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RANDOM SAMPLING AND RECONSTRUCTION OF SIGNALS WITH FINITE RATE OF INNOVATION

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Abstract. In this paper, we mainly study the random sampling and reconstruction of signals living in the subspace $V^p(\Phi, \Lambda)$ of $L^p(\mathbb{R}^d)$, which is generated by a family of molecules Φ located on a relatively separated subset $\Lambda \subset \mathbb{R}^d$. The space $V^p(\Phi, \Lambda)$ is used to model signals with finite rate of innovation, such as stream of pulses in GPS applications, cellular radio and ultra wide-band communication. The sampling set is independently and randomly drawn from a general probability distribution over $\mathbb{R}^d$. Under some proper conditions for the generators $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$ and the probability density function ρ, we first approximate $V^p(\Phi, \Lambda)$ by a finite dimensional subspace $V^p_N(\Phi, \Lambda)$ on any bounded domains. Then, we prove that the random sampling stability holds with high probability for all signals in $V^p(\Phi, \Lambda)$ whose energy concentrate on a cube when the sampling size is large enough. Finally, a reconstruction algorithm based on random samples is given for signals in $V^p_N(\Phi, \Lambda)$.

1. Introduction

Random sampling plays an important role in many fields, such as image processing [5], compressed sensing [7] and learning theory [13]. Random sampling has been generally studied for multivariate trigonometric polynomials [1], bandlimited signals [2,3], signals that satisfy some locality properties in short-time Fourier transform [16], signals with bounded derivatives [19] and signals in a shift-invariant space [8,9,18,20]. Moreover, random samples were almost all taken from a uniform distribution on a bounded domain $[-K, K]^d$, although

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the probability density function in [9] is assumed to have support in $[-K, K]^d$ and has nonzero lower bound.

Recently, random sampling in signal spaces with finite rate of innovation was firstly studied in [10], where the used techniques depend on the properties in the Hilbert space $L^2(\mathbb{R}^d)$. In this paper, we mainly study the random sampling and reconstruction of signals with finite rate of innovation in a Banach space $L^p(\mathbb{R}^d)$. Moreover, the random samples are drawn over $\mathbb{R}^d$ from a general probability distribution, which may have more and more important applications in real world [5].

The space with finite rate of innovation (FRI) is used to model signals with finite degree of freedom in unit time, which was firstly introduced in [17] and were further studied in [14, 15] from the mathematical viewpoint.

Let Λ be a relatively separated subset of $\mathbb{R}^d$, which means that

$$D(\Lambda) := \sup_{x\in\mathbb{R}^d} \sum_{\lambda\in\Lambda} \chi_{\lambda+[0,1]^d}(x) < \infty, \tag{1}$$

where χ denotes the characteristic function. The space $L^p(\mathbb{R}^d)$ consists of all functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\|f\|_{L^p(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |f(x)|^p dx \Big)^{1/p} < \infty \tag{2}$$

for $1 \le p < \infty$ and $\|f\|_{L^\infty(\mathbb{R}^d)} = \operatorname{ess}\sup_{x\in\mathbb{R}^d} |f(x)| < \infty$.

Given a relatively separated set Λ, the subspaces of $L^p(\mathbb{R}^d)$ with finite rate of innovation are defined as

$$V^p(\Phi,\Lambda) = \Big\{ \sum_{\lambda\in\Lambda} c(\lambda)\phi_\lambda : \big\|(c(\lambda))_{\lambda\in\Lambda}\big\|_{\ell^p(\Lambda)} < \infty \Big\}. \tag{3}$$

Here, $\big\|(c(\lambda))_{\lambda\in\Lambda}\big\|_{\ell^p(\Lambda)} = \Big(\sum_{\lambda\in\Lambda} |c(\lambda)|^p \Big)^{1/p}$ for $1 \le p < \infty$ and $\big\|(c(\lambda))_{\lambda\in\Lambda}\big\|_{\ell^\infty(\Lambda)} = \sup_{\lambda\in\Lambda} |c(\lambda)|$.

The FRI model spaces $V^p(\Phi,\Lambda)$ can contain many signal spaces, such as bandlimited spaces, finitely generated shift-invariant spaces, nonuniform spline spaces for modeling electrocardiogram signals, twisted shift-invariant spaces in Gabor system, and so on. Moreover, they have been widely applied to many scientific fields, such as curve fitting and radar imaging [11, 12].

In this paper, we always suppose that the generators $\Phi = \{\phi_\lambda : \lambda \in \Lambda\}$ satisfy the following two assumptions:

(A1) $\|\Phi\|_{\infty,q,u} < \infty$ for some $1 \le q \le \infty$ and weight function $u(x) = u_\alpha(x) = (1+|x|)^\alpha$ with $\alpha > \max\big\{d(1-1/q), d(1-1/p)\big\}$. Here,

$$\|\Phi\|_{p,q,u} = \sup_{\lambda\in\Lambda} \Big\| \Big(\big\|\phi_\lambda(\cdot)u(\cdot-\lambda)\big\|_{L^p(k+[0,1]^d)} \Big)_{k\in\mathbb{Z}^d} \Big\|_{\ell^q(\mathbb{Z}^d)}$$

$$+\sup_{k\in\mathbb{Z}^d}\left\|\Big(\|\phi_\lambda(\cdot)u(\cdot-\lambda)\|_{L^p(k+[0,1]^d)}\Big)_{\lambda\in\Lambda}\right\|_{\ell^q(\Lambda)}. \tag{4}$$

(A2) Φ is a Riesz basis of $V^2(\Phi,\Lambda)$ which means that there exists a positive constant $C_0\ge 1$ such that

$$C_0^{-1}\|c\|_{\ell^2(\Lambda)}\le\Big\|\sum_{\lambda\in\Lambda}c(\lambda)\phi_\lambda\Big\|_{L^2(\mathbb{R}^d)}\le C_0\|c\|_{\ell^2(\Lambda)} \tag{5}$$

holds for all $c=\big(c(\lambda)\big)_{\lambda\in\Lambda}\in\ell^2(\Lambda)$ and $\{\phi_\lambda:\lambda\in\Lambda\}$ is a basis of $V^2(\Phi,\Lambda)$.

Let $0<\delta<1$ and $C_K=[-K,K]^d$ for $K>0$. Define a compact subset of $V^p(\Phi,\Lambda)$ by

$$V_K^p(\Phi,\Lambda)=\left\{f\in V^p(\Phi,\Lambda):\int_{C_K}|f(x)|^p dx\ge(1-\delta)\int_{\mathbb{R}^d}|f(x)|^p dx\right\}, \tag{6}$$

which contains all functions in $V^p(\Phi,\Lambda)$ whose energy concentrates on the cube C_K.

