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AREA PROPERTIES ASSOCIATED WITH STRICTLY CONVEX CURVES

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Abstract. Archimedes proved that for a point P on a parabola X and a chord AB of X parallel to the tangent of X at P, the area of the region bounded by the parabola X and the chord AB is four thirds of the area of the triangle $\triangle ABP$. This property was proved to be a characteristic of parabolas, so called the Archimedean characterization of parabolas. In this article, we study strictly convex curves in the plane $\mathbb{R}^2$. As a result, first using a functional equation we establish a characterization theorem for quadrics. With the help of this characterization we give another proof of the Archimedean characterization of parabolas. Finally, we present two related conditions which are necessary and sufficient for a strictly convex curve in the plane to be an open arc of a parabola.

1. Introduction

A regular plane curve $X : I \to \mathbb{R}^2$ defined on an open interval is called *convex* if, for all $t \in I$, the trace $X(I)$ lies entirely on one side of the closed half-plane determined by the tangent line at $X(t)$ [1].

From now on, we will say that a convex curve X in the plane $\mathbb{R}^2$ is *strictly convex* if the curve is smooth (that is, of class $C^{(3)}$) and is of positive curvature κ with respect to the unit normal N pointing to the convex side. Hence, in this case we have $\kappa(s) = \langle X''(s), N(X(s))\rangle > 0$, where $X(s)$ is an arc-length parametrization of X.

For a smooth function $f : I \to \mathbb{R}$ defined on an open interval, we will also say that f is *strictly convex* if the graph of f has positive curvature κ with respect to the upward unit normal N. This condition is equivalent to the positivity of the second derivative of the function f on I.

Suppose that X is a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. For a fixed point $P \in X$ and a

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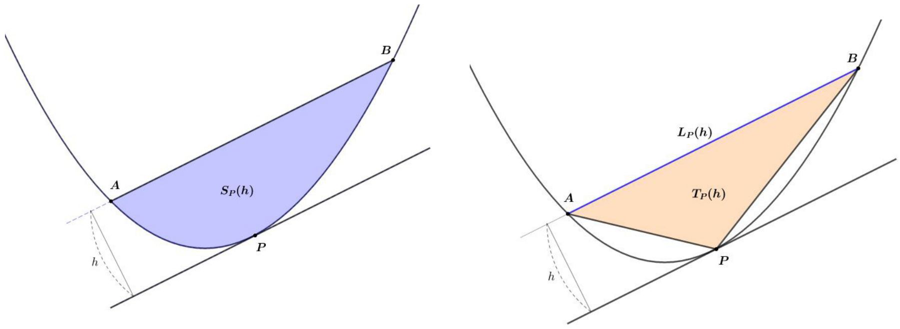

FIGURE 1. $S_P(h)$, $L_P(h)$ and $T_P(h)$ at a point $P \in X$.

sufficiently small $h > 0$, we consider the line m passing through $P + hN(P)$ which is parallel to the tangent ℓ of X at P and the points A and B where the line m intersects the curve X.

We consider $L_P(h)$, $T_P(h)$ and $S_P(h)$ defined by the length $|AB|$, the area $|\triangle PAB|$ and the area of the region bounded by the curve X and chord AB. Then, obviously we have

$$T_P(h) = \frac{1}{2} h L_P(h)$$

and

$$S'_P(h) = L_P(h),$$

where we put $S'_P(h) = \frac{d}{dh} S_P(h)$ [6].

Lemma 1.1. *Suppose that X is a strictly locally convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. Then we have*

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}, \tag{1.1}$$

$$\lim_{h\to 0} \frac{1}{h\sqrt{h}} S_P(h) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}}, \tag{1.2}$$

$$\lim_{h\to 0} \frac{1}{h\sqrt{h}} T_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}, \tag{1.3}$$

where $\kappa(P)$ is the curvature of X at P with respect to the unit normal N.

