has been proved in Theorem 3.16 that if the system is topologically transitive, then uniform chain transitivity is independent of the compatible uniformity. In Theorem 3.17, we show that if (X, f) is a topologically transitive, uniformly equicontinuous on a Hausdorff uniform space $(X, \mathscr{U})$, then f is a homeomorphism. We show in Theorem 3.19 that Hausdorff pseudo orbital specification and uniform pseudo orbital specification are equivalent on compact Hausdorff spaces and further they are equivalent to the standard pseudo orbital specification on compact metric spaces. Similar result for weak specification has been proved in Theorem 3.20. Lastly, we prove in Theorem 3.21 that a mixing dynamical system on a Hausdorff uniform space with at least two points is uniformly sensitive.

2. Preliminaries

In 1937, Andre Weil [27] introduced the concept of uniform space. A uniform space is a pair $(X, \mathscr{U})$, where X is a non-empty set and $\mathscr{U}$ is a uniformity on X which is a non-empty family of subsets of $X \times X$ satisfying the following conditions:

a) Each member E of $\mathscr{U}$ contains the diagonal Δ, where $\Delta = \{(x,x) \in X \times X : x \in X\}$.
b) $E^{-1} \in \mathscr{U}$ if $E \in \mathscr{U}$, where $E^{-1} = \{(y,x) \in X \times X : (x,y) \in E\}$.
c) For any $E \in \mathscr{U}$, there exists some $D \in \mathscr{U}$ such that $D \circ D \subset E$, where $D \circ D = \{(x,y) :$ there exists $z \in X$ such that $(x,z) \in D$ and $(z,y) \in D\}$.
d) For any $D, E \in \mathscr{U}$, $D \cap E \in \mathscr{U}$.
e) For any $D \in \mathscr{U}$ and $D \subset E \subset X \times X$, then $E \in \mathscr{U}$.

The members of $\mathscr{U}$ are called entourages. $E \subset X \times X$ is said to be symmetric if $E = E^{-1}$. For any entourage $E \in \mathscr{U}$, we can find a symmetric entourage $D \in \mathscr{U}$ such that $D \circ D \subset E$. The uniform topology $\mathscr{T}$ is defined by $\mathscr{T} = \{T \subset X :$ for each $x \in T$ there exists $E \in \mathscr{U}$ with $E[x] \subset T\}$, where $E[x] = \{y \in X : (x,y) \in E\}$ is called the cross section of E at x. We say that a topological space X is uniformizable if there exists a uniformity on X such that the associated uniform topology is the given topology. A topological space is uniformizable if and only if it is completely regular. Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be two uniform spaces. A function $f : X \to Y$ is said to be uniformly continuous if for each $E \in \mathscr{V}$, there exists some $D \in \mathscr{U}$ such that $(f(x), f(y)) \in E$ whenever $(x,y) \in D$. Throughout this paper, we consider X to be a Hausdorff space unless otherwise stated and we use the notation D^n to denote

$$\underbrace{D \circ D \circ \cdots \circ D}_{n\text{-times}},$$

where n is a positive integer.

For a dynamical system (X, f), the set $O(x, f) = \{f^n(x) : n \in \mathbb{Z}_+\}$ is called the orbit of a point $x \in X$ under the dynamical system (X, f), where $\mathbb{Z}_+$ denotes the set of non-negative integers. Thus, a sequence $\{y_i\}_{i \in \mathbb{Z}_+}$ in X is an orbit if $f(y_i) = y_{i+1}$, $\forall\ i \geq 0$. A point $x \in X$ is said to be a periodic point of f if $f^p(x) = x$ for some $p > 0$ and $f^m(x) \neq x$ for all $m \in \{1, 2, \ldots, p-1\}$. A point $x \in X$ is said to be a transitive point if its orbit $O(x, f)$ is dense in X. A dynamical system (X, f) is said to be point transitive if it has a transitive point. A dynamical system (X, f) is said to be topologically transitive if for every pair of non-empty open sets $U, V \subset X$, there is some $n \geq 0$ such that $f^n(U) \cap V \neq \phi$ and it is said to be topologically mixing if for every pair of non-empty open sets $U, V \subset X$, there is $N > 0$ such that $f^n(U) \cap V \neq \phi$, $\forall$ $n \geq N$.

The notion of shadowing in a compact metric space X with metric d was introduced independently by Anosov [5] and Bowen [9] in 1970s. It is motivated by computer simulations of orbits and plays a crucial role in the general qualitative theory of dynamical systems. In systems with shadowing property, the actual orbits follow the pseudo orbits. A finite sequence $\{x_0 = x, x_1, \ldots, x_n = y\}$ is said to be a δ-chain from x to y of length n if $d(f(x_i), x_{i+1}) < \delta$ for any $i \in \{0, 1, 2, \ldots, n-1\}$. A δ-pseudo orbit is an infinite δ-chain. We say that a δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ is ϵ-shadowed by a point $y \in X$ if $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}_+$. A dynamical system (X, f) on a metric space is said have shadowing property if for every $\epsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo orbit is ϵ-shadowed by some point in X.

The open cover or Hausdorff approach of shadowing on Hausdorff topological space has been introduced by Good and Macías in [17]. They have also given the Hausdorff and uniform version of h-shadowing. It has been proved that these versions coincide in compact Hausdorff space and are equivalent to the metric definitions in compact metric spaces. The Hausdorff and uniform version of shadowing and h-shadowing are given below.

Definition ([17])**.** Let (X, f) be a dynamical system on a Hausdorff space X. Let $x, y \in X$ be two points and let $\{x_0 = x, x_1, \ldots, x_n = y\}$ be a finite sequence in X.

i) If $\mathscr{A}$ is a finite open cover of X, then $\{x_0, x_1, \ldots, x_n\}$ is said to be an $\mathscr{A}$-chain from x to y of length n if for each $i \in \{0, 1, \ldots, n-1\}$, we have $\{f(x_i), x_{i+1}\} \subset A_j$ for some $A_j \in \mathscr{A}$. An $\mathscr{A}$-pseudo orbit is an infinite $\mathscr{A}$-chain.
ii) Let $\mathscr{A}$, $\mathscr{B}$ be finite open covers of X, we say that a $\mathscr{B}$-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ in X is $\mathscr{A}$-shadowed by a point $y \in X$ if for each $i \in \mathbb{Z}_+$, we have $\{f^i(y), x_i\} \subset A_j$ for some $A_j \in \mathscr{A}$.

Definition ([17])**.** A dynamical system (X, f) on a Hausdorff space X is said to have

a) *Hausdorff shadowing property* if for every finite open cover $\mathscr{A}$ of X, there exists a finite open cover $\mathscr{B}$ of X such that every $\mathscr{B}$-pseudo orbit is $\mathscr{A}$-shadowed by some point in X.
b) *Hausdorff h-shadowing property* if for every finite open cover $\mathscr{A}$ of X, there exists a finite open cover $\mathscr{B}$ of X such that for every $\mathscr{B}$-chain $\{x_0, x_1, \ldots, x_m\}$ there exists a point $y \in X$ such that $\{f^i(y), x_i\} \subset A_j$ for each $i \in \{0, 1, 2, \ldots, m-1\}$, for some $A_j \in \mathscr{A}$ and $f^m(y) = x_m$.

Definition ([17])**.** Let (X, f) be a dynamical system on a uniform space $(X, \mathscr{U})$. Let $x, y \in X$ be two points and let $\{x_0 = x, x_1, \ldots, x_n = y\}$ be a finite sequence in X.

i) If $D \in \mathscr{U}$ is an entourage, then $\{x_0, x_1, \ldots, x_n\}$ is said to be a D-chain from x to y of length n if $(f(x_i), x_{i+1}) \in D$ for any $i \in \{0, 1, 2, \ldots, n-1\}$. A D-pseudo orbit is an infinite D-chain.

ii) Let $D, E \in \mathscr{U}$ be entourages of X. We say that a D-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}_+}$ is E-shadowed by a point $y \in X$ if $(f^i(y), x_i) \in E$ for all $i \in \mathbb{Z}_+$.

Definition ([17])**.** A dynamical system (X, f) on a Hausdorff uniform space $(X, \mathscr{U})$ is said to have

a) *uniform shadowing property* if for every entourage $E \in \mathscr{U}$, there exists an entourage $D \in \mathscr{U}$ such that every D-pseudo orbit is E-shadowed by some point in X.
b) *uniform h-shadowing property* if for every entourage $E \in \mathscr{U}$, there exists an entourage $D \in \mathscr{U}$ such that for every D-chain $\{x_0, x_1, \dots, x_m\}$ there exists a point $y \in X$ such that $(f^i(y), x_i) \in E$ for each $i \in \{0, 1, 2, \dots, m-1\}$ and $f^m(y) = x_m$.

The notion of equicontinuity has been studied in various approaches (see [6, 19–21]). In [26], Wang introduced the notions of Hausdorff equicontinuity and uniform equicontinuity which are given below:

Definition ([26])**.** Let (X, f) be a dynamical system.

a) (X, f) is said to be Haudorff equicontinuous if for every finite open cover $\mathscr{A}$ of X, there exists a finite open cover $\mathscr{B}$ of X such that whenever $\{x, y\} \subset B$ for some $B \in \mathscr{B}$ and for each $n \geq 0$ we have $\{f^n(x), f^n(y)\} \subset A$ for some $A \in \mathscr{A}$.
b) Suppose X is a uniform space. Then, (X, f) is said to be uniformly equicontinuous if for every symmetric entourage $E \in \mathscr{U}$, there exists a symmetric entourage $D \in \mathscr{U}$ such that $(f^n(x), f^n(y)) \in E$ for all $n \geq 0$ whenever $(x, y) \in D$.

Wang [26] has shown that these definitions coincide in compact Hausdorff spaces and are equivalent to the standard definition in compact metric spaces. Also, we use the concept of Lebesgue number in the proofs of various results and Example 3.7. Suppose X is a metric space and $\mathscr{A}$ is an open cover of X. A number $\delta > 0$ is said to be a Lebesgue number of $\mathscr{A}$ if every subset of X with diameter less than δ is a subset of some member of $\mathscr{A}$.

3. Theorems

The notion of positive expansivity was extended to uniform spaces by Das et al. in [12]. It has been extended to Hausdorff topological spaces by Good and Macías in [17]. In Theorem 3.1, we show that these two approaches are equivalent in compact Hausdorff spaces and further they are equivalent to the standard definition in compact metric spaces.

Definition. Let (X, f) be a dynamical system.

(1) (X, f) is said to be Hausdorff positively expansive if there exists a finite cover $\mathscr{A}$ such that for any pair of distinct points $x, y \in X$, there is $n \geq 0$ such that $\{f^n(x), f^n(y)\} \nsubseteq A$ for any $A \in \mathscr{A}$. Such an open cover $\mathscr{A}$

is called a Hausdorff positive expansivity cover for f. Equivalently, if for all $n \geq 0$, we have $\{f^n(x), f^n(y)\} \subseteq A$ for any $A \in \mathscr{A}$, then $x = y$.