This paper is organized as follows. In Section 2, we show that $V^p(\Phi,\Lambda)$ can be approximated by a finite dimensional subspace $V_N^p(\Phi,\Lambda)$ on any bounded domains. A result about the covering number for the normalized $V_N^p(\Phi,\Lambda)$ is shown in Section 3. In Section 4, we prove that the sampling inequality holds with high probability for all functions in $V_K^p(\Phi,\Lambda)$. In Section 5, we estimate the condition number for random matrix and provide a reconstruction algorithm based on random samples for functions in $V_N^p(\Phi,\Lambda)$.

2. Approximation to $V^p(\Phi,\Lambda)$

In this section, we will show that $V^p(\Phi,\Lambda)$ can be approximated by a finite dimensional subspace on any bounded domains.

For a given positive integer N, define a finite dimensional subspace

$$V_N^p(\Phi,\Lambda)=\left\{\sum_{\lambda\in\Lambda\cap[-N,N]^d}c(\lambda)\phi_\lambda:c(\lambda)\in\mathbb{R}\right\} \tag{7}$$

of $V^p(\Phi,\Lambda)$ and its normalization

$$V_N^{p,*}(\Phi,\Lambda)=\left\{f\in V_N^p(\Phi,\Lambda):\|f\|_{L^p(\mathbb{R}^d)}=1\right\}. \tag{8}$$

Lemma 2.1 ([15])**.** *Let $1\le q\le\infty$ and $\alpha>d(1-1/q)$. Suppose that Λ is a relatively separated subset of $\mathbb{R}^d$ and $\|\Phi\|_{\infty,q,u}<\infty$. If Φ is a Riesz basis of $V^2(\Phi,\Lambda)$, then Φ is a p-Riesz basis of the space $V^p(\Phi,\Lambda)$ for any $1\le p\le\infty$, that is, there exist positive constants c_p and C_p such that*

$$c_p\|c\|_{\ell^p(\Lambda)}\le\Big\|\sum_{\lambda\in\Lambda}c(\lambda)\phi_\lambda\Big\|_{L^p(\mathbb{R}^d)}\le C_p\|c\|_{\ell^p(\Lambda)} \tag{9}$$

holds for any $(c(\lambda))_{\lambda\in\Lambda}\in\ell^p(\Lambda)$.

In the following, we will show that $V^p(\Phi,\Lambda)$ can be approximated by $V^p_N(\Phi,\Lambda)$ on any bounded domains $C_R=[-R,R]^d$ for $R>0$.

Lemma 2.2. *Let $1\le p\le\infty$ and p' be the conjugate number of p. Suppose that Φ satisfy the assumptions (A1) and (A2). If $f\in V^p(\Phi,\Lambda)$ and $\|f\|_{L^p(\mathbb{R}^d)}=1$, then for given $\varepsilon_1>0$ and $\varepsilon_2>0$, there exists an $f_N\in V^p_N(\Phi,\Lambda)$ such that*

$$\|f-f_N\|_{L^p(C_R)}\le\varepsilon_1 \tag{10}$$

when

$$\begin{aligned}N\ge R+\Big[\frac{\|\Phi\|_{\infty,q,u}(2R)^{d/p}}{c_p\varepsilon_1}\Big(\frac{d2^dD(\Lambda)(1+R)^{d-1}}{\alpha p'-d}\Big)^{1/p'}\Big]^{\frac{p'}{\alpha p'-d}}\\ =: N_1(\varepsilon_1,R)\end{aligned} \tag{11}$$

and

$$\|f-f_N\|_{L^\infty(C_R)}\le\varepsilon_2 \tag{12}$$

when

$$N\ge R+\Big[\frac{\|\Phi\|_{\infty,q,u}}{c_p\varepsilon_2}\Big(\frac{d2^dD(\Lambda)(1+R)^{d-1}}{\alpha p'-d}\Big)^{1/p'}\Big]^{\frac{p'}{\alpha p'-d}}=:N_2(\varepsilon_2,R). \tag{13}$$

Proof. It follows from the assumption $\|\Phi\|_{\infty,q,u}<\infty$ that

$$|\phi_\lambda(x)|\le\frac{\|\Phi\|_{\infty,q,u}}{(1+|x-\lambda|)^\alpha},\ \forall x\in\mathbb{R}^d,\ \lambda\in\Lambda. \tag{14}$$

For $f=\sum_{\lambda\in\Lambda}c(\lambda)\phi_\lambda\in V^p(\Phi,\Lambda)$, take $N>R$ and choose

$$f_N=\sum_{\lambda\in\Lambda\cap[-N,N]^d}c(\lambda)\phi_\lambda\in V^p_N(\Phi,\Lambda). \tag{15}$$

Then by Lemma 2.1, one has

$$\begin{aligned}\|f-f_N\|^p_{L^p(C_R)}&=\int_{C_R}\Big|\sum_{\lambda\in\Lambda\cap\{\mathbb{R}^d\setminus[-N,N]^d\}}c(\lambda)\phi_\lambda(x)\Big|^p dx\\ &\le\frac{1}{c_p^p}\int_{C_R}\Big(\sum_{\lambda\in\Lambda\cap\{\mathbb{R}^d\setminus[-N,N]^d\}}|\phi_\lambda(x)|^{p'}\Big)^{\frac{p}{p'}}dx\\ &\le\frac{\|\Phi\|^p_{\infty,q,u}}{c_p^p}\int_{C_R}\Big(\sum_{\lambda\in\Lambda\cap\{\mathbb{R}^d\setminus[-N,N]^d\}}\frac{1}{(1+|x-\lambda|)^{\alpha p'}}\Big)^{\frac{p}{p'}}dx.\end{aligned} \tag{16}$$

For $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_d)\in\Lambda$, let $|\lambda|=\max\{|\lambda_1|,|\lambda_2|,\dots,|\lambda_d|\}$. Since Λ is a relatively separated set of $\mathbb{R}^d$, for any $m\in\mathbb{N}$, one has

$$\sharp\{\lambda\in\Lambda: m<|\lambda|\le m+1\}\le D(\Lambda)\big((2m+2)^d-(2m)^d\big). \tag{17}$$