Proof. It follows from [6] that (1.1) holds. Hence (1.2) and (1.3) follow directly. □

It is well known that parabolas satisfy the following properties.

Proposition 1.2. *Suppose that X is an open arc of a parabola. For an arbitrary point $P \in X$ and a positive number h, we have*

$$\widetilde{L_P}(h) := \frac{1}{\sqrt{h}} L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}, \tag{1.4}$$

$$\widetilde{S_P}(h) := \frac{1}{h\sqrt{h}} S_P(h) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}}, \tag{1.5}$$

$$\widetilde{T_P}(h) := \frac{1}{h\sqrt{h}} T_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}} \tag{1.6}$$

and

$$S_P(h) = \frac{4}{3} T_P(h). \tag{1.7}$$

Proof. It is straightforward to show that (1.4) holds. Then, (1.5) and (1.6) can be derived directly. Hence (1.7) follows. For a proof of (1.7), see [12]. □

In fact, Archimedes showed that parabolas satisfy (1.7) ([12]).

In [6], it was proved that (1.7) is a characteristic property of parabolas and some characterizations of parabolas was established, which are the converses of well-known properties of parabolas originally due to Archimedes ([12]). See [8, 10] for another characterizations for parabolas. For the higher dimensional analogues of some results in [6], see [4, 5].

In Section 2, first of all, we prove the following characterization of quadrics.

Theorem 1.3. *Suppose that X denotes the graph of a strictly convex $C^{(3)}$ function $f : I \to \mathbb{R}$ defined on an open interval I. Then X is an open arc of a quadric, that is, X is an open arc of one of parabolas, ellipses and hyperbolas if and only if it satisfies the following functional equation*

$$\Phi_f(u, v) := f''(u) B_f(u, v)^3 + f''(v) A_f(u, v)^3 = 0, \quad u, v \in I, \tag{1.8}$$

where we define

$$\begin{aligned} A_f(u, v) &= f'(u)(u - v) - (f(u) - f(v)), \\ B_f(u, v) &= f'(v)(u - v) - (f(u) - f(v)). \end{aligned} \tag{1.9}$$

With the help of Theorem 1.3 in Section 3 we give another proof of the following Archimedean characterization theorem for parabolas, which was originally established in [6].

Theorem 1.4. *Suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then X satisfies (1.7) for all $P \in X$ and sufficiently small positive number h if and only if it is an open arc of a parabola.*

It follows from Theorem 1.4 that:

Proposition 1.5. *Suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then X is an open arc of a parabola if and only if it satisfies*

$$\widetilde{S_P}(h) \div \widetilde{T_P}(h) = \alpha(P), \tag{1.10}$$

where $\alpha(P)$ is a function of the point $P \in X$ only.

Proof. It follows from Lemma 1.1 that

$$\alpha(P) = \lim_{h\to 0} S_P(h)/T_P(h) = 4/3.$$

Hence we have $S_P(h)/T_P(h) = 4/3$. Thus Archimedean characterization theorem (Theorem 1.4) shows that X is an open arc of a parabola. □

Finally, in Section 4 we prove the following characterization theorem for parabolas.

Theorem 1.6. *Suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then X is an open arc of a parabola if and only if it satisfies one of the following conditions*:

1) *There exists a function $\beta(P)$ of only $P \in X$ satisfying*

$$\widetilde{S_P}(h) + \widetilde{T_P}(h) = \beta(P). \tag{1.11}$$

2) *There exists a function $\gamma(P)$ of only $P \in X$ satisfying*

$$\widetilde{S_P}(h) \times \widetilde{T_P}(h) = \gamma(P). \tag{1.12}$$

Remark 1.7. Suppose that X is a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. If one of the following is a function of only h;

$$\widetilde{S_P}(h) + \widetilde{T_P}(h),\ \widetilde{S_P}(h) - \widetilde{T_P}(h),\ \widetilde{S_P}(h) \times \widetilde{T_P}(h), \tag{1.13}$$

then Lemma 1.1 shows that the curvature κ is constant. Hence it is an open arc of a circle. Conversely, it is obvious that for a circle X the functions in (1.13) are all functions of only h.