(2) Suppose X is a uniform space. (X, f) is said to be uniform positively expansive if there exists an entourage $E \in \mathscr{U}$ such that for any pair of distinct points $x, y \in X$, there is $n \geq 0$ such that $(f^n(x), f^n(y)) \notin E$. Such an entourage E is called a uniform positive expansivity entourage for f. Equivalently, if for all $n \geq 0$, we have $(f^n(x), f^n(y)) \in E$ for any $E \in \mathscr{U}$, then $x = y$.

(3) Suppose X is a metric space. (X, f) is said to be positively expansive if there exists a constant $\epsilon > 0$ such that for any pair of distinct points $x, y \in X$, there is $n \geq 0$ such that $d(f^n(x), f^n(y)) > \epsilon$. Such a constant ϵ is called a positive expansivity constant for f. Equivalently, if for all $n \geq 0$ and for any $\epsilon > 0$, we have $d(f^n(x), f^n(y)) \leq \epsilon$, then $x = y$.

Theorem 3.1. *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. Then the following statements are equivalent*:

(i) (X, f) *is Hausdorff positively expansive.*

(ii) (X, f) *is uniform positively expansive.*

If X is metric, then (i) *and* (ii) *are equivalent to*

(iii) (X, f) *is positively expansive.*

Proof. (i) $\Longrightarrow$ (ii). Assume that (i) holds. Let $\mathscr{A}$ be a Hausdorff positive expansivity cover for f. Since X is compact, we have $\bigcup_{A \in \mathscr{A}} A \times A = E \in \mathscr{U}$ (by [17, Lemma 2.6]). From (i), for any pair of distinct points $x, y \in X$, there exists $n \geq 0$ such that $\{f^n(x), f^n(y)\} \nsubseteq A$ for any $A \in \mathscr{A}$ which implies that $(f^n(x), f^n(y)) \notin A \times A$ for any $A \in \mathscr{A}$. It follows that $(f^n(x), f^n(y)) \notin \bigcup_{A \in \mathscr{A}} A \times A = E$. This proves (ii).

(ii) $\Longrightarrow$ (i). Assume that (ii) holds. Let $E \in \mathscr{U}$ be a uniform positive expansivity entourage for f. Then, for any pair of distinct points $x, y \in X$, there is $n \geq 0$ such that $(f^n(x), f^n(y)) \notin E$. Let $D \in \mathscr{U}$ be a symmetric entourage such that $D^2 \subset E$. $\{int_X D[z] : z \in X\}$ is an open cover of X. Since X is compact, there exist $z_1, z_2, \ldots, z_m$ in X such that $\mathscr{A} = \{int_X D[z_1], int_X D[z_2], \ldots, int_X D[z_m]\}$ is a finite open cover of X. If possible, assume that $\{f^n(x), f^n(y)\} \subset int_X D[z_j]$ for some $j \in \{1, 2, \ldots, m\}$. Then, $\{f^n(x), f^n(y)\} \subset D[z_j]$ which implies that $(f^n(x), f^n(y)) \in D^2 \subset E$. This is a contradiction. Therefore, $\{f^n(x), f^n(y)\} \nsubseteq int_X D[z_j]$ for any $j \in \{1, 2, \ldots, m\}$. Thus, (i) holds.

For the rest of the proof, let us assume that X is a metric space.

(i) $\Longrightarrow$ (iii). Suppose (i) holds. Let $\mathscr{A}$ be a Hausdorff positive expansivity cover for f. Then, for any pair of distinct points $x, y \in X$, there exists $n \geq 0$ such that $\{f^n(x), f^n(y)\} \nsubseteq A$ for any $A \in \mathscr{A}$. Let $\epsilon > 0$ be a Lebesgue number for $\mathscr{A}$. Then, $f^n(x) \notin B(f^n(y), \epsilon)$. It follows that $d(f^n(x), f^n(y)) > \epsilon$. Thus, (iii) holds.

(iii) $\Longrightarrow$ (i). Suppose (iii) holds. Let $\epsilon > 0$ be the positive expansivity constant for f. Now, for any pair of distinct points $x, y \in X$, there is $n \geq 0$ such that $d(f^n(x), f^n(y)) > \epsilon$. $\{B(z, \frac{\epsilon}{2}) : z \in X\}$ is an open cover of X. Since X is compact, there exist $z_1, z_2, \ldots, z_m$ such that $\mathscr{A} = \{B(z_1, \frac{\epsilon}{2}), B(z_2, \frac{\epsilon}{2}), \ldots, B(z_m, \frac{\epsilon}{2})\}$ is a finite open cover of X. Suppose, if possible $\{f^n(x), f^n(y)\} \subset B(z_j, \frac{\epsilon}{2})$ for some $j \in \{1, 2, \ldots, m\}$. It follows that $d(f^n(x), f^n(y)) < \epsilon$. This is a contradiction. Therefore, $\{f^n(x), f^n(y)\} \nsubseteq B(z_j, \frac{\epsilon}{2})$ for $j \in \{1, 2, \ldots, m\}$. This proves (i). □

In Example 3.2, we consider a system which is positively expansive but not Hausdorff positively expansive on a non-compact space. We also verify that positive expansivity, uniform positive expansivity and Hausdorff positive expansivity are equivalent on compact spaces in Example 3.3.

Example 3.2. Let $X = \mathbb{R}$ with the usual metric topology. Let $f : X \to X$ be defined by $f(x) = 2x$. Then, f is positively expansive and but not Hausdorff positively expansive.

Proof. f is positively expansive (see [22, Example 7]). Now, we show that f is not Hausdorff positively expansive. For any finite open cover $\mathscr{A}$ of $\mathbb{R}$, we can find some $M > 0$ such that $(M, \infty) \in \mathscr{A}$. Let $x, y \in X$ be two distinct points. Suppose both x and y are positive, then $\lim_{n\to\infty} f^n(x) = \infty$ and $\lim_{n\to\infty} f^n(y) = \infty$. So, there exists $n \geq 0$ such that $\{f^n(x), f^n(y)\} \subset (M, \infty)$. Therefore, f is not Hausdorff positively expansive. □

Example 3.3. Let $X = [0, 1]/\{1 \sim 0\}$ with distance d, where $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$. Let $f : X \to X$ be the doubling map defined by

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x < \frac{1}{2}, \\ 2x - 1, & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

Then, f is positively expansive, uniform positively expansive and Hausdorff positively expansive.

Proof. It is clear that f is positively expansive. Next, we show that f is uniform positively expansive. From the definition, $d(f(x), f(y)) = 2d(x, y)$ if $d(x, y) < \frac{1}{4}$. Let $E = \{(x, y) \in X \times X : d(x, y) < \frac{1}{4}\}$. Then, E is an entourage of X. Let $x, y \in X$ be a pair of distinct points. If $(x, y) \notin E$, then we are done. If not, then $d(f(x), f(y)) = 2d(x, y)$. If $(f(x), f(y)) \notin E$, then we are done. If not, then $d(f^2(x), f^2(y)) = 2^2 d(x, y)$. Continuing, we can find some $n > 0$ such that $(f^n(x), f^n(y)) \notin E$. Hence, f is uniform positively expansive with uniform positive expansivity entourage E.

Lastly, we show that f is Hausdorff positively expansive. We have, $\mathscr{A} = \{[0, \frac{1}{8}) \cup (\frac{7}{8}, 1], (\frac{1}{10}, \frac{7}{20}), (\frac{3}{10}, \frac{11}{20}), (\frac{1}{2}, \frac{3}{4}), (\frac{7}{10}, \frac{19}{20})\}$ is a finite open cover of X. Let $x, y \in X$ be a pair of distinct points.

Case (i): Let $x, y \in A$ be a pair of distinct points, where $A \in \mathscr{A}$. Then, $d(x, y) < \frac{1}{4}$. This implies that $d(f^n(x), f^n(y)) > \frac{1}{4}$ for some $n > 0$ as $d(x, y) \neq 0$. Thus, for some $n > 0$, we have $\{f^n(x), f^n(y)\} \nsubseteq A$ for any $A \in \mathscr{A}$.

Case (ii): Let $x \in A_p$, $y \in A_q$ for some $A_p, A_q \in \mathscr{A}$.

(a) If $x, y \notin A_p \cap A_q$, then $\{x, y\} \nsubseteq A$ for any $A \in \mathscr{A}$.

(b) If $x, y \in A_p \cap A_q$, then $d(x, y) < \frac{1}{4}$. By (i) for some $n > 0$, we have $\{f^n(x), f^n(y)\} \nsubseteq A$ for any $A \in \mathscr{A}$. Thus, for any pair of distinct points $x, y \in X$, there exists some $n \geq 0$ such that $\{f^n(x), f^n(y)\} \nsubseteq A$ for any $A \in \mathscr{A}$. Hence, f is Hausdorff positively expansive with Hausdorff positive expansivity cover $\mathscr{A}$. □

In [12, Proposition 3.5], it is proved that a topologically positive expansive dynamical system (X, f) on a uniform space X with a uniform shadowing property has uniform h-shadowing. We extend this result on a Hausdorff space.

Theorem 3.4. *Let (X, f) be a Hausdorff positively expansive dynamical system. If (X, f) has a Hausdorff shadowing property, then (X, f) has Hausdorff h-shadowing.*

Proof. Let $\mathscr{A}$ be a Hausdorff positive expansivity cover for f. Since (X, f) has a Hausdorff shadowing property, there exists a finite open cover $\mathscr{B}$ of X such that any $\mathscr{B}$-pseudo orbit is $\mathscr{A}$-shadowed by some point in X. Let $\{x_0, x_1, \ldots, x_n\}$ be a $\mathscr{B}$-chain. Then, it can be extended to a $\mathscr{B}$-pseudo orbit $\{x_0, x_1, \ldots, x_n, f(x_n), f^2(x_n), \ldots\}$. If $z \in X$ $\mathscr{A}$-shadows this $\mathscr{B}$-pseudo orbit, then for each i, we have $\{f^i(z), x_i\} \subset A$ for some $A \in \mathscr{A}$. In particular, for each $j \geq 0$, we have $\{f^j(f^n(z)), f^j(x_n)\} \subset A$ for some $A \in \mathscr{A}$. Since $\mathscr{A}$ is a Hausdorff positive expansivity cover for f, we have $f^n(z) = x_n$. Hence, (X, f) has Hausdorff h-shadowing. □

In [8, Theorem 6.1], it has been proved that if (X, d) is a compact metric space and $f : X \to X$ is a homeomorphism, then f has h-shadowing if and only if f has shadowing and f^{-1} is equicontinuous. In the following, we extend this result in compact Hausdorff uniform spaces.