Then we can obtain

$$\begin{aligned}
&\sum_{\lambda\in\Lambda\cap\{\mathbb{R}^d\setminus[-N,N]^d\}} \frac{1}{(1+|x-\lambda|)^{\alpha p'}} \\
&= \sum_{\lambda\in\Lambda,|\lambda|>N} \frac{1}{(1+|x-\lambda|)^{\alpha p'}} \\
&\le \sum_{m=N}^{\infty} \sum_{\lambda\in\Lambda,m<|\lambda|\le m+1} \frac{1}{(1+|\lambda|-R)^{\alpha p'}} \\
&\le D(\Lambda) \sum_{m=N}^{\infty} \frac{(2m+2)^d-(2m)^d}{(1+m-R)^{\alpha p'}} \\
&\le d2^d D(\Lambda) \sum_{m=N}^{\infty} \frac{(m+1)^{d-1}}{(1+m-R)^{\alpha p'}} \\
&\le d2^d D(\Lambda) \sum_{m=N}^{\infty} \frac{(1+R)^{d-1}}{(1+m-R)^{\alpha p'-d+1}} \\
&\le d2^d D(\Lambda)(1+R)^{d-1} \int_{N-R-1}^{\infty} \frac{1}{(1+y)^{\alpha p'-d+1}} dy \\
&= d2^d D(\Lambda)(1+R)^{d-1} \frac{1}{\alpha p'-d} (N-R)^{d-\alpha p'}. \qquad (18)
\end{aligned}$$

This together with (16) obtains

$$\begin{aligned}
&\|f-f_N\|_{L^p(C_R)} \\
&\le \frac{\|\Phi\|_{\infty,q,u}(2R)^{d/p}}{c_p} \Big(\frac{d2^d D(\Lambda)(1+R)^{d-1}}{\alpha p'-d}\Big)^{1/p'} (N-R)^{d/p'-\alpha}. \qquad (19)
\end{aligned}$$

Finally, we obtain (10) from (19). The desired result (12) follows from

$$\|f-f_N\|_{L^\infty(C_R)} \le \frac{\|\Phi\|_{\infty,q,u}}{c_p} \Big(\frac{d2^d D(\Lambda)(1+R)^{d-1}}{\alpha p'-d}\Big)^{1/p'} (N-R)^{d/p'-\alpha}. \qquad \square$$

3. Covering number for $V_N^{p,*}(\Phi,\Lambda)$

In this section, we discuss the covering number of $V_N^{p,*}(\Phi,\Lambda)$ with respect to the norm $\|\cdot\|_{L^\infty(\mathbb{R}^d)}$. Let S be a metric space and $\eta>0$, the covering number $\mathcal{N}(S,\eta)$ is defined to be the minimal integer $m\in\mathbb{N}$ such that there exist m disks with radius η covering S.

Lemma 3.1 ([6])**.** *Suppose that $\mathbb{E}$ is a finite dimensional Banach space with $\dim\mathbb{E}=s$. Let $B_\varepsilon:=\{x\in\mathbb{E}:\|x\|\le\varepsilon\}$ be the closed ball of radius ε centered at the origin. Then*

$$\mathcal{N}(B_\varepsilon,\eta) \le \Big(\frac{2\varepsilon}{\eta}+1\Big)^s.$$

Note that

$$\dim\Big(V_N^p(\Phi,\Lambda)\Big) \le \sharp\big\{\lambda\in\Lambda : \lambda\in[-N,N]^d\big\} \le (2N)^d D(\Lambda). \tag{20}$$

Then by Lemma 3.1, we have the following result.

Lemma 3.2. *Let $V_N^{p,*}(\Phi,\Lambda)$ be defined by (8). Then for any $\eta>0$, the covering number of $V_N^{p,*}(\Phi,\Lambda)$ concerning the norm $\|\cdot\|_{L^p(\mathbb{R}^d)}$ is bounded by*

$$\mathcal{N}\big(V_N^{p,*}(\Phi,\Lambda),\eta\big) \le \exp\bigg((2N)^d D(\Lambda)\ln\Big(\frac{2}{\eta}+1\Big)\bigg).$$

Lemma 3.3. *Suppose that Φ satisfy the assumptions (A1) and (A2). Then for every $f\in V^p(\Phi,\Lambda)$, we have*

$$\|f\|_{L^\infty(\mathbb{R}^d)} \le C^*\|f\|_{L^p(\mathbb{R}^d)}, \tag{21}$$

where

$$C^* = \frac{\|\Phi\|_{\infty,q,u}}{c_p}\bigg(D(\Lambda)+D(\Lambda)\sum_{m=1}^{\infty}\frac{(2m+2)^d-(2m)^d}{m^{\alpha p'}}\bigg)^{1/p'}. \tag{22}$$

Proof. Suppose that $f=\sum_{\lambda\in\Lambda}c(\lambda)\phi_\lambda\in V^p(\Phi,\Lambda)$. Then it follows from (14) and Lemma 2.1 that

$$\begin{aligned}
\|f\|_{L^\infty(\mathbb{R}^d)} &\le \sup_{x\in\mathbb{R}^d}\sum_{\lambda\in\Lambda}|c(\lambda)\phi_\lambda| \\
&\le \|\Phi\|_{\infty,q,u}\|c\|_{\ell^p}\sup_{x\in\mathbb{R}^d}\bigg(\sum_{\lambda\in\Lambda}\frac{1}{(1+|x-\lambda|)^{\alpha p'}}\bigg)^{1/p'} \\
&\le \frac{\|\Phi\|_{\infty,q,u}}{c_p}\|f\|_{L^p(\mathbb{R}^d)}\sup_{x\in\mathbb{R}^d}\bigg(\sum_{\lambda\in\Lambda}\frac{1}{(1+|x-\lambda|)^{\alpha p'}}\bigg)^{1/p'}. \qquad (23)
\end{aligned}$$

Note that $\sup_{k\in\mathbb{Z}^d}\sharp\big(\Lambda\cap(k+[0,1]^d)\big)\le D(\Lambda)$. Then