In view of the above discussions, for further study we raise two questions as follows.

Question 1.8. Let X denote a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Suppose that there exists a function $\delta(P)$ of only $P \in X$ satisfying

$$\widetilde{S_P}(h) - \widetilde{T_P}(h) = \delta(P). \tag{1.14}$$

Then is it an open arc of a parabola?

Question 1.9. Let X denote a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Suppose that there exists a function $\eta(h)$ of only h satisfying

$$\widetilde{S_P}(h) \div \widetilde{T_P}(h) = \eta(h). \tag{1.15}$$

Then is it an open arc of a circle or a parabola?

In [3], using curvature function κ and support function h of a plane curve, a characterization theorem of ellipses and hyperbolas was established. See also [2,7] for some characterization theorems of ellipses and hyperbolas. For further characterization theorems for parabolas, for examples, see [8, 10, 11]. For the higher dimensional analogues of some results in [3], see [9].

Throughout this article, all curves are of class $C^{(3)}$ and connected, unless otherwise mentioned.

2. A characterization of quadrics

In this section, we prove Theorem 1.3 stated in Section 1. In order to prove Theorem 1.3, we need the following lemma.

Lemma 2.1. *For a $C^{(2)}$ function $f : I \to \mathbb{R}$ defined on an open interval I, we have the following.*

1) *The functional equation $\Phi_f(u, v) = 0$ is translation invariant.*
2) *The functional equation $\Phi_f(u, v) = 0$ is rotation invariant.*

Proof. 1) If we put $g(x) = f(x + a) + b$, then we have

$$A_g(u, v) = A_f(u + a, v + a),\ B_g(u, v) = B_f(u + a, v + a). \tag{2.1}$$

This shows that $\Phi_g(u, v) = \Phi_f(u + a, v + a)$, which completes the proof of 1).

2) We consider the rotation around the origin defined by

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}, \ ad - bc = 1.$$

In case $bf'(x) + a \neq 0$, the function $\overline{y} = g(\overline{x})$ is given by $g(\overline{x}) = cx + df(x)$, where $\overline{x} = ax + bf(x)$. Hence we get

$$g'(\overline{x}) = \frac{df'(x) + c}{bf'(x) + a}, \quad g''(\overline{x}) = \frac{f''(x)}{[bf'(x) + a]^3}. \tag{2.2}$$

This shows

$$\Phi_g(\overline{u}, \overline{v}) = \frac{\Phi_f(u, v)}{[bf'(u) + a]^3 [bf'(v) + a]^3}, \tag{2.3}$$

which completes the proof of 2). □

Suppose that X is an open arc of a quadric. Then, with the help of Lemma 2.1 it suffices to show that the standard form given by

$$f(x) = ax^2, \quad \text{or} \quad \frac{b}{a}\sqrt{a^2 \pm x^2}$$

satisfies the equation $\Phi_f(u, v) = 0$, which can be easily checked.

Conversely, suppose that X denotes the graph of a strictly convex $C^{(3)}$ function defined on an open interval which satisfies the equation $\Phi(u, v) = 0$. Around an arbitrary point P of X, by a suitable translation and rotation around the origin if necessary it is the graph of a $C^{(3)}$ function $f : I \to \mathbb{R}$ defined on an open interval I containing zero with $P = (0, 0)$, $f(0) = f'(0) = 0$

and $f''(x) > 0$ on I. It follows from Lemma 2.1 that the function f satisfies the equation $\Phi_f(u,v) = 0$.