Theorem 3.5. *Let $(X, \mathscr{U})$ be a compact Hausdorff uniform space and $f : X \to X$ a homeomorphism. Then f has uniform h-shadowing if and only if f has uniform shadowing and f^{-1} is uniformly equicontinuous.*

Proof. First, suppose f has uniform h-shadowing. Then, f has uniform shadowing. Let $E \in \mathscr{U}$ be a symmetric entourage and let $D \in \mathscr{U}$ be an entourage which satisfies the definition of h-shadowing with respect to E. Without loss of generality, we assume that $D \subset E$. Since $D \in \mathscr{U}$, there exists a symmetric entourage $V \in \mathscr{U}$ such that $V^2 \subset D$. Let x, y be two points such that $(x, y) \in V$. For $n > 0$, $\{f^{-n}(x), f^{-n+1}(x), \ldots, f^{-1}(x), y\}$ is a V-chain. It follows that $\{f^{-n}(x), f^{-n+1}(x), \ldots, f^{-1}(x), y\}$ is a D-chain. By h-shadowing, there is $z \in X$ such that $(f^i(z), f^{-n+i}(x)) \in E$ for $i \in \{0, 1, 2, \ldots, n-1\}$ and $f^n(z) = y$. It follows that $(f^{-n}(y), f^{-n}(x)) \in E$ for all $n \geq 0$. Thus, f^{-1} is equicontinuous.

Conversely, let us assume that f has uniform shadowing and f^{-1} is uniformly equicontinuous. Let E be an entourage and D a symmetric entourage such that $D^2 \subset E$. Let V be a symmetric entourage which satisfies the definition of equicontinuity with respect to D. Without loss of generality, we assume that $V \subset D$. By shadowing, there exists an entourage U such that every U-pseudo orbit is V-shadowed by some point in X. Let $\{x_0, x_1, \ldots, x_n\}$ be a U-chain in X. Then, it can be extended to a U-pseudo orbit $\{x_0, x_1, \ldots, x_n, f(x_n), f^2(x_n), \cdots\}$. If $z \in X$ V-shadows this U-pseudo orbit, then $(f^i(z), x_i) \in V$ for all $i \geq 0$. By equicontinuity of f^{-1}, $(f^{-i}(x_n), f^{-i}(f^n(z))) \in D$ for all $i \geq 0$. Put $y = f^{-n}(x_n)$, then for $i < n$, $(f^i(y), f^i(z)) = (f^{-(n-i)}(x_n), f^{-(n-i)}(f^n(z))) \in D$. It follows that $(f^i(y), x_i) \in D^2 \subset E$ for $i < n$ and $f^n(y) = x_n$. Hence f has uniform h-shadowing. □

The following result shows that Hausdorff shadowing property is independent of the equivalent metric and the compatible uniformity, i.e., it is preserved by the topological conjugacy. Similar proofs for other dynamical notions are given in Appendix (Section 4).

Theorem 3.6. *Let (X, f) and (Y, g) be two topologically conjugated dynamical systems. Then f has Hausdorff shadowing property if and only if g does.*

Proof. Let $h : X \to Y$ be a conjugacy. First suppose f has Hausdorff shadowing property. Let $\mathscr{A}$ be a finite open cover of Y. $h^{-1}(\mathscr{A}) = \{h^{-1}(A) : A \in \mathscr{A}\}$ is a finite open cover of X. Therefore, there exists a finite open cover $\mathscr{B}$ of X such that every $\mathscr{B}$-pseudo orbit is $h^{-1}(\mathscr{A})$-shadowed by some point in X. $h(\mathscr{B}) = \{h(B) : B \in \mathscr{B}\}$ is a finite open cover of Y. Let $\{y_0, y_1, y_2, \ldots\}$ be a $h(\mathscr{B})$-pseudo orbit. Then, for all $i \geq 0$, $\{g(y_i), y_{i+1}\} \subset h(B)$ for some $h(B) \in h(\mathscr{B})$. It follows that $g(y_i) \in h(B)$ and $h^{-1}(g(y_i)) \in B$. Now $f(h^{-1}(y_i)) \in B$ and $h^{-1}(y_{i+1}) \in B$, therefore $\{f(h^{-1}(y_i)), h^{-1}(y_{i+1})\} \subset B$. Put $x_i = h^{-1}(y_i)$ for all $i \geq 0$, then, $\{x_0, x_1, x_2, \ldots\}$ is a $\mathscr{B}$-pseudo orbit. Therefore, there exists a point $x \in X$ such that for all $i \geq 0$, we have $\{f^i(x), x_i\} \subset h^{-1}(A)$ for some $h^{-1}(A) \in h^{-1}(\mathscr{A})$. It follows that $h(f^i(x)) \in A$ and $g^i(h(x)) \in A$. Also $h(x_i) \in A$, so for all $i \geq 0$, we have $\{g^i(h(x)), y_i\} \subset A$ for some $A \in \mathscr{A}$. Thus, g has Hausdorff shadowing property. Similarly, we can prove the converse part. Hence, f has Hausdorff shadowing property if and only if g does. □

We know that different metrics give rise to different uniformities. In Examples 3.7 and 3.9, we give examples of dynamical systems verifying that Hausdorff shadowing is independent of the equivalent metric whereas uniform shadowing and standard shadowing are dependent on the uniformity and metric, respectively.

Example 3.7. Let $X = (0, a)$, $a > 0$ with the usual metric. Let $f : X \to X$ be defined by $f(x) = kx,\ 0 < k < \frac{1}{2}$. Then f has shadowing, uniform shadowing and Hausdorff shadowing property.

To prove this example we need the following result.

Proposition 3.8. *Every finite cover of an interval of $\mathbb{R}$ has a Lebesgue number.*

Proof. Let X be an interval of $\mathbb{R}$ and let $a, b \in \mathbb{R}$.

Case (I). Let $X = [a, b]$. Then we are done.

Case (II). Let $X = (a, b)$. Let $\mathscr{A}$ be a finite open cover of X. Let $\mathscr{A}' = \{A_1, A_2, \ldots, A_n\}$ be a finite subcover of least cardinality n in their natural order. Let $\mathscr{B} = \{A_p \cap A_q : A_p \cap A_q \neq \phi$ where $A_p, A_q \in \mathscr{A}'\}$. Let $\lambda(\mathscr{B}) = \{z \in \lambda(B) : B \in \mathscr{B}\}$, where $\lambda(B)$ denotes the boundary of B. Let $\epsilon = \min\{d(x, y) : x, y \in \mathscr{B}\}$ and let $\delta = \frac{\epsilon}{4}$. Now, we show that δ is a Lebesgue number of $\mathscr{A}$. We have, X is a totally bounded subset of $\mathbb{R}$, therefore, there exist points $x_1, x_2, \ldots, x_m$ such that $X \subseteq \bigcup_{i=1}^{m} B(x_i, \delta)$. Let $E \subset X$ be such that $diam(E) < \delta$. Let $p \in E$ be a fixed point. Then $p \in B(x_i, \delta)$ for some $i \in \{1, 2, \ldots, m\}$. Let $q \in E$ be any point. Then, $d(x_i, q) \leq d(x_i, p) + d(p, q) < 2\delta < \frac{\epsilon}{2}$. This implies that

$$E \subset B\left(x_i, \frac{\epsilon}{2}\right). \tag{1}$$

Now, we claim that for any $x \in X$, there exists some $1 \leq j \leq n$ such that $B(x, \frac{\epsilon}{2}) \subseteq A_j$. To prove this claim, we consider two cases $x \in \lambda(\mathscr{B})$ and $x \in X - \lambda(\mathscr{B})$. For the first case, as $x \in \lambda(\mathscr{B})$, $x \in \lambda(A_p)$ for some $1 \leq p \leq n$. This implies that either $x \in A_{p-1}$ or $x \in A_{p+1}$. Let $y, y' \in A_p$ be such that $y \in \lambda(A_{p+1})$ and $y' \in \lambda(A_{p-1})$. If $x \in A_{p+1}$, then $d(x, y) > \frac{\epsilon}{2}$, so $B(x, \frac{\epsilon}{2}) \subseteq A_{p+1}$. Similarly, $B(x, \frac{\epsilon}{2}) \subseteq A_{p-1}$ if $x \in A_{p-1}$. For the second case, three subcases arises.

Subcase (a). $x \in A_p \cap A_{p+1}$, $p = 1, 2, \ldots, n-1$. Let $y \in \lambda(A_p)$ and $y' \in \lambda(A_{p+1})$ be such that $y, y' \in \lambda(A_p \cap A_{p+1})$. If $d(x, y) < \frac{\epsilon}{2}$, then $d(x, y') > \frac{\epsilon}{2}$. So, $B(x, \frac{\epsilon}{2}) \subseteq A_{p+1}$. If not, $d(x, y) > \frac{\epsilon}{2}$, therefore, $B(x, \frac{\epsilon}{2}) \subseteq A_p$.

Subcase (b). $x \in (A_p - A_{p+1}) \cap (A_p - A_{p-1})$, $p = 2, \ldots, n-1$. Then, $x \in A_p$. Let $y \in \lambda(A_p)$. Then, obviously, $d(x, y) > \frac{\epsilon}{2}$. Therefore, $B(x, \frac{\epsilon}{2}) \subseteq A_p$.

Subcase (c). $x \in A_1 - A_2$ or $x \in A_n - A_{n-1}$. By similar reasoning as in Subcase (b), we have $B(x, \frac{\epsilon}{2}) \subseteq A_1$ or $B(x, \frac{\epsilon}{2}) \subseteq A_n$. Thus, for any $x \in X$, there exists some $j \in \{1, 2, \ldots, n\}$ such that $B(x, \frac{\epsilon}{2}) \subseteq A_j$. From (1) we have, $E \subset B(x_i, \frac{\epsilon}{2}) \subseteq A_j$ for some $j \in \{1, 2, \ldots, n\}$. Thus, δ is a Lebesgue number of $\mathscr{A}$ with respect to X.

Case (III). Let $X = (a, \infty)$. Let $\mathscr{A}$ be a finite open cover of X. There exists $N \in \mathbb{R}$ such that $(N, \infty) \in \mathscr{A}$. Let $M \in \mathbb{R}$ be the greatest real number such that $(M, \infty) \in \mathscr{A}$. Let $\mathscr{A}' = \{A_1, A_2, \ldots, A_n\}$ be a finite subcover of least cardinality n in their natural order such that $A_n = (M, \infty)$. There exists $(b, c) = A \in \mathscr{A}'$ such that $b < M < c$. Also, $\mathscr{A}$ is a finite open cover of (a, c). By Case (II), there exists a Lebesgue number of $\mathscr{A}$ with respect to (a, c), say δ. We claim that δ is a Lebesgue number of $\mathscr{A}$ with respect to (a, ∞). Let $E \subset X$ be such that $diam(E) < \delta$. If $E \subset (a, c)$, then there exists some $A \in \mathscr{A}$ such that $E \subset A$. If not, $E \subset (M, \infty)$. Thus, δ is a Lebesgue number of $\mathscr{A}$ with respect to X.