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^d}\bigg(\sum_{\lambda\in\Lambda}\frac{1}{(1+|x-\lambda|)^{\alpha p'}}\bigg)^{1/p'} \\
&\le \sup_{k\in\mathbb{Z}^d}\sup_{x\in k+[0,1]^d}\bigg(\sum_{\lambda\in\Lambda\cap(k+[0,1]^d)}\frac{1}{(1+|x-\lambda|)^{\alpha p'}} \\
&\qquad\qquad + \sum_{\lambda\in\Lambda\cap\{\mathbb{R}^d\setminus(k+[0,1]^d)\}}\frac{1}{(1+|x-\lambda|)^{\alpha p'}}\bigg)^{1/p'} \\
&\le \sup_{k\in\mathbb{Z}^d}\bigg(D(\Lambda)+\sum_{\lambda\in\Lambda,|\lambda-k|>1}\frac{1}{|\lambda-k|^{\alpha p'}}\bigg)^{1/p'}
\end{aligned}$$

$$\le \sup_{k\in\mathbb{Z}^d}\left(D(\Lambda)+\sum_{m=1}^{\infty}\sum_{\lambda\in\Lambda, m<|\lambda-k|\le m+1}\frac{1}{m^{\alpha p'}}\right)^{1/p'}$$

$$\le \left(D(\Lambda)+D(\Lambda)\sum_{m=1}^{\infty}\frac{(2m+2)^d-(2m)^d}{m^{\alpha p'}}\right)^{1/p'}. \tag{24}$$

Since $\alpha > d(1-1/p)$, we have $\sum_{m=1}^{\infty}\frac{(2m+2)^d-(2m)^d}{m^{\alpha p'}} < \infty$. Finally, it follows from (23) and (24) that (21) holds. □

Lemma 3.4. *Suppose that Φ satisfy the assumptions $(A1)$ and $(A2)$. Then the covering number of $V_N^{p,*}(\Phi,\Lambda)$ with respect to $\|\cdot\|_{L^\infty(\mathbb{R}^d)}$ is bounded by*

$$\mathcal{N}\big(V_N^{p,*}(\Phi,\Lambda),\eta\big) \le \exp\left((2N)^d D(\Lambda)\ln\Big(\frac{2C^*}{\eta}+1\Big)\right).$$

Proof. By Lemma 3.2, the covering number of $V_N^{p,*}(\Phi,\Lambda)$ with respect to $\|\cdot\|_{L^p(\mathbb{R}^d)}$ satisfies

$$\mathcal{N}\Big(V_N^{p,*}(\Phi),\frac{\eta}{C^*}\Big) \le \exp\left((2N)^d D(\Lambda)\ln\Big(\frac{2C^*}{\eta}+1\Big)\right). \tag{25}$$

Let $\mathcal{F}$ be the corresponding $\frac{\eta}{C^*}$-net for $V_N^{p,*}(\Phi,\Lambda)$. It means that for every $f \in V_N^{p,*}(\Phi,\Lambda)$, there exists an $\widetilde{f}\in\mathcal{F}$ such that $\|f-\widetilde{f}\|_{L^p(\mathbb{R}^d)} \le \frac{\eta}{C^*}$. By Lemma 3.3, we have

$$\|f-\widetilde{f}\|_{L^\infty(\mathbb{R}^d)} \le C^*\|f-\widetilde{f}\|_{L^p(\mathbb{R}^d)} \le \eta.$$

Therefore, $\mathcal{F}$ is also an η-net of $V_N^{p,*}(\Phi,\Lambda)$ with respect to the norm $\|\cdot\|_{L^\infty(\mathbb{R}^d)}$. Since

$$\sharp(\mathcal{F}) \le \exp\left((2N)^d D(\Lambda)\ln\Big(\frac{2C^*}{\eta}+1\Big)\right),$$

the desired result is proved. □

4. Random sampling inequality of $V_K^p(\Phi,\Lambda)$

Let $X=\{x_j : j\in\mathbb{N}\}$ be a sequence of independent random variables that are drawn from a general probability distribution over $\mathbb{R}^d$ with density function ρ satisfying

$$0 < c_\rho = \operatorname{ess}\inf_{x\in C_K}\rho(x) \text{ and } C_\rho = \operatorname{ess}\sup_{x\in\mathbb{R}^d}\rho(x) < \infty. \tag{26}$$

Then for any $f\in V^p(\Phi,\Lambda)$, we introduce the random variables

$$X_j(f) = |f(x_j)|^p - \int_{\mathbb{R}^d}\rho(x)|f(x)|^p dx. \tag{27}$$

It is easy to see that $X_j(f)$ is a sequence of independent random variables with expectation $\mathbb{E}[X_j(f)]=0$. Next, we will give some estimates for $X_j(f)$.

Lemma 4.1. *Let $\rho(x)$ be a probability density function over $\mathbb{R}^d$ satisfying* (26). *Then for any $f,g\in V^p(\Phi,\Lambda)$, the following inequalities hold*:

(1) $\|X_j(f)\|_{\ell^\infty} \le \|f\|^p_{L^\infty(\mathbb{R}^d)}$.

(2) $\|X_j(f) - X_j(g)\|_{\ell^\infty} \le 2p\Big(\max\big\{\|f\|_{L^\infty(\mathbb{R}^d)}, \|g\|_{L^\infty(\mathbb{R}^d)}\big\}\Big)^{p-1}$
$\|f-g\|_{L^\infty(\mathbb{R}^d)}$.

(3) $Var(X_j(f)) \le C_\rho \|f\|^p_{L^\infty(\mathbb{R}^d)}\|f\|^p_{L^p(\mathbb{R}^d)}$.

(4) $Var\big(X_j(f) - X_j(g)\big) \le pC_\rho\Big(\max\big\{\|f\|_{L^\infty(\mathbb{R}^d)}, \|g\|_{L^\infty(\mathbb{R}^d)}\big\}\Big)^{p-1}$
$\|f-g\|_{L^\infty(\mathbb{R}^d)}\big(\|f\|^p_{L^p(\mathbb{R}^d)} + \|g\|^p_{L^p(\mathbb{R}^d)}\big)$.

Proof. (1) Direct computation obtains

$$\|X_j(f)\|_{\ell^\infty} \le \sup_{x\in\mathbb{R}^d} \max\left\{|f(x)|^p, \int_{\mathbb{R}^d} \rho(x)|f(x)|^p dx\right\} \le \|f\|^p_{L^\infty(\mathbb{R}^d)}.$$

(2) By mean value theorem, one has

$$\begin{aligned}
&\|X_j(f) - X_j(g)\|_{\ell^\infty}\\
\le\ &\sup_{x\in\mathbb{R}^d}\left(\Big||f(x)|^p - |g(x)|^p\Big| + \int_{\mathbb{R}^d}\rho(x)\Big||f(x)|^p - |g(x)|^p\Big|dx\right)\\
\le\ &2\sup_{x\in\mathbb{R}^d}\Big||f(x)|^p - |g(x)|^p\Big|\\
=\ &2p\Big(\max\big\{\|f\|_{L^\infty(\mathbb{R}^d)}, \|g\|_{L^\infty(\mathbb{R}^d)}\big\}\Big)^{p-1}\|f-g\|_{L^\infty(\mathbb{R}^d)}.
\end{aligned}$$