First, we put $u = 0$ into the equation $\Phi_f(u,v) = 0$. Then we have $A = f(v)$ and $B = f(v) - vf'(v)$. Hence we get

$$a[f(v) - vf'(v)]^3 + f''(v)f(v)^3 = 0, \tag{2.4}$$

where we put $a = f''(0)$.

Second, we differentiate $\Phi_f(u,v) = 0$ with respect to u and put again $u = 0$. Then for $b = f'''(0)$ we obtain

$$-3avf(v)^2 f''(v) + 3af'(v)[f(v) - vf'(v)]^2 + b[f(v) - vf'(v)]^3 = 0. \tag{2.5}$$

Combining (2.4) with (2.5), we get

$$\frac{[f(v)-vf'(v)]^2}{f(v)} \left(3a^2vf(v)+bf(v)^2+f'(v)[-bvf(v)+3af(v)-3a^2v^2]\right)=0. \tag{2.6}$$

Since the function f is strictly convex, by replacing v with x, (2.6) yields

$$3a^2xf(x) + bf(x)^2 + f'(x)[-bxf(x) + 3af(x) - 3a^2x^2] = 0. \tag{2.7}$$

Putting $y = f(x)$, from (2.8) we have

$$\left(3a^2xy + by^2\right) dx + \left(3ay - bxy - 3a^2x^2\right) dy = 0. \tag{2.8}$$

Note that y^{-3} is an integrating factor of (2.8). Hence, multiplying (2.8) by y^{-3} and then integrating gives

$$3a^2x^2 + 2bxy - 2cy^2 - 6ay = 0, \tag{2.9}$$

where c is a constant. It follows from (2.9) that around an arbitrary point $P \in X$, the curve X is locally an open arc of a quadric given by

$$f(x) = \begin{cases} \frac{bx-3a+\sqrt{(b^2+6a^2c)x^2-6abx+9a^2}}{2c}, & \text{if } c \neq 0, \\ \frac{3a^2x^2}{2(3a-bx)}, & \text{if } c = 0, \end{cases} \tag{2.10}$$

which is defined on an interval I containing zero. It is straightforward to show that the function f given by (2.10) satisfies the equation $\Phi_f(u,v) = 0$ on the interval I with initial conditions $f(0) = f'(0) = 0$ and $a = f''(0)$, $b = f'''(0)$.

Finally with the aid of the following lemma, using the initial conditions in the same manner as in the proof of Theorem 3 in [6] we can show that the curve X is globally the graph of a function, which is an open arc of a quadric. This completes the proof of Theorem 1.3.

Lemma 2.2. *For three numbers a, b and c with $a > 0$ we denote by $f_{a,b,c}$ the function given in* (2.10)*, which is defined on an interval I containing zero. We consider the function f given by*

$$f(x) = \begin{cases} f_1(x) = f_{a,b,c_1}(x), & \text{if } x \leq 0, \\ f_2(x) = f_{a,b,c_2}(x), & \text{if } x \geq 0, \end{cases} \tag{2.11}$$

then f is a $C^{(3)}$ function on an interval J containing zero. Suppose that the function f satisfies the functional equation $\Phi_f(u,v) = 0$ on J. Then we have $c_1 = c_2$.

Proof. Since the proof of Lemma 2.2 is straightforward and tedious, we omit it. □

3. Archimedean characterization theorem

In this section, using Theorem 1.3 we give another proof of Archimedean characterization theorem for parabolas (Theorem 1.4 in Section 1), which was originally established in [6].

It follows from Proposition 1.2 that any open arcs of parabolas satisfy for all $P \in X$ and sufficiently small $h > 0$

$$S_P(h) = \frac{4}{3} T_P(h). \tag{3.1}$$

Conversely, suppose that X denotes a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ which satisfies (3.1) for all $P \in X$ and sufficiently small $h > 0$. Around an arbitrary point Q of X, by a suitable translation and rotation around the origin if necessary it is the graph of a $C^{(3)}$ function $f : I \to \mathbb{R}$ defined on an open interval I containing zero with $Q = (0,0)$, $f(0) = f'(0) = 0$ and $f''(x) > 0$ on I.