Case (IV). Let $X = (-\infty, a)$. If $\mathscr{A}$ is any finite open cover of X, then by proceeding as in Case (III), we can find a Lebesgue number of $\mathscr{A}$. □

One can easily deduce that every finite open cover of $\mathbb{R}$ has a Lebesgue number. Now, we give the proof of Example 3.7.

Proof. Let $\epsilon > 0$ and $\delta = k\epsilon$. Let $\{x_0, x_1, x_2, \ldots\}$ be a δ-pseudo orbit. By induction, we show that $x = x_0$ ϵ-shadows this δ-pseudo orbit. For $i = 0$, it is done. Let us assume that $d(f^i(x), x_i) < \epsilon$. Then, $d(f^{i+1}(x), x_{i+1}) = |k^{i+1}x - x_{i+1}| \leq |k^{i+1}x - kx_i| + |kx_i - x_{i+1}| < k\epsilon + \delta < \epsilon$. Thus, f has shadowing property.

Now, we show that f has uniform shadowing property. Let $U_\epsilon = \{(x, y) : |x - y| < \epsilon\}$ be an entourage of X, where $\epsilon > 0$. Let $\delta = k\epsilon$, then $U_\delta = \{(x, y) : |x - y| < \delta\}$ is an entourage of X. Let $\{x_0, x_1, x_2, \ldots\}$ be a U_δ-pseudo orbit. By induction, we show that $x = x_0$ U_ϵ-shadows this U_δ-pseudo orbit. For $i = 0$, it is obvious. Let us assume that $(f^i(x), x_i) \in U_\epsilon$. Then, $|k^{i+1}x - x_{i+1}| \leq |k^{i+1}x - kx_i| + |kx_i + x_{i+1}| < k\epsilon + \delta < \epsilon$. Therefore, $(f^{i+1}(x), x_{i+1}) \in U_\epsilon$. Thus, f has uniform shadowing property. Lastly, we show that f has Hausdorff shadowing property. Let $\mathscr{A}$ be a finite open cover of X, and let $\epsilon > 0$ be a Lebesgue number of $\mathscr{A}$. Since f has shadowing property, there exists $\delta > 0$ such that any δ-pseudo orbit $\{x_0, x_1, \ldots\}$ is ϵ-shadowed by $x = x_0$. Now, $X = (0, a) = \big(0, \frac{\delta}{2}\big) \cup \big[\frac{\delta}{2}, a - \frac{\delta}{2}\big] \cup \big(a - \frac{\delta}{2}, a\big)$. We have, $\big\{B(y, \frac{\delta}{2}) : y \in [\frac{\delta}{2}, a - \frac{\delta}{2}]\big\}$ is an open cover of $\big[\frac{\delta}{2}, a - \frac{\delta}{2}\big]$. Since it is compact, there exist finite number of points $y_1, y_2, \ldots, y_m$ in $\big[\frac{\delta}{2}, a - \frac{\delta}{2}\big]$ such that $\big\{B(y_1, \frac{\delta}{2}), B(y_2, \frac{\delta}{2}), \ldots, B(y_m, \frac{\delta}{2})\big\}$ is a finite open cover of $\big[\frac{\delta}{2}, a - \frac{\delta}{2}\big]$. Then, $\mathscr{B} = \big\{(0, \frac{\delta}{2}), B(y_1, \frac{\delta}{2}), B(y_2, \frac{\delta}{2}), \ldots, B(y_m, \frac{\delta}{2}), (a - \frac{\delta}{2}, a)\big\}$ is a finite open cover of X. Let $\{x_0, x_1, \ldots\}$ be a $\mathscr{B}$-pseudo orbit. Then, for all $i \geq 0$, we have $\{f(x_i), x_{i+1}\} \subset B$ for some $B \in \mathscr{B}$. It follows that $|f(x_i) - x_{i+1}| < \delta$ and $|f^i(x) - x_i| < \epsilon$, therefore $\{f^i(x), x_i\} \subset A$ for all $i \geq 0$, for some $A \in \mathscr{A}$. Hence, f has Hausdorff shadowing property. □

Example 3.9. Let $X = (0, a)$, $a > 0$ with the inverse metric d defined as $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$. Let $f : X \to X$ be defined by $f(x) = kx$, $0 < k < \frac{1}{2}$. Then, f doesn't have shadowing and uniform shadowing property.

Proof. First, we show that f doesn't have shadowing property. Let $\epsilon > 0$. Suppose if possible, let $\delta > 0$ be such that every δ-pseudo orbit is ϵ-shadowed by some point x in X. Let $\{x_0, x_1, x_2, \ldots\}$ be a δ-pseudo orbit. By our assumption, $d(f^i(x), x_i) < \epsilon$ $\forall i \geq 0$. But, as $i \to \infty$, $f^i(x) = k^i x \to 0$. So, $d(f^i(x), x_i) = |\frac{1}{f^i(x)} - \frac{1}{x_i}| \to \infty$ as $i \to \infty$ which is a contradiction. Therefore, f doesn't have shadowing property.

Now, we show that f doesn't have uniform shadowing property. Let $U_\epsilon = \{(x, y) : |\frac{1}{x} - \frac{1}{y}| < \epsilon\}$ be an entourage of X where $\epsilon > 0$. Suppose if possible, there exists an entourage D of X such that every D-pseudo orbit is U_ϵ-shadowed by some point x in X. Let $\{x_0, x_1, x_2, \ldots\}$ be a D-pseudo orbit. By our assumption, $(f^i(x), x_i) \in U_\epsilon$ $\forall i \geq 0$. But as $i \to \infty$, $|\frac{1}{f^i(x)} - \frac{1}{x_i}| \to \infty$. Therefore, $(f^i(x), x_i) \notin U_\epsilon$ as $i \to \infty$. Thus, f doesn't have uniform shadowing property. We know that the identity function $id : X \to X$ from the usual metric onto the inverse metric is a conjugacy between (X, f) (X with usual metric) and (X, f) (X with inverse metric). Moreover every finite open cover of X with respect to usual metric is a finite open cover with respect to inverse metric and vice versa. Therefore (X, f) (X with inverse metric) has Hausdorff shadowing. □

In [10], Brian has explicitly defined the notion of chain transitivity in terms of open covers. Also, Ahmadi introduced the concept of uniform chain transitivity in [1]. Here, we show that these definitions coincide in compact Hausdorff spaces and are equivalent to the standard definition in compact metric spaces. We also show that uniform chain transitivity is invariant under a topological conjugacy.

Definition. Let (X, f) be a dynamical system. Let $x, y \in X$ be two points.

(1) f is said to be Hausdorff chain transitive if for any finite open cover $\mathscr{A}$ of X, there exists an $\mathscr{A}$-chain from x to y.
(2) Suppose X is a uniform space, then f is said to be uniformly chain transitive if for any entourage $D \in \mathscr{U}$, there exists a D-chain from x to y.
(3) Suppose X is a metric space, then f is said to be chain transitive if for any $\delta > 0$, there exists a δ-chain from x to y.

Definition. Let (X, f) be a dynamical system. Let $x, y \in X$ be two points.

(1) f is said to be Hausdorff chain mixing if for any finite open cover $\mathscr{A}$ of X, there exists a positive integer N such that for all $n > N$, there exists an $\mathscr{A}$-chain from x to y of length exactly n.
(2) Suppose X is a uniform space, then f is said to be uniformly chain mixing if for any entourage $D \in \mathscr{U}$, there exists a positive integer N such that for all $n > N$, there exists a D-chain from x to y of length exactly n.
(3) Suppose X is a metric space, then f is said to be chain mixing if for any $\delta > 0$, there exists a positive integer N such that for all $n > N$, there exists a δ-chain from x to y of length exactly n.

Theorem 3.10. *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. Then the following statements are equivalent*:

(i) *f is Hausdorff chain transitive.*
(ii) *f is uniformly chain transitive.*
If X is metric, then (i) *and* (ii) *are equivalent to*
(iii) *f is chain transitive.*

Proof. (i) $\Longrightarrow$ (ii). Let $E \in \mathscr{U}$ be an entourage and $x, y \in X$ be two points. Let $D \in \mathscr{U}$ be a symmetric entourage such that $D^2 \subset E$. Then, $\{int_X D[z] : z \in X\}$ is an open cover of X. Since X is compact, there are finite points $z_1, z_2, \ldots, z_m$ such that $\{int_X D[z_1], int_X D[z_2], \ldots, int_X D[z_m]\}$ is a finite cover of X. By (i), there exists a finite sequence $\{x_0 = x, x_1, \ldots, x_n = y\}$ such that for each $i \in$

$\{0, 1, \ldots, n-1\}$, we have $\{f(x_i), x_{i+1}\} \subset int_X D[z_j]$ for some $j \in \{1, 2, \ldots, m\}$. It follows that $(f(x_i), x_{i+1}) \in D^2 \subset E$ for each $i \in \{0, 1, \ldots, n-1\}$. Hence (ii) holds true.

(ii) $\Longrightarrow$ (i). Let $\mathscr{A} = \{A_1, A_2, \ldots, A_t\}$ be a finite cover of X and let $x, y \in X$ be two points. Then, $\bigcup_{q=1}^{t} A_q \times A_q = E \in \mathscr{U}$ (by [17, Lemma 2.6]). By (ii), there exists a finite sequence $\{x_0 = x, x_1, \ldots, x_n = y\}$ such that $(f(x_i), x_{i+1}) \in E$ for each $i \in \{0, 1, \ldots, n-1\}$. This implies that for each $i \in \{0, 1, \ldots, n-1\}$, there exists some $q \in \{1, 2, \ldots, t\}$ such that $(f(x_i), x_{i+1}) \in A_q \times A_q$. It follows that $\{f(x_i), x_{i+1}\} \subset A_q$. Hence (i) holds true.

For the rest of the proof, let us assume that X is a metric space.

(i) $\Longrightarrow$ (iii). Let $\epsilon > 0$. Let $x, y \in X$ be two points. The collection $\{B(z, \frac{\epsilon}{2}) : z \in X\}$ is an open cover of X. Since X is compact, there exists a finite set $\{z_1, z_2, \ldots, z_m\}$ such that $\mathscr{A} = \{B(z_1, \frac{\epsilon}{2}), B(z_2, \frac{\epsilon}{2}), \ldots, B(z_m, \frac{\epsilon}{2})\}$ is a finite open cover of X. By (i), there exists a finite sequence $\{x_0 = x, x_1, \ldots, x_n = y\}$ such that for each $i \in \{0, 1, \ldots, n-1\}$, we have $\{f(x_i), x_{i+1}\} \subset B(z_j, \frac{\epsilon}{2})$ for some $j \in \{1, 2, \ldots, m\}$. It follows that $d(f(x_i), x_{i+1}) < \epsilon$. Hence (iii) holds true.