(3) Since $\mathbb{E}[X_j(f)] = 0$, we have

$$\begin{aligned}
Var(X_j(f)) &= \mathbb{E}[(X_j(f))^2]\\
&= \mathbb{E}\big[|f(x_j)|^{2p}\big] - \Big(\int_{\mathbb{R}^d}\rho(x)|f(x)|^p dx\Big)^2\\
&\le \int_{\mathbb{R}^d}\rho(x)|f(x)|^{2p}dx\\
&\le C_\rho\|f\|^p_{L^\infty(\mathbb{R}^d)}\|f\|^p_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

(4) Using the similar method as (3), we have

$$\begin{aligned}
&Var\big(X_j(f) - X_j(g)\big)\\
=\ &\mathbb{E}\big[\big(X_j(f) - X_j(g)\big)^2\big]\\
\le\ &C_\rho\int_{\mathbb{R}^d}\Big(|f(x)|^p - |g(x)|^p\Big)^2 dx\\
\le\ &C_\rho\int_{\mathbb{R}^d}\Big||f(x)|^p - |g(x)|^p\Big|\Big(|f(x)|^p + |g(x)|^p\Big)dx\\
\le\ &C_\rho\sup_{x\in\mathbb{R}^d}\Big||f(x)|^p - |g(x)|^p\Big|\Big(\|f\|^p_{L^p(\mathbb{R}^d)} + \|g\|^p_{L^p(\mathbb{R}^d)}\Big)\\
\le\ &pC_\rho\Big(\max\big\{\|f\|_{L^\infty(\mathbb{R}^d)}, \|g\|_{L^\infty(\mathbb{R}^d)}\big\}\Big)^{p-1}\|f-g\|_{L^\infty(\mathbb{R}^d)}
\end{aligned}$$

$$\times \big(\|f\|^p_{L^p(\mathbb{R}^d)} + \|g\|^p_{L^p(\mathbb{R}^d)}\big). \qquad \square$$

In the following lemma, we will show that a uniform large deviation inequality holds for functions in $V_N^{p,*}(\Phi,\Lambda)$ by Bernstein's inequality.

Lemma 4.2 (Bernstein's inequality [4])**.** *Let $X_1, X_2, \ldots, X_n$ be independent random variables with expected values $\mathbb{E}(X_j) = 0$ for $j = 1, 2, \ldots, n$. Assume that $Var(X_j) \le \sigma^2$ and $|X_j| \le M$ almost surely for all j. Then for any $\lambda \ge 0$,*

$$Prob\bigg(\Big|\sum_{j=1}^n X_j\Big| \ge \lambda\bigg) \le 2\exp\bigg(- \frac{\lambda^2}{2n\sigma^2 + \frac{2}{3}M\lambda}\bigg).$$

Lemma 4.3. *Let $\{x_j : j \in \mathbb{N}\}$ be a sequence of independent random variables that are drawn from a general probability distribution over $\mathbb{R}^d$ with density function ρ satisfying* (26)*. If $f \in V_N^{p,*}(\Phi,\Lambda)$, then for $n \in \mathbb{N}$ and $\lambda \ge 0$,*

$$Prob\bigg(\sup_{f\in V_N^{p,*}(\Phi,\Lambda)} \Big|\sum_{j=1}^n X_j(f)\Big| \ge \lambda\bigg) \le A\exp\bigg(- B\frac{\lambda^2}{12nC_\rho + 2\lambda}\bigg),$$

where A is of order $\exp(CN^d)$ *with* $B = \min\{\frac{\sqrt{2}}{2592p(C^*)^{p-1}}, \frac{3}{2(C^*)^p}\}$ *and C depending on Λ and Φ.*

Proof. For given $\ell \in \mathbb{N}$, we construct a $2^{-\ell}$-covering for $V_N^{p,*}(\Phi,\Lambda)$ with respect to the norm $\|\cdot\|_{L^\infty(\mathbb{R}^d)}$. Let $\mathcal{C}_\ell$ be the corresponding $2^{-\ell}$-net for $\ell = 1, 2, \ldots$. Then, $\mathcal{C}_\ell$ has cardinality at most $\mathcal{N}\big(V_N^{p,*}(\Phi,\Lambda), 2^{-\ell}\big)$.

For given $f \in V_N^{p,*}(\Phi,\Lambda)$, let f_ℓ be the function in $\mathcal{C}_\ell$ that is closest to f with respect to the norm $\|\cdot\|_{L^\infty(\mathbb{R}^d)}$. Then, $\|f - f_\ell\|_{L^\infty(\mathbb{R}^d)} \le 2^{-\ell} \to 0$ when $\ell \to \infty$. Moreover, by Lemma 3.3 and the item (2) of Lemma 4.1, we have

$$X_j(f) = X_j(f_1) + (X_j(f_2) - X_j(f_1)) + (X_j(f_3) - X_j(f_2)) + \cdots .$$

If $\sup_{f\in V_N^{p,*}(\Phi,\Lambda)} \Big|\sum_{j=1}^n X_j(f)\Big| \ge \lambda$, the event ω_ℓ must hold for some $\ell \ge 1$, where

$$\omega_1 = \bigg\{\text{there exists } f_1 \in \mathcal{C}_1 \text{ such that} \Big|\sum_{j=1}^n X_j(f_1)\Big| \ge \frac{\lambda}{2}\bigg\}$$

and for $\ell \ge 2$,

$$\omega_\ell = \bigg\{\text{there exist } f_\ell \in \mathcal{C}_\ell \text{ and } f_{\ell-1} \in \mathcal{C}_{\ell-1} \text{ with } \|f_\ell - f_{\ell-1}\|_{L^\infty(\mathbb{R}^d)} \le 3\cdot 2^{-\ell},$$
$$\text{such that } \Big|\sum_{j=1}^n \big(X_j(f_\ell) - X_j(f_{\ell-1})\big)\Big| \ge \frac{\lambda}{2\ell^2}\bigg\}.$$

If this is not the case, then with $f_0 = 0$, we have

$$\Big|\sum_{j=1}^n X_j(f)\Big| \le \sum_{\ell=1}^\infty \Big|\sum_{j=1}^n (X_j(f_\ell) - X_j(f_{\ell-1}))\Big| \le \sum_{\ell=1}^\infty \frac{\lambda}{2\ell^2} = \frac{\pi^2\lambda}{12} \le \lambda.$$