For distinct $u, v \in I$, we put $A = (u, f(u))$ and $B = (v, f(v))$. If we denote by $P = (x, f(x))$ with $x = x(u,v)$ the point where the tangent line to the curve is parallel to the chord AB, then we have

$$(u - v) f'(x(u,v)) = f(u) - f(v). \tag{3.2}$$

Let us differentiate (3.2) with respect to u and v, respectively. Then, we get

$$x_u(u,v) = \frac{f'(u) - f'(x(u,v))}{(u-v) f''(x(u,v))} \tag{3.3}$$

and

$$x_v(u,v) = \frac{f'(x(u,v)) - f'(v)}{(u-v) f''(x(u,v))}. \tag{3.4}$$

We denote by h the distance from the point P to the line through A and B. Then we have

$$2\epsilon S_P(h) = (f(u) + f(v))(v - u) - 2\int_u^v f(w)dw \tag{3.5}$$

and

$$2\epsilon T_P(h) = (x - u)(f(v) - f(u)) - (v - u)(f(x) - f(u)), \tag{3.6}$$

where $\epsilon = 1$ if $u < v$ and $\epsilon = -1$ if $u > v$. Hence it follows from (3.1) that

$$\begin{aligned} &3(f(u)+f(v))(v-u) - 6\int_u^v f(w)dw \\ &= 4\{(x-u)(f(v)-f(u)) - (v-u)(f(x)-f(u))\}. \end{aligned} \tag{3.7}$$

Let us differentiate (3.7) with respect to u and v, respectively. Then, we obtain

$$f'(u)(4x-3u-v) = 4f(x) - 3f(u) - f(v) \tag{3.8}$$

and

$$f'(v)(4x-u-3v) = 4f(x) - f(u) - 3f(v), \tag{3.9}$$

which implies respectively

$$4f(x) = 3f(u) + f(v) + f'(u)(4x-3u-v) \tag{3.10}$$

and

$$4f(x) = f(u) + 3f(v) + f'(v)(4x-u-3v). \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$4x\left(f'(u)-f'(v)\right) = f'(u)(3u+v) - f'(v)(u+3v) - 2\left(f(u)-f(v)\right). \tag{3.12}$$

Now we differentiate (3.10) and (3.11) with respect to u and v, respectively. Then we get

$$x_u(u,v) = \frac{f''(u)(4x-3u-v)}{4\left(f'(x)-f'(u)\right)} \tag{3.13}$$

and

$$x_v(u,v) = \frac{f''(v)(4x-u-3v)}{4\left(f'(x)-f'(v)\right)}. \tag{3.14}$$

Combining (3.3) and (3.13), we get

$$f''(x) = \frac{-4\left(f'(x)-f'(u)\right)^2}{(u-v)f''(u)(4x-3u-v)}. \tag{3.15}$$

From (3.4) and (3.14), we also get

$$f''(x) = \frac{4\left(f'(x)-f'(v)\right)^2}{(u-v)f''(v)(4x-u-3v)}. \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\begin{aligned} &f''(u)(4x-3u-v)\left(f'(x)-f'(v)\right)^2 \\ &+ f''(v)(4x-u-3v)\left(f'(x)-f'(u)\right)^2 = 0. \end{aligned} \tag{3.17}$$

We substitute $4x$ in (3.12) and $f'(x)$ in (3.2) into (3.17). Then we obtain

$$\frac{\Phi_f(u,v)}{(u-v)^2\left(f'(u)-f'(v)\right)} = 0, \tag{3.18}$$

where $\Phi_f(u,v)$ is defined in (1.8). Hence Theorem 1.3 implies that X is an open arc of a quadric (if necessary, we may use Lemma 2.2 again). That is, X is an open arc of one of parabolas, ellipses and hyperbolas.