(iii) $\Longrightarrow$ (i). Let $\mathscr{A}$ be a finite open cover of X. Let $x, y \in X$ be two points. Let $\epsilon > 0$ be a Lebesgue number for $\mathscr{A}$. By (iii), there exists a finite sequence $\{x_0 = x, x_1, \ldots, x_n = y\}$ such that $d(f(x_i), x_{i+1}) < \epsilon$ for each $i \in \{0, 1, \ldots, n-1\}$. It follows that $f(x_i) \in B(x_{i+1}, \epsilon) \subset A_j$ for some $A_j \in \mathscr{A}$. Hence (i) holds true. □

Corollary 3.11. *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. Then the followings are equivalent*:

(i) *f is Hausdorff chain mixing.*
(ii) *f is uniformly chain mixing.*
If X is metric, then (i) *and* (ii) *are equivalent to*
(iii) *f is chain mixing.*

Theorem 3.12. *Let (X, f) and (Y, g) be two conjugated dynamical systems, where $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ are compact Hausdorff uniform spaces. Then f is uniformly chain transitive if and only if g does.*

Proof. Let $h : X \to Y$ be a topological conjugacy. First, suppose f is uniformly chain transitive. Let $E \in \mathscr{V}$ be an entourage. Let x, y be two points in Y. Since h is uniformly continuous, there exists an entourage $D \in \mathscr{U}$ such that for any $u, v \in X$,

$$(u, v) \in D \implies (h(u), h(v)) \in E. \tag{2}$$

Let $\{x_0 = h^{-1}(x), x_1, \ldots, x_{n-1}, x_n = h^{-1}(y)\}$ be a D-chain from $h^{-1}(x)$ to $h^{-1}(y)$ for f. Then, $(f(x_i), x_{i+1}) \in D$ for all $i \in \{0, 1, 2, \ldots, n-1\}$. By (2), $(h(f(x_i)), h(x_{i+1})) \in E$ which implies that $(g(h(x_i)), h(x_{i+1})) \in E$ for all $i \in \{0, 1, 2, \ldots, n-1\}$. It follows that $(g(x), h(x_1)) \in E$ and $(g(h(x_{n-1})), y) \in E$. Put $y_i = h(x_i)$ for $i = 0, 1, 2, \ldots, n$. It follows that $\{y_0 = x, y_1, \ldots, y_n = y\}$ is an E-chain from x to y for g. Hence g is uniformly chain transitive. Similarly, we can prove the converse part. □

Corollary 3.13. *Let (X, f) and (Y, g) be two conjugated dynamical systems, where $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ are compact Hausdorff uniform spaces. Then f is uniformly chain mixing if and only if g does.*

The notion of uniform rigidity is defined in terms metric but there are definitions of uniform rigidity in terms of open cover and uniformity. Here, we give the definitions of uniform rigidity in terms of open cover for Hausdorff topological spaces and in terms of entourages for uniform spaces. In Theorem 3.14, we show that these two approaches are equivalent in compact Hausdorff spaces while both are equivalent to the standard definition in compact metric spaces.

Definition. Let (X, f) be a dynamical system.

(1) The system (X, f) is said to be Hausdorff uniformly rigid, if for any finite open cover $\mathscr{A}$, there exists a positive integer n such that for each $x \in X$, there exists $A \in \mathscr{A}$ such that $\{x, f^n(x)\} \subseteq A$.
(2) Suppose X is a uniform space. The system (X, f) is said to be Uniform uniformly rigid if for each entourage $E \in \mathscr{U}$, there exists $n \geq 1$ such that $(x, f^n(x)) \in E$, $\forall\, x \in X$.
(3) Suppose X is a metric space. The system (X, f) is uniformly rigid if for each $\epsilon > 0$, there exists $n > 0$ such that $d(x, f^n(x)) < \epsilon$, $\forall\, x \in X$.

Theorem 3.14. *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. Then the following statements are equivalent*:

(i) *(X, f) is Hausdorff uniformly rigid.*
(ii) *(X, f) is Uniform uniformly rigid.*
If X is metric, then (i) *and* (ii) *are equivalent to*
(iii) *(X, f) is uniformly rigid.*

Proof. (i) $\Longrightarrow$ (ii). Let (X, f) be a Hausdorff uniformly rigid dynamical system. Let $E \in \mathscr{U}$ be an entourage. Let D be a symmetric entourage such that $D^2 \subset E$. Then $\{int_X D[z] : z \in X\}$ is an open cover for X. Since X is compact, there are finite points $z_1, z_2, \ldots, z_m$ such that $\{int_X D[z_1], int_X D[z_2], \ldots, int_X D[z_m]\}$ is a finite open cover of X. Since (X, f) is Hausdorff uniformly rigid, there exists a positive integer n such that for each $x \in X$, we have $\{x, f^n(x)\} \subset int_X D[z_i]$ for some $i \in \{0, 1, \ldots, m\}$. It follows that $(x, f^n(x)) \in E$ for all $x \in X$. Hence (ii) holds true.

(ii) $\Longrightarrow$ (i). Suppose (X, f) is Uniform uniformly rigid. Let $\mathscr{A} = \{A_1, A_2, \ldots, A_t\}$ be a finite open cover of X. Now $\bigcup_{q=1}^{t}(A_q \times A_q) = E \in \mathscr{U}$ (by [17, Lemma 2.6]). Since (X, f) is Uniform uniformly rigid, there exists a positive integer n such that $(x, f^n(x)) \in E$, $\forall\, x \in X$. It follows that for each $x \in X$, there exists some q such that $\{x, f^n(x)\} \subset A_q$. Hence (i) holds true.

For the rest of the proof, let us assume that X is a metric space.

(i) $\implies$ (iii). Suppose (X, f) is Hausdorff uniformly rigid. Let $\epsilon > 0$. Consider the open cover $\{B(x, \frac{\epsilon}{2}) : x \in X\}$. Since X is compact, there exists a finite set $\{z_1, z_2, \ldots, z_m\}$ such that $\bigcup_{i=1}^{m} B(z_i, \epsilon) = X$. Then, $\mathscr{A} = \{B(z_1, \frac{\epsilon}{2}), \ldots, B(z_m, \frac{\epsilon}{2})\}$ is a finite open cover of X. By (i) there exists a positive integer n such that for each $x \in X$, we have $\{x, f^n(x)\} \subseteq B(z_i, \frac{\epsilon}{2})$ for some $i \in \{0, 1, \ldots, m\}$. It follows that $d(x, f^n(x)) < \epsilon$ for all $x \in X$. Hence (iii) holds true.

(iii) $\implies$ (i). Assume that (X, f) is uniformly rigid. Let $\mathscr{A}$ be a finite open cover of X. There exists $\epsilon > 0$ such that for each $x \in X$, we have $B(x, \frac{\epsilon}{2}) \subset A$ for some $A \in \mathscr{A}$. By (iii), there exists $n \geq 0$ such that $f^n(x) \in B(x, \frac{\epsilon}{2})$ for all $x \in X$. It follows that for each $x \in X$, there exists some $A \in \mathscr{A}$ such that $\{x, f^n(x)\} \subseteq A$. Hence (i) holds true. □

In [18, Theorem 2.33], it is shown that a point transitive, uniformly equicontinuous system on a compact metric space is uniformly rigid. In the following Theorem 3.15, we are extending the result on compact Hausdorff uniform space.

Theorem 3.15. *Let (X, f) be a point transitive, uniformly equicontinuous system on a compact Hausdorff space X. Then (X, f) is Uniform uniformly rigid.*

Proof. Let $x \in X$ be a transitive point and $U \in \mathscr{U}$ be any entourage, where $\mathscr{U}$ is the unique uniformity in X whose uniform topology is the topology of X. Let $V \in \mathscr{U}$ be a symmetric entourage such that $V^3 \subset U$. Let $W \subset V$ be a symmetric entourage which satisfies the definition of uniform equicontinuity with respect to V. Since x is a transitive point, $f^n(x) \in W[x]$ for some $n \geq 1$. By the definition of uniform equicontinuity

$$(f^{n+m}(x), f^m(x)) \in V, \ \forall\, m \in \mathbb{N}. \tag{3}$$

We know that f^n is uniformly continuous. Let $D \in \mathscr{U}$ be a symmetric entourage such that $D^2 \subset W$ and $(f^n \times f^n)(D) \subset W$.

Let $y \in X$ be any point. Since x is a transitive point, there exists some $m > 0$ such that $f^m(x) \in D[y]$. Then,

$$(f^{n+m}(x), f^n(y)) \in W. \tag{4}$$

Combining equations (3) and (4) we get

$$(f^m(x), f^n(y)) \in V \circ W. \tag{5}$$

Now,

$$(y, f^m(x)) \in D. \tag{6}$$

From equations (5) and (6), we get $(y, f^n(y)) \in D \circ V \circ W \subset V^3 \subset U$. Hence, (X, f) is Uniform uniformly rigid. □

The notion of topological transitivity by its very name is a topological notion and therefore it is independent of the compatible uniformity of the uniform topology. Whereas, the notion of uniform chain transitivity is not a topological notion. It depends on the particular compatible uniformity of the uniform topology. However the following theorem shows that topologically transitive system on a uniform space is uniformly chain transitive with respect to any compatible uniformity of the uniform topology.

Theorem 3.16. *Let (X, f) be a topologically transitive system on a uniform space $(X, \mathscr{U})$. Then (X, f) is uniformly chain transitive.*

Proof. Let x, y be two points in X. Let $E \in \mathscr{U}$ be an entourage. Since $E \in \mathscr{U}$, there exists a symmetric entourage $D \in \mathscr{U}$ such that $D^2 \subset E$. It is obvious that $D = \Delta \circ D \subset D^2 \subset E$. Now, $int_X D[f(x)]$ and $int_X D[y]$ are two non-empty open subsets of X. By topological transitivity, there exists $n > 0$ such that $f^n(int_X D[f(x)]) \cap int_X D[y] \neq \phi$. There exists $z \in int_X D[f(x)]$ such that $f^n(z) \in int_X D[y]$ which implies that $z \in D[f(x)]$ and $f^n(z) \in D[y]$. It follows that $(f(x), z) \in D$ and $(f^n(z), y) \in D$. Therefore $\{x, z, f(z), f^2(z), \ldots, f^{n-1}(z), y\}$ is an E-chain from x to y. Hence, (X, f) is uniformly chain transitive. □

Here, in Theorem 3.17, we show that if (X, f) is a topologically transitive, uniformly equicontinuous system on a Hausdorff uniform space $(X, \mathscr{U})$, then f is a homeomorphism.