Next, we estimate the probability of each ω_ℓ. By Lemmas 3.3, 4.1 and 4.2, for every fixed $f \in \mathcal{C}_1$,

$$Prob\Big(\Big|\sum_{j=1}^{n} X_j(f)\Big| \geq \frac{\lambda}{2}\Big) \leq 2\exp\Bigg(-\frac{(\frac{\lambda}{2})^2}{2nVar(X_j(f)) + \frac{2}{3}\|X_j(f)\|_{\ell^\infty}\cdot\frac{\lambda}{2}}\Bigg)$$
$$\leq 2\exp\Bigg(-\frac{\lambda^2}{8nC_\rho(C^*)^p + \frac{4}{3}\lambda(C^*)^p}\Bigg).$$

By Lemma 3.4, there are at most

$$\mathcal{N}\Big(V_N^{p,*}(\Phi,\Lambda), \frac{1}{2}\Big) \leq \exp\Big((2N)^d D(\Lambda)\ln(4C^*+1)\Big)$$

functions in $\mathcal{C}_1$. Thus, the probability of ω_1 is bounded by

$$Prob(\omega_1) \leq 2\exp\Big((2N)^d D(\Lambda)\ln(4C^*+1)\Big)\exp\Bigg(-\frac{\lambda^2}{8nC_\rho(C^*)^p + \frac{4}{3}\lambda(C^*)^p}\Bigg)$$
$$= 2\exp\Big(2^d N^d D(\Lambda)\ln(4C^*+1)\Big)\exp\Bigg(-\frac{\lambda^2}{\frac{2}{3}(C^*)^p(12nC_\rho+2\lambda)}\Bigg). \tag{28}$$

For $\ell \geq 2$, we estimate the probability of ω_ℓ in a similar way. For $f \in \mathcal{C}_\ell$, $g \in \mathcal{C}_{\ell-1}$ and $\|f-g\|_{L^\infty(\mathbb{R}^d)} \leq 3\cdot 2^{-\ell}$, we have

$$Prob\Big(\Big|\sum_{j=1}^{n}(X_j(f) - X_j(g))\Big| \geq \frac{\lambda}{2\ell^2}\Big)$$
$$\leq 2\exp\Bigg(-\frac{(\frac{\lambda}{2\ell^2})^2}{2nVar(X_j(f)-X_j(g)) + \frac{2}{3}\|X_j(f)-X_j(g)\|_{\ell^\infty}\cdot\frac{\lambda}{2\ell^2}}\Bigg)$$
$$\leq 2\exp\Big(-\frac{\upsilon 2^\ell}{\ell^4}\Big),$$

where $\upsilon = \frac{\lambda^2}{4p(C^*)^{p-1}(12nC_\rho+2\lambda)}$. There are at most $\mathcal{N}\big(V_N^{p,*}(\Phi,\Lambda), 2^{-\ell}\big)$ functions in $\mathcal{C}_\ell$ and $\mathcal{N}\big(V_N^{p,*}(\Phi,\Lambda), 2^{-\ell+1}\big)$ functions in $\mathcal{C}_{\ell-1}$. Therefore, we have

$$Prob\Big(\bigcup_{\ell=2}^{\infty}\omega_\ell\Big) \leq \sum_{\ell=2}^{\infty}\mathcal{N}\big(V_N^{p,*}(\Phi,\Lambda), 2^{-\ell}\big)\mathcal{N}\big(V_N^{p,*}(\Phi,\Lambda), 2^{-\ell+1}\big)2\exp\Big(-\frac{\upsilon 2^\ell}{\ell^4}\Big)$$
$$\leq 2(2C^*+1)^{2(2N)^d D(\Lambda)}\sum_{\ell=2}^{\infty}\exp\Big((2\ln 2)(2N)^d D(\Lambda)\ell - \frac{\upsilon 2^\ell}{\ell^4}\Big)$$
$$=: C_1\sum_{\ell=2}^{\infty}\exp\Big(C_2\ell - \frac{\upsilon 2^\ell}{\ell^4}\Big)$$
$$= C_1\sum_{\ell=2}^{\infty}\exp\Big(-\upsilon 2^{\frac{\ell}{2}}\Big(\frac{2^{\frac{\ell}{2}}}{\ell^4} - \frac{C_2\ell}{2^{\frac{\ell}{2}}\upsilon}\Big)\Big),$$

where $C_1 = 2(2C^*+1)^{2(2N)^d D(\Lambda)}$ and $C_2 = (2\ln 2)(2N)^d D(\Lambda)$.

Let $C_3 := \min_{\ell \ge 2} \frac{2^{\frac{\ell}{2}}}{\ell^4} = \frac{1}{324}$ and $C_4 := \max_{\ell \ge 2} \frac{8p(C^*)^{p-1}\ell \ln 2}{2^{\frac{\ell}{2}}} = 6\sqrt{2}p(C^*)^{p-1}\ln 2$. Then

$$\begin{aligned}\frac{2^{\frac{\ell}{2}}}{\ell^4} - \frac{C_2\ell}{2^{\frac{\ell}{2}}\upsilon} &= \frac{2^{\frac{\ell}{2}}}{\ell^4} - \frac{8\ell p(C^*)^{p-1}(2N)^d D(\Lambda)(12nC_\rho + 2\lambda)\ln 2}{2^{\frac{\ell}{2}}\lambda^2}\\ &\ge \frac{1}{324} - \frac{C_4(2N)^d D(\Lambda)(12nC_\rho + 2\lambda)}{\lambda^2}.\end{aligned}$$

We first consider the case that

$$\frac{1}{324} - \frac{C_4(2N)^d D(\Lambda)(12nC_\rho + 2\lambda)}{\lambda^2} > \frac{1}{648}. \tag{29}$$

Since $p, a > 0$, we has $\sum_{\ell=2}^{\infty} e^{-pa^\ell} \le \frac{e^{-ap}}{pa\ln a}$ ([13]), then

$$\begin{aligned}Prob\Big(\bigcup_{\ell=2}^{\infty}\omega_\ell\Big) &\le \frac{C_1\exp\left(-\sqrt{2}\upsilon\left(\frac{1}{324} - \frac{C_4(2N)^d D(\Lambda)(12nC_\rho+2\lambda)}{\lambda^2}\right)\right)}{\sqrt{2}\ln\sqrt{2}\cdot\upsilon\left(\frac{1}{324} - \frac{C_4(2N)^d D(\Lambda)(12nC_\rho+2\lambda)}{\lambda^2}\right)}\\ &= \frac{2(2C^*+1)^{2(2N)^d D(\Lambda)}}{\sqrt{2}\ln\sqrt{2}\cdot\upsilon\left(\frac{1}{324} - \frac{C_4(2N)^d D(\Lambda)(12nC_\rho+2\lambda)}{\lambda^2}\right)}\\ &\quad\times\exp\left(-\sqrt{2}\upsilon\left(\frac{1}{324} - \frac{C_4(2N)^d D(\Lambda)(12nC_\rho + 2\lambda)}{\lambda^2}\right)\right).\end{aligned}$$