It is well-known (or straightforward to show) that every ellipse satisfies $S_P(h)/T_P(h) > 4/3$ and every hyperbola satisfies $S_P(h)/T_P(h) < 4/3$. This completes the proof of Theorem 1.4.

4. Proof of Theorem 1.6

In this section, we prove Theorem 1.6 stated in Section 1. We consider a strictly convex $C^{(3)}$ curve X in the plane $\mathbb{R}^2$.

First suppose that there exists a function $\beta(P)$ of only P satisfying (1.11). Then, Lemma 1.1 shows that the function $\beta(P)$ is given by

$$\beta(P) = \lim_{h\to 0}\left(\widetilde{S_P}(h) + \widetilde{T_P}(h)\right) = \frac{7\sqrt{2}}{3\sqrt{\kappa(P)}}, \tag{4.1}$$

and hence (1.11) becomes

$$S_P(h) + T_P(h) = \beta(P)h\sqrt{h}. \tag{4.2}$$

Since $S'_P(h) = L_P(h)$ and $2T_P(h) = hL_P(h)$, by differentiating (4.2) with respect to h, we get

$$L'_P(h) + \frac{3}{h}L_P(h) = 3\beta(P)\frac{1}{\sqrt{h}}. \tag{4.3}$$

After multiplying (4.3) by an integration factor h^3 and integrating gives

$$h^3L_P(h) = \frac{6}{7}\beta(P)h^3\sqrt{h} + c(P), \tag{4.4}$$

where $c(P)$ is a constant independent on h. Tending $h \to 0$, we see that $c(P) = 0$. Hence we obtain

$$L_P(h) = \frac{6}{7}\beta(P)\sqrt{h} = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}\sqrt{h}. \tag{4.5}$$

Thus the curve X satisfies (1.5) and (1.6) and hence (1.7). Therefore Theorem 1.4 completes the proof of 1) of Theorem 1.6.

Now suppose that there exists a function $\gamma(P)$ of only P satisfying (1.12). Then, it follows from Lemma 1.1 that the function $\gamma(P)$ is given by

$$\gamma(P) = \lim_{h\to 0}\left(\widetilde{S_P}(h) \times \widetilde{T_P}(h)\right) = \frac{8}{3\kappa(P)}, \tag{4.6}$$

and hence (1.12) implies

$$S_P(h) = \gamma(P)h^3/T_P(h). \tag{4.7}$$

Note that $S'_P(h) = L_P(h)$ and $2T_P(h) = hL_P(h)$. Hence differentiating (4.7) with respect to h gives

$$2\gamma(P)h^2L'_P(h) - 4\gamma(P)hL_P(h) = -L_P(h)^3, \tag{4.8}$$

which becomes

$$\frac{dL_P(h)}{dh} - \frac{2}{h}L_P(h) = -\frac{1}{2\gamma(P)h^2}L_P(h)^3. \tag{4.9}$$

We put $v = L_P(h)^{-2}$. Then we obtain

$$\frac{dv}{dh} + \frac{4}{h}v = \frac{1}{\gamma(P)h^2}. \tag{4.10}$$

After multiplying (4.10) by an integration factor h^4 and integrating, we get

$$L_P(h)^{-2} = v = \frac{3\gamma(P)c(P) + h^3}{3\gamma(P)h^4}, \tag{4.11}$$

where $c(P)$ is a constant independent on h.

Suppose that the constant $c(P)$ is not zero. Then (4.11) shows that

$$\lim_{h\to 0} L_P(h)^2/h = 0,$$

which contradicts Lemma 1.1. Hence we see that $c(P) = 0$. Therefore we obtain

$$L_P(h) = \sqrt{3\gamma(P)}\sqrt{h} = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}\sqrt{h}. \tag{4.12}$$

Thus, as in the proof of 1), Theorem 1.4 completes the proof of 2) of Theorem 1.6.

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