Theorem 3.17. *Let (X, f) be a topologically transitive, uniformly equicontinuous system on a compact Hausdorff uniform space $(X, \mathscr{U})$. Then f is a homeomorphism.*

Proof. Let $f(x) = f(y)$. Let $E \in \mathscr{U}$ be an entourage. Let $D \in \mathscr{U}$ be a symmetric entourage such that $D^3 \subset E$. By Theorem 3.15, (X, f) is uniformly rigid. Therefore, there exists $n \geq 1$ such that $(x, f^n(x)) \in D$ and $(y, f^n(y)) \in D$. Now, $f^n(x) = f^n(y)$, it follows that $(x, y) \in D \circ \Delta \circ D \subset E$. This implies that $(x, y) \in E$ for any $E \in \mathscr{U}$, therefore, $x = y$. This shows that f is injective. Suppose f is not surjective, then $X - f(X) \neq \phi$. By Theorem 3.16, (X, f) is chain transitive. Let x be a point in $X - f(X)$ and $E \in \mathscr{U}$ an entourage such that $E[x] \subset X - f(X)$. It follows that there is no E-chain from x to itself. This is a contradiction, therefore f is surjective. Since X is compact, f^{-1} is continuous. Hence, f is a homeomorphism. □

Ruelle introduced the concept of the weak specification in [23]. The pseudo orbital specification was introduced by Fakhari et al. in [15]. Das et al. [12] introduced the uniform approach of weak specification and pseudo orbital specification. Here, we define the weak specification and pseudo orbital specification in terms of open covers for Hausdorff topological spaces. We show that the open cover approach and uniform approach are equivalent in a compact Hausdorff space while both are equivalent to the usual definition in a compact metric space.

Theorem 3.18 ([14, 8.3.G])**.** *Let X be a compact Hausdorff space and let $\mathscr{U}$ be the unique uniformity of X that induces its topology. Then for every open cover $\mathscr{A}$ of X, there exists $V \in \mathscr{U}$ such that $\mathfrak{C}(V)$ refines $\mathscr{A}$.*

Definition. Let (X, f) be a dynamical system.

(1) (X, f) is said to have Hausdorff pseudo orbital specification if for every finite open cover $\mathscr{A}$ of X, there exist a finite open cover $\mathscr{B}$ of X and a positive integer M such that for any non-negative integers $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$ and $\mathscr{B}$-chains $\xi_1, \xi_2, \ldots, \xi_n$, where $\xi_j = \{x_{(i,j)}\}_{a_j \leq i \leq b_j}$ for $j \in \{1, 2, \ldots, n\}$, there is $y \in X$ such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $\{f^i(y), x_{(i,j)}\} \subset A_q$ for some $A_q \in \mathscr{A}$.

(2) Suppose X is a uniform space. (X, f) is said to have uniform pseudo orbital specification if for every entourage $E \in \mathscr{A}$, there exist an entourage $D \in \mathscr{U}$ and a positive integer M such that for any non-negative integers $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$ and D-chains $\xi_1, \xi_2, \ldots, \xi_n$, where $\xi_j = \{x_{(i,j)}\}_{a_j \leq i \leq b_j}$ for $j \in \{1, 2, \ldots, n\}$, there is $y \in X$ such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $(f^i(y), x_{(i,j)}) \in E$.

(3) Suppose X is a metric space. Then, (X, f) is said to have pseudo orbital specification if for every $\epsilon > 0$, there are $\delta > 0$ and a positive integer M such that for any non-negative integers $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$ and δ-chains $\xi_1, \xi_2, \ldots, \xi_n$, where $\xi_j = \{x_{(i,j)}\}_{a_j \leq i \leq b_j}$ for $j \in \{1, 2, \ldots, n\}$, there is $y \in X$ such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $d(f^i(y), x_{(i,j)}) < \epsilon$.

Theorem 3.19. *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. Then the following statements are equivalent*:

(i) *(X, f) has Hausdorff pseudo orbital specification.*

(ii) *(X, f) has uniform pseudo orbital specification.*

If X is metric, then (i) *and* (ii) *are equivalent to*

(iii) *(X, f) has pseudo orbital specification.*

Proof. (i) $\Longrightarrow$ (ii). Suppose (X, f) has Hausdorff pseudo orbital specification. Let $E \in \mathscr{U}$ be an entourage. Let $D \in \mathscr{U}$ be a symmetric entourage such that $D^2 \subset E$. Then, $\{int_X D[z] : z \in X\}$ is an open cover of X. Since X is compact, there exist $z_1, z_2, \ldots, z_m$ in X such that $\mathscr{A} = \{int_X D[z_1], int_X D[z_2], \ldots, int_X D[z_m]\}$ is a finite open cover of X. By (i), there exist a finite open cover $\mathscr{B}$ of X and a positive integer M which satisfies the conditions of Hausdorff pseudo orbital specification with respect to $\mathscr{A}$. By Theorem 3.18, there exists $V \in \mathscr{U}$ such that $\mathfrak{C}(V)$ refines $\mathscr{B}$. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be any non-negative integers such that $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$. Let $\xi_1, \xi_2, \ldots, \xi_n$ be V-chains, where $\xi_j = \{x_{(i,j)}\}_{a_j \leq i \leq b_j}$ for $j \in \{1, 2, \ldots, n\}$. Then, for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$

we have $(f(x_{(i,j)}), x_{(i+1,j)}) \in V$. Since $\mathfrak{C}(V)$ refines $\mathscr{B}$, there exists some $B_q \in \mathscr{B}$ such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$ we have $\{f(x_{(i,j)}), x_{(i+1,j)}\} \subset B_q$. Therefore, $\xi_1, \xi_2, \ldots, \xi_n$ are $\mathscr{B}$-chains. By (i), there exists a point y in X such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $\{f^i(y), x_{(i,j)}\} \subset int_X D[z_p]$ for some $p \in \{1, 2, \ldots, m\}$. It follows that $(f^i(y), x_{(i,j)}) \in E$. Hence (ii) holds true.

(ii) $\Longrightarrow$ (i). Suppose (X, f) has uniform pseudo orbital specification. Let $\mathscr{A} = \{A_1, A_2, \ldots, A_t\}$ be a finite open cover of X. Since X is compact, $\bigcup_{q=1}^{t} A_q \times A_q = E \in \mathscr{U}$ (by [17, Lemma 2.6]). By (ii), there exist an entourage $D \in \mathscr{U}$ and a positive integer M which satisfies the conditions of uniform pseudo orbital specification with respect to E. Let $V \in \mathscr{U}$ be a symmetric entourage such that $V^2 \subset D$. Then, $\{int_X V[z] : z \in X\}$ is an open cover of X. Since X is compact, there exist $z_1, z_2, \ldots, z_m$ in X such that $\mathscr{B} = \{int_X V[z_1], int_X V[z_2], \ldots, int_X V[z_m]\}$ is a finite open cover of X. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be any non-negative integers such that $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$. Let $\xi_1, \xi_2, \ldots, \xi_n$ be $\mathscr{B}$-chains where $\xi_j = \{x_{(i,j)}\}_{a_j \leq i \leq b_j}$ for $j \in \{1, 2, \ldots, n\}$. Then, for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $\{f(x_{(i,j)}), x_{(i+1,j)}\} \subset int_X V[z_p]$ for some $p \in \{1, 2, \ldots, m\}$ which implies that $\{f(x_{(i,j)}), x_{(i+1,j)}\} \subset V[z_p]$. It follows that $(f(x_{(i,j)}), x_{(i+1,j)}) \in V^2 \subset D$. Thus, $\xi_1, \xi_2, \ldots, \xi_n$ are D-chains. Therefore, by (ii), there exists a point y in X such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $(f^i(y), x_{(i,j)}) \in E$. It follows that $\{f^i(y), x_{(i,j)}\} \subset A_q$ for some $q \in \{1, 2, \ldots, t\}$. Hence (i) holds true.

For the rest of the proof, let us assume that X is a metric space.

(i) $\Longrightarrow$ (iii). Assume that (X, f) has Hausdorff pseudo orbital specification. Let $\epsilon > 0$. Then, $\{B(z, \frac{\epsilon}{2}) : z \in X\}$ is an open cover of X. Since X is compact, there exist $z_1, z_2, \ldots, z_m$ in X such that $\mathscr{A} = \{B(z_1, \frac{\epsilon}{2}), B(z_2, \frac{\epsilon}{2}), \ldots, B(z_m, \frac{\epsilon}{2})\}$ is a finite open cover of X. By (i), there exist a finite open cover $\mathscr{B}$ and a positive integer M which satisfies the condition of Hausdorff pseudo orbital specification. Let $\delta > 0$ be a Lebesgue number for $\mathscr{B}$. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be non-negative integers such that $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$ and let $\xi_1, \xi_2, \ldots, \xi_n$ be δ-chains, where $\xi_j = \{x_{(i,j)}\}_{a_j \leq i \leq b_j}$ for $j \in \{1, 2, \ldots, n\}$. Then, for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $d(f(x_{(i,j)}), x_{(i+1,j)}) < \delta$. Again, as δ is a Lesbesgue number for $\mathscr{B}$, there exists $p \in \{1, 2, \ldots, m\}$ such that $\{f(x_{(i,j)}), x_{(i+1,j)}\} \subset B(z_p, \frac{\epsilon}{2})\}$. Thus, $\xi_1, \xi_2, \ldots, \xi_n$ are $\mathscr{B}$-chains. Therefore, by (i), there exists a point y in X such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $\{f^i(y), x_{(i,j)}\} \subset B(z_p, \frac{\epsilon}{2})$ for some $p \in \{1, 2, \ldots, m\}$. This implies that $d(f^i(y), x_{(i,j)}) < \epsilon$. Hence (iii) holds true.

(iii) $\Longrightarrow$ (i). Assume that (X, f) has pseudo orbital specification. Let $\mathscr{A}$ be a finite open cover of X. Let $\epsilon > 0$ be a Lebesgue number for $\mathscr{A}$. By (iii), there exist $\delta > 0$ and a positive integer M which satisfies the conditions of pseudo orbital specification with respect to ϵ. Now, $\{B(z, \frac{\delta}{2}) : z \in X\}$

is an open cover of X. Since X is compact, there exist $z_1, z_2, \ldots, z_m$ in X such that $\mathscr{B} = \{B(z_1, \frac{\delta}{2}), B(z_2, \frac{\delta}{2}), \ldots, B(z_m, \frac{\delta}{2})\}$ is a finite open cover of X. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be any non-negative integers such that $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$ and let $\xi_1, \xi_2, \ldots, \xi_n$ be $\mathscr{B}$-chains, where $\xi_j = \{x_{(i,j)}\}_{a_j \leq i \leq b_j}$ for $j \in \{1, 2, \ldots, n\}$. Then, for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $\{f(x_{(i,j)}), x_{(i+1,j)}\} \subset B(z_p, \frac{\delta}{2})$ for some $p \in \{1, 2, \ldots, m\}$. It follows that $d(f(x_{(i,j)}), x_{(i+1,j)}) < \delta$. Thus, $\xi_1, \xi_2, \ldots, \xi_n$ are δ-chains. Therefore, by (iii), there exists a point y in X such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $d(f^i(y), x_{(i,j)}) < \epsilon$. It implies that $f^i(y) \in B(x_{(i,j)}, \epsilon) \subset A_q$ for some $A_q \in \mathscr{A}$. It follows that $\{f^i(y), x_{(i,j)}\} \subset A_q$ for some $A_q \in \mathscr{A}$. Hence (i) holds true. □

Definition. Let (X, f) be a dynamical system.