Under the condition (29), we have

$$\begin{aligned}&\sqrt{2}\ln\sqrt{2}\cdot\upsilon\left(\frac{1}{324} - \frac{C_4(2N)^d D(\Lambda)(12nC_\rho + 2\lambda)}{\lambda^2}\right)\\ \ge\ & \frac{\sqrt{2}\ln\sqrt{2}C_4(2N)^d D(\Lambda)}{4p(C^*)^{p-1}}\\ \ge\ & 3D(\Lambda)\ln\sqrt{2}\ln 2.\end{aligned}$$

This together with the probability of ω_1 in (28) obtains

$$\begin{aligned}Prob\Big(\sup_{f\in V_N^{p,*}(\Phi,\Lambda)}\Big|\sum_{j=1}^n X_j(f)\Big| \ge \lambda\Big) &\le Prob\Big(\bigcup_{\ell=1}^{\infty}\omega_\ell\Big)\\ &\le A\exp\left(-B\frac{\lambda^2}{12nC_\rho + 2\lambda}\right).\end{aligned}$$

Here, A is of order $\exp\big(CN^d\big)$ with $C = 2^{d+1}D(\Lambda)\ln(2C^*+1)$ and $B = \min\left\{\frac{\sqrt{2}}{2592p(C^*)^{p-1}}, \frac{3}{2(C^*)^p}\right\}$. Finally, we consider the case that

$$\frac{1}{324} - \frac{C_4(2N)^d D(\Lambda)(12nC_\rho + 2\lambda)}{\lambda^2} \le \frac{1}{648}.$$

In this case, we can choose $C \geq 648 C_4 B 2^d D(\Lambda)$ such that

$$A \exp\bigg(-B \frac{\lambda^2}{12 n C_\rho + 2\lambda}\bigg) \geq 1.$$

This completes the proof. □

Lemma 4.4. *Let $X = \{x_j : j \in \mathbb{N}\}$ be a sequence of independent random variables that are drawn from a general probability distribution over $\mathbb{R}^d$ with density function ρ satisfying* (26). *Then for any $\gamma > 0$, the sampling inequality*

$$nc_\rho\bigg(\|f\|^p_{L^p(C_K)} - \gamma \|f\|^p_{L^p(\mathbb{R}^d)}\bigg) \leq \sum_{j=1}^n |f(x_j)|^p \leq n\big(c_\rho \gamma + C_\rho\big)\|f\|^p_{L^p(\mathbb{R}^d)} \tag{30}$$

holds for function $f \in V^p_N(\Phi, \Lambda)$ with probability at least

$$1 - A \exp\bigg(-B \frac{\gamma^2 n c_\rho^2}{12 C_\rho + 2\gamma c_\rho}\bigg),$$

where A and B are as in Lemma 4.3.

Proof. It is obvious that every $f \in V^p_N(\Phi, \Lambda)$ satisfies the inequality (30) if and only if $f/\|f\|_{L^p(\mathbb{R}^d)}$ does. So we assume that $\|f\|_{L^p(\mathbb{R}^d)} = 1$, then $f \in V^{p,*}_N(\Phi, \Lambda)$. The event

$$D = \bigg\{ \sup_{f \in V^{p,*}_N(\Phi,\Lambda)} \bigg| \sum_{j=1}^n X_j(f) \bigg| > \gamma n c_\rho \bigg\}$$

is the complement of

$$\begin{aligned}
\widetilde{D} = \bigg\{ & n \int_{\mathbb{R}^d} \rho(x)|f(x)|^p dx - \gamma n c_\rho \leq \sum_{j=1}^n |f(x_j)|^p \\
& \leq \gamma n c_\rho + n \int_{\mathbb{R}^d} \rho(x)|f(x)|^p dx, \quad \forall f \in V^{p,*}_N(\Phi, \Lambda)\bigg\} \\
\subseteq \bigg\{ & nc_\rho\bigg(\|f\|^p_{L^p(C_K)} - \gamma \|f\|^p_{L^p(\mathbb{R}^d)}\bigg) \leq \sum_{j=1}^n |f(x_j)|^p \\
& \leq n\big(c_\rho\gamma + C_\rho\big)\|f\|^p_{L^p(\mathbb{R}^d)}, \quad \forall f \in V^p_N(\Phi, \Lambda)\bigg\} = \overline{D}.
\end{aligned}$$

Using Lemma 4.3, the sampling inequality (30) holds for all $f \in V^p_N(\Phi, \Lambda)$ with probability

$$Prob(\overline{D}) \geq Prob(\widetilde{D}) = 1 - Prob(D) \geq 1 - A \exp\bigg(-B \frac{\gamma^2 n c_\rho^2}{12 C_\rho + 2\gamma c_\rho}\bigg). \quad \square$$

In the following, we will show that if the sampling size is sufficiently large, the sampling inequality holds with overwhelming probability for all functions in $V^p_K(\Phi, \Lambda)$.

Theorem 4.5. *Let $X = \{x_j : j \in \mathbb{N}\}$ be a sequence of independent random variables that are drawn from a general probability distribution over $\mathbb{R}^d$ with density function ρ satisfying (26). Then for any $0 < \varepsilon_1, \varepsilon_2, \gamma < 1$ which satisfy*

$$L(\varepsilon_1,\varepsilon_2,\gamma) =: c_\rho\Big(1-\delta-p(1+\varepsilon_1)^{p-1}\varepsilon_1-\gamma\big(\frac{C_p}{c_p}\big)^p\Big)-p(C^*)^{p-1}\varepsilon_2>0, \tag{31}$$

the sampling inequality

$$nL(\varepsilon_1,\varepsilon_2,\gamma)\|f\|^p_{L^p(\mathbb{R}^d)} \le \sum_{j=1}^n |f(x_j)|^p \le nU(\varepsilon_2,\gamma)\|f\|^p_{L^p(\mathbb{R}^d)} \tag{32}$$

holds uniformly for all functions $f \in V_K^p(\Phi,\Lambda)$ with probability at least

$$1 - A\exp\bigg(-B\frac{\gamma^2 n c_\rho^2}{12C_\rho+2\gamma c_\rho}\bigg).$$

Here, $U(\varepsilon_2,\gamma) = (c_\rho\gamma + C_\rho)\big(\frac{C_p}{c_p}\big)^p + p(C^*)^{p-1}\varepsilon_2$, A and B are the constants in Lemma 4.3 corresponding to $N = \max\{N_1(\varepsilon_1,R), N_2(\varepsilon_2,R)\}$ with $R > K$ being a constant such that $\{x_j : j = 1,2,\dots,n\} \subset C_R$.