(1) (X, f) is said to have Hausdorff weak specification if for every finite open cover $\mathscr{A}$ of X, there exists a positive integer M such that for any finite sequence of points $x_1, x_2, \ldots, x_n$ in X and any non-negative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$, there is $y \in X$ such that for integers $i \in [a_j, b_j]$ and for $j \in \{1, 2, \ldots, n\}$, $\{f^i(y), f^i(x_j)\} \subset A_q$ for some $A_q \in \mathscr{A}$.

(2) Suppose X is a uniform space, then (X, f) is said to have uniform weak specification if for every entourage $E \in \mathscr{A}$, there exists a positive integer M such that for any finite sequence of points $x_1, x_2, \ldots, x_n$ in X and any non-negative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$, there is $y \in X$ such that $(f^i(y), f^i(x_j)) \in E$ for integers $i \in [a_j, b_j]$ and for $j \in \{1, 2, \ldots, n\}$.

(3) Suppose X is a metric space, then, (X, f) is said to have weak specification if for every $\epsilon > 0$, there is a positive integer M such that for any finite sequence of points $x_1, x_2, \ldots, x_n$ in X and any non-negative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$, there is $y \in X$ such that $d(f^i(y), f^i(x_j)) < \epsilon$, for integers $i \in [a_j, b_j]$ and for $j \in \{1, 2, \ldots, n\}$.

Theorem 3.20. *Let (X, f) be a dynamical system where X is a compact Hausdorff space, then the following statements are equivalent*:

(i) *(X, f) has Hausdorff weak specification.*

(ii) *(X, f) has uniform weak specification.*

If X is metric, then (i) *and* (ii) *are equivalent to*

(iii) *(X, f) has weak specification.*

Proof. (i) $\Longrightarrow$ (ii). Suppose (X, f) has Hausdorff weak specification. Let $E \in \mathscr{U}$ be an entourage. Then, there exists a symmetric entourage $D \in \mathscr{U}$ such that $D^2 \subset E$. Now, $\{int_X D[z] : z \in X\}$ is an open cover for X. Since X is compact, there exist $z_1, z_2, \ldots, z_m$ such that $\mathscr{A} = \{int_X D[z_1], int_X D[z_2], \ldots, int_X D[z_m]\}$ is a finite open cover of X. By (i), there exists a positive integer M such that for any finite sequence of points $x_1, x_2, \ldots, x_n$ in X and any non-negative integers

$a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$, there is $y \in X$ such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $\{f^i(y), f^i(x_j)\} \subset int_X D[z_p]$ for some $p \in \{1, 2, \ldots, m\}$. This implies that $(f^i(y), f^i(x_j)) \in D^2 \subset E$ for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$. Hence (X, f) has uniform weak specification.

(ii) $\Longrightarrow$ (i). Suppose (X, f) has uniform weak specification. Let $\mathscr{A} = \{A_1, A_2, \ldots, A_t\}$ be a finite open cover of X. Now $\bigcup_{q=1}^{t} A_q \times A_q = E \in \mathscr{U}$ (by [17, Lemma 2.6]). By (ii), there exists a positive integer M such that for any finite sequence of points $x_1, x_2, \ldots, x_n$ in X and any non-negative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$, there is $y \in X$ such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $(f^i(y), f^i(x_j)) \in E$. It follows that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $\{f^i(y), f^i(x_j)\} \subset A_q$ for some $q \in \{1, 2, \ldots, t\}$. Hence (X, f) has Hausdorff weak specification.

For the rest of the proof, let us assume that X is a metric space.

(i) $\Longrightarrow$ (iii). Suppose (X, f) has Hausdorff weak specification. Let $\epsilon > 0$. Then, $\{B(z, \frac{\epsilon}{2}) : z \in X\}$ is an open cover of X. Since X is compact, there exist $z_1, z_2, \ldots, z_m$ in X such that $\mathscr{A} = \{B(z_1, \frac{\epsilon}{2}), B(z_2, \frac{\epsilon}{2}), \ldots, B(z_m, \frac{\epsilon}{2})\}$ is a finite open cover of X. By (i), there exists a positive integer M such that for any finite sequence of points $x_1, x_2, \ldots, x_n$ in X and any non-negative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$, there is $y \in X$ such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $\{f^i(y), f^i(x_j)\} \subset B(z_p, \frac{\epsilon}{2})$ for some $p \in \{1, 2, \ldots, m\}$. It follows that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $d(f^i(y), f^i(x_j)) < \epsilon$. Hence (X, f) has weak specification.

(iii) $\Longrightarrow$ (i). Suppose (X, f) has Hausdorff weak specification. Let $\mathscr{A}$ be a finite open cover of X. Let $\epsilon > 0$ be a Lebesgue number for $\mathscr{A}$. By (iii), there is a positive integer M such that for any finite sequence of points $x_1, x_2, \ldots, x_n$ in X and any non-negative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \ldots, n-1\}$, there is $y \in X$ such that for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $d(f^i(y), f^i(x_j)) < \epsilon$. Therefore, for integers $i \in [a_j, b_j]$ and $j \in \{1, 2, \ldots, n\}$, we have $\{f^i(y), f^i(x_j)\} \subset A_q$ for some $A_q \in \mathscr{A}$. Hence (X, f) has Hausdorff weak specification. □

The uniform approach of sensitivity has been introduced in [17]. A dynamical system (X, f) on a Hausdorff uniform space $(X, \mathscr{U})$ is said to be uniformly sensitive if there is a symmetric entourage $D \in \mathscr{U}$ such that for any non-empty open subset U of X there are $x, y \in U$ and $n > 0$ with $(f^n(x), f^n(y)) \notin D$. Here, we extend [18, Proposition 2.38] to Hausdorff uniform spaces.

Theorem 3.21. *Let (X, f) be a mixing dynamical system on a Hausdorff uniform space $(X, \mathscr{U})$ with at least two points. Then (X, f) is uniformly sensitive.*

Proof. Let u, v be any two distinct points in X and $E \in \mathscr{U}$ be any entourage such that $(u, v) \notin E$. Let $D \in \mathscr{U}$ be a symmetric entourage such that $D^3 \subset E$.

Then, $int_X D[u]$ and $int_X D[v]$ are non-empty open subsets of X. Let U be any non-empty open subset of X. Since (X, f) is topologically mixing, there exists $N > 0$ such that for all $n \geq N$, we have $f^n(U) \cap int_X D[x] \neq \phi$ and $f^n(U) \cap int_X D[y] \neq \phi$. Therefore, there exist $x, y \in U$ such that $f^n(x) \in int_X D[u]$ and $f^n(y) \in int_X D[v]$. It implies that $f^n(x) \in D[u]$ and $f^n(y) \in D[v]$ which follows that $(u, f^n(x)) \in D$ and $(f^n(y), v) \in D$. If possible, let $(f^n(x), f^n(y)) \in D$. Then $(u, v) \in E$, which is a contradiction. Hence, (X, f) is uniformly sensitive. □

4. Appendix

In this section, we are going to show that the Hausdorff definitions of various notions of dynamical system are independent of the equivalent metrics and compatible uniformities, i.e., they are invariant under a topological conjugacy.

Theorem 4.1. *Let (X, f) and (Y, g) be two topologically conjugated dynamical systems, where X and Y are Hausdorff spaces. Then f is Hausdorff chain transitive if and only if g does.*

Proof. Let $h : X \to Y$ be a topological conjugacy. Suppose that f is Hausdorff chain transitive. Let $x, y \in Y$ be two points and let $\mathscr{A}$ be a finite open cover of Y. $h^{-1}(\mathscr{A}) = \{h^{-1}(A) : A \in \mathscr{A}\}$ is a finite open cover of X and $h^{-1}(x), h^{-1}(y) \in X$. Therefore, there exists a $h^{-1}(\mathscr{A})$-chain say $\{x_0 = h^{-1}(x), x_1, \ldots, x_n = h^{-1}(y)\}$ from $h^{-1}(x)$ to $h^{-1}(y)$. It follows that for each $i \in \{0, 1, 2, \ldots, n-1\}$ we have $\{f(x_i), x_{i+1})\} \subset h^{-1}(A)$ for some $h^{-1}(A) \in h^{-1}(\mathscr{A})$. Take $y_i = h(x_i)$, then it follows that for each $i \in \{0, 1, 2, \ldots, n-1\}$ we have $\{g(y_i), y_{i+1}\} \subset A$ for some $A \in \mathscr{A}$. Therefore $\{y_0, y_1, \ldots, y_n\}$ is an $\mathscr{A}$-chain from x to y. Thus, g is Hausdorff chain transitive. Similarly, we can prove the converse part. Hence, f is Hausdorff chain transitive if and only if g does. □

Corollary 4.2. *Let (X, f) and (Y, g) be two topologically conjugated dynamical systems. Then f is Hausdorff chain mixing if and only if g does.*

Theorem 4.3. *Let (X, f) and (Y, g) be two topologically conjugated dynamical systems. Then (X, f) is Hausdorff uniformly rigid if and only if (Y, g) does.*

Proof. Let $h : X \to Y$ be a topological conjugacy. Suppose (X, f) is Hausdorff uniformly rigid. Let $\mathscr{A}$ be a finite open cover of Y. Then, $h^{-1}(\mathscr{A}) = \{h^{-1}(A) : A \in \mathscr{A}\}$ is a finite open cover of X. Therefore, there exists $n > 0$ such that for all $x \in X$, there exists $h^{-1}(A) \in h^{-1}(\mathscr{A})$ such that $\{x, f^n(x)\} \subset h^{-1}(A)$. This implies that $\{h(x), g^n(h(x))\} \subset A$ for some $A \in \mathscr{A}$. Thus (Y, g) is Hausdorff uniformly rigid. Similarly, we can prove the converse part. Hence, (X, f) is Hausdorff uniformly rigid if and only if (Y, g) does. □

Theorem 4.4. *Let (X, f) and (Y, g) be two topologically conjugated dynamical systems. Then (X, f) has Hausdorff pseudo orbital specification if and only if (Y, g) does.*

Proof. Let $h : X \to Y$ be a topological conjugacy. Suppose (X, f) has Hausdorff pseudo orbital specification. Let $\mathscr{A}$ be a finite open cover of Y. Then, $h^{-1}(\mathscr{A}) = \{h^{-1}(A) : A \in \mathscr{A}\}$ is a finite open cover of X. There exist a finite open cover $\mathscr{B}$ of X and a positive integer M which satisfies the definition of Hausdorff pseudo orbital specification with respect to $\mathscr{A}$. $h(\mathscr{B})$ is a finite open cover of Y. Let $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ be non-negative integers with $a_{j+1} - b_j > M$ for $j \in \{1, 2, \dots, n-1\}$ and let $\{\xi_1, \xi_2, \dots, \xi_n\}$ be $h(\mathscr{B})$-chains, where $\xi_j = \{x_{(i,j)}\}$ for integers $i \in [a_j, b_j]$ and for $j \in \{1, 2, \dots, n\}$. For $j \in \{1, 2, \dots, n\}$ and integers $i \in [a_j, b_j]$, we have $\{g(x_{(i,j)}), x_{(i+1,j)}\} \subset h(B)$ for some $h(B) \in h(\mathscr{B})$. This implies that $\{f(h^{-1}(x_{(i,j)})), h^{-1}(x_{(i+1,j)})\} \subset B$ for some $B \in \mathscr{B}$. Take $\eta_j = \{h^{-1}(x_{(i,j)})\}$ for integers $i \in [a_j, b_j]$ and for $j \in \{1, 2, \dots, n\}$. Then, $\{\eta_1, \eta_2, \dots, \eta_n\}$ are $\mathscr{B}$-chains in X. Therefore, there exists $x \in X$ such that for integers $i \in [a_j, b_j]$ and for $j \in \{1, 2, \dots, n\}$, we have $\{f^i(x), h^{-1}(x_{(i,j)})\} \subset h^{-1}(A)$ for some $h^{-1}(A) \in h^{-1}(\mathscr{A})$. It follows that for integers $i \in [a_j, b_j]$ and for $j \in \{1, 2, \dots, n\}$, we have $\{g^i(h(x)), x_{(i,j)}\} \subset A$ for some $A \in \mathscr{A}$. Thus, (Y, g) has Hausdorff pseudo orbital specification. Similarly, we can prove the converse part. Hence, (X, f) has Hausdorff pseudo orbital specification if and only if (Y, g) does. □

Corollary 4.5. *Let (X, f) and (Y, g) be two topologically conjugated dynamical systems. Then (X, f) has Hausdorff weak specification if and only if (Y, g) does.*

Theorem 4.6. *Let (X, f) and (Y, g) be two topologically conjugated dynamical systems. Then (X, f) is Hausdorff positively expansive if and only if (Y, g) does.*

Proof. Let $h : X \to Y$ be a topological conjugacy. Suppose (X, f) is Hausdorff positively expansive. Let $x, y \in Y$ be a pair of distinct points and let $\mathscr{A}$ be a finite open cover of Y. $h^{-1}(\mathscr{A}) = \{h^{-1}(A) : A \in \mathscr{A}\}$ is a finite open cover of X and $h^{-1}(x), h^{-1}(y)$ is a pair of distinct points in X. Therefore, there exists $n \geq 0$ such that $\{f^n(h^{-1}(x)), f^n(h^{-1}(y))\} \nsubseteq h^{-1}(A)$ for any $h^{-1}(A) \in h^{-1}(\mathscr{A})$. This implies that $\{g^n(x), g^n(y)\} \nsubseteq A$ for any $A \in \mathscr{A}$. Thus, (Y, g) is Hausdorff positively expansive. Similarly, we can prove the converse part. Hence, (X, f) is Hausdorff positively expansive if and only if (Y, g) does. □

Theorem 4.7. *Let (X, f) and (Y, g) be two topologically conjugated dynamical systems. Then (X, f) is Hausdorff equicontinuous if and only if (Y, g) does.*

Proof. Let $h : X \to Y$ be a topological conjugacy. Suppose (X, f) is Hausdorff equicontinuous. Let $\mathscr{A}$ be a finite open cover of Y. $h^{-1}(\mathscr{A}) = \{h^{-1}(A) : A \in \mathscr{A}\}$ is a finite open cover of X. Therefore, there exists a finite open cover $\mathscr{B}$ which satisfies the definition of Hausdorff equicontinuity with respect to $\mathscr{A}$. $h(\mathscr{B})$ is a finite open cover of Y. Let $x, y \in h(B)$ be two points, where $h(B) \in h(\mathscr{B})$, $B \in \mathscr{B}$. Now, $\{h^{-1}(x), h^{-1}(y)\} \subset B$, therefore, for all $n \geq 0$, we have $\{f^n(h^{-1}(x)), f^n(h^{-1}(y))\} \subset h^{-1}(A)$ for some $h^{-1}(A) \in h^{-1}(\mathscr{A})$. This implies that $\{g^n(x), g^n(y)\} \subset A$ for some $A \in \mathscr{A}$. Therefore (Y, g) is Hausdorff equicontinuous. Similarly, we can prove the converse part. Hence, (X, f) is Hausdorff equicontinuous if and only if (Y, g) does. □

Corollary 4.8. *Let (X, f) and (Y, g) be two topologically conjugated dynamical systems. Then (X, f) is Hausdorff sensitive if and only if (Y, g) does.*

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References

[1] S. A. Ahmadi, *Shadowing, ergodic shadowing and uniform spaces*, Filomat **31** (2017), no. 16, 5117–5124. https://doi.org/10.2298/fil1716117a

[2] S. A. Ahmadi, X. Wu, and G. Chen, *Topological chain and shadowing properties of dynamical systems on uniform spaces*, Topology Appl. **275** (2020), 107153, 11 pp. https://doi.org/10.1016/j.topol.2020.107153

[3] S. A. Ahmadi, X. Wu, Z. Feng, X. Ma, and T. Lu, *On the entropy points and shadowing in uniform spaces*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **28** (2018), no. 12, 1850155, 10 pp. https://doi.org/10.1142/S0218127418501559

[4] D. Alcaraz and M. Sanchis, *A note on extensions of dynamical systems from uniform spaces*, Topology Appl. **137** (2004), no. 1-3, 3–12. https://doi.org/10.1016/S0166-8641(03)00194-9

[5] D. V. Anosov, *On a class of invariant sets of smooth dynamical systems*, Proceedings of 5th International Conference on Nonlinear Oscillations **2**, Kiev, (1970), 39–45.

[6] J. Auslander, G. Greschonig, and A. Nagar, *Reflections on equicontinuity*, Proc. Amer. Math. Soc. **142** (2014), no. 9, 3129–3137. https://doi.org/10.1090/S0002-9939-2014-12034-X

[7] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, *On Devaney's definition of chaos*, Amer. Math. Monthly **99** (1992), no. 4, 332–334. https://doi.org/10.2307/2324899

[8] A. D. Barwell, C. Good, and P. Oprocha, *Shadowing and expansivity in subspaces*, Fund. Math. **219** (2012), no. 3, 223–243. https://doi.org/10.4064/fm219-3-2

[9] R. Bowen, *ω-limit sets for axiom* A *diffeomorphisms*, J. Differential Equations **18** (1975), no. 2, 333–339. https://doi.org/10.1016/0022-0396(75)90065-0

[10] W. Brian, *Abstract ω-limit sets*, J. Symb. Log. **83** (2018), no. 2, 477–495. https://doi.org/10.1017/jsl.2018.11

[11] T. Ceccherini-Silberstein and M. Coornaert, *Sensitivity and Devaney's chaos in uniform spaces*, J. Dyn. Control Syst. **19** (2013), no. 3, 349–357. https://doi.org/10.1007/s10883-013-9182-7

[12] P. Das and T. Das, *Various types of shadowing and specification on uniform spaces*, J. Dyn. Control Syst. **24** (2018), no. 2, 253–267. https://doi.org/10.1007/s10883-017-9388-1

[13] T. Das, K. Lee, D. Richeson, and J. Wiseman, *Spectral decomposition for topologically Anosov homeomorphisms on noncompact and non-metrizable spaces*, Topology Appl. **160** (2013), no. 1, 149–158. https://doi.org/10.1016/j.topol.2012.10.010

[14] R. Engelking, *General topology*, translated from the Polish by the author, second edition, Sigma Series in Pure Mathematics, 6, Heldermann Verlag, Berlin, 1989.

[15] A. Fakhari and F. H. Ghane, *On shadowing*: *ordinary and ergodic*, J. Math. Anal. Appl. **364** (2010), no. 1, 151–155. https://doi.org/10.1016/j.jmaa.2009.11.004

[16] E. Glasner and B. Weiss, *Sensitive dependence on initial conditions*, Nonlinearity **6** (1993), no. 6, 1067–1075. http://stacks.iop.org/0951-7715/6/1067

[17] C. Good and S. Macías, *What is topological about topological dynamics?*, Discrete Contin. Dyn. Syst. **38** (2018), no. 3, 1007–1031. https://doi.org/10.3934/dcds.2018043

[18] P. Kůrka, *Topological and symbolic dynamics*, Cours Spécialisés, 11, Société Mathématique de France, Paris, 2003.
[19] K. B. Mangang, *Equicontinuity of the limit function of a sequence of equicontinuous functions*, J. Indian Math. Soc. (N.S.) **81** (2014), no. 1-2, 115–121.
[20] K. B. Mangang, *Minimality and equicontinuity of a sequence of maps in iterative way*, Intern. J. Math. Stat. Invent. **2** (2014), 4–6.
[21] K. B. Mangang, *Equicontinuity and sensitivity of sequence dynamical systems*, Glob. J. Pure Appl. Math. **13** (2017), 4223–4228.
[22] D. Richeson and J. Wiseman, *Positively expansive dynamical systems*, Topology Appl. **154** (2007), no. 3, 604–613. https://doi.org/10.1016/j.topol.2006.08.009
[23] D. Ruelle, *Statistical mechanics on a compact set with Z^v action satisfying expansiveness and specification*, Trans. Amer. Math. Soc. **187** (1973), 237–251. https://doi.org/10.2307/1996437
[24] S. Shah, R. Das, and T. Das, *Specification property for topological spaces*, J. Dyn. Control Syst. **22** (2016), no. 4, 615–622. https://doi.org/10.1007/s10883-015-9275-6
[25] S. Silverman, *On maps with dense orbits and the definition of chaos*, Rocky Mountain J. Math. **22** (1992), no. 1, 353–375. https://doi.org/10.1216/rmjm/1181072815
[26] H. Wang, *Equicontinuity, shadowing and distality in general topological spaces*, Czechoslovak Math. J. **70(145)** (2020), no. 3, 711–726. https://doi.org/10.21136/CMJ.2020.0488-18
[27] A. Weil, *Sur les espaces à structure uniforme et sur la topologie gènèrale*, Herman & Cie, Paris, 1938.
[28] X. Wu, Y. Luo, X. Ma, and T. Lu, *Rigidity and sensitivity on uniform spaces*, Topology Appl. **252** (2019), 145–157. https://doi.org/10.1016/j.topol.2018.11.014
[29] X. Wu, X. Ma, Z. Zhu, and T. Lu, *Topological ergodic shadowing and chaos on uniform spaces*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **28** (2018), no. 3, 1850043, 9 pp. https://doi.org/10.1142/S0218127418500438

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