Proof. It is obvious that every $f \in V_K^p(\Phi,\Lambda)$ satisfies the inequality (32) if and only if $f/\|f\|_{L^p(\mathbb{R}^d)}$ does. Hence, we assume that $\|f\|_{L^p(\mathbb{R}^d)} = 1$.

For random variables $\{x_j : j = 1,2,\dots,n\}$, there exists an $R > K$ such that $\{x_j : j = 1,2,\dots,n\} \subset C_R$. By Lemma 2.2, for any $\varepsilon_1, \varepsilon_2 > 0$ satisfying (31), there exist $N = \max\{N_1(\varepsilon_1,R), N_2(\varepsilon_2,R)\}$ and $f_N \in V_N^p(\Phi,\Lambda)$ such that

$$\|f-f_N\|_{L^p(C_K)} \le \|f-f_N\|_{L^p(C_R)} \le \varepsilon_1 \text{ and } \|f-f_N\|_{L^\infty(C_R)} \le \varepsilon_2. \tag{33}$$

This together with mean value theorem obtains

$$\Big|\|f\|^p_{L^p(C_K)} - \|f_N\|^p_{L^p(C_K)}\Big| \le p(1+\varepsilon_1)^{p-1}\varepsilon_1 \tag{34}$$

and

$$\begin{aligned}\Big||f(x_j)|^p - |f_N(x_j)|^p\Big| &\le p\big(\max\{|f(x_j)|, |f_N(x_j)|\}\big)^{p-1}|f(x_j)-f_N(x_j)|\\ &\le p(C^*)^{p-1}\varepsilon_2.\end{aligned} \tag{35}$$

It follows from (35) that

$$\begin{aligned}\sum_{j=1}^n |f_N(x_j)|^p - np(C^*)^{p-1}\varepsilon_2 &\le \sum_{j=1}^n |f(x_j)|^p\\ &\le \sum_{j=1}^n |f_N(x_j)|^p + np(C^*)^{p-1}\varepsilon_2.\end{aligned} \tag{36}$$

For the above $f_N \in V_N^p(\Phi,\Lambda)$, we know from Lemma 4.4 that

$$nc_\rho\bigg(\|f_N\|^p_{L^p(C_K)} - \gamma\|f_N\|^p_{L^p(\mathbb{R}^d)}\bigg) \le \sum_{j=1}^n |f_N(x_j)|^p \le n\big(c_\rho\gamma + C_\rho\big)\|f_N\|^p_{L^p(\mathbb{R}^d)} \tag{37}$$

holds with probability at least

$$1 - A\exp\Big(- B\frac{\gamma^2 n c_\rho^2}{12C_\rho + 2\gamma c_\rho}\Big). \tag{38}$$

Then, it follows from (34), (36) and (37) that

$$\begin{aligned} & nc_\rho\Big(\|f\|^p_{L^p(C_K)} - p(1+\varepsilon_1)^{p-1}\varepsilon_1 - \gamma\|f_N\|^p_{L^p(\mathbb{R}^d)}\Big) - np(C^*)^{p-1}\varepsilon_2 \\ \le & \sum_{j=1}^n |f(x_j)|^p \\ \le & \, n\big(c_\rho\gamma + C_\rho\big)\|f_N\|^p_{L^p(\mathbb{R}^d)} + np(C^*)^{p-1}\varepsilon_2 \end{aligned} \tag{39}$$

holds with the same probability as (38). Since $f \in V_K^p(\Phi, \Lambda)$, we have

$$(1-\delta)\|f\|^p_{L^p(\mathbb{R}^d)} \le \|f\|^p_{L^p(C_K)}. \tag{40}$$

Moreover, we know from Lemma 2.1 that

$$\|f_N\|_{L^p(\mathbb{R}^d)} \le C_p\|c\|_{\ell^p} \le \frac{C_p}{c_p}\|f\|_{L^p(\mathbb{R}^d)}. \tag{41}$$

Note that $\|f\|_{L^p(\mathbb{R}^d)} = 1$. Then, the sampling inequality (32) follows from (39)-(41). □

5. Reconstruction algorithm in $V_N^p(\Phi, \Lambda)$

In this section, we consider the reconstruction of functions in $V_N^p(\Phi, \Lambda)$ from the corresponding random samples.

For a linear operator $\mathcal{L}$ defined on $\ell^p(\Lambda \cap [-N, N]^d)$, the p-norm condition number of $\mathcal{L}$ is defined by

$$\kappa(\mathcal{L}, p) := \max_{a\in\ell^p(\Lambda\cap[-N,N]^d), a\neq 0} \frac{\|\mathcal{L}a\|_{\ell^p}}{\|a\|_{\ell^p}}\Big(\min_{a\in\ell^p(\Lambda\cap[-N,N]^d), a\neq 0}\frac{\|\mathcal{L}a\|_{\ell^p}}{\|a\|_{\ell^p}}\Big)^{-1}. \tag{42}$$

Now, we will estimate the condition number of random matrix

$$U = (u_{j,\lambda})_{j=1,2,\dots,n;\lambda\in\Lambda\cap[-N,N]^d}, \tag{43}$$

where $u_{j,\lambda} = \phi_\lambda(x_j)$. In fact, U is a matrix with n rows and the column number is less than $(2N)^d D(\Lambda)$.

Theorem 5.1. *Let random variables $X = \{x_j : j \in \mathbb{N}\}$ and density function ρ be as in Theorem 4.5. Suppose that there exists $\alpha_p > 0$ such that for all $c \in \ell^p(\Lambda \cap [-N, N]^d)$,*

$$\Big\|\sum_{\lambda\in\Lambda\cap[-N,N]^d} c(\lambda)\phi_\lambda\Big\|_{L^p(C_K)} \ge \alpha_p\|c\|_{\ell^p}. \tag{44}$$

Then for any $0<\gamma<\left(\frac{\alpha_p}{C_p}\right)^p$, *the p-norm condition number*

$$\kappa(U,p)\le\left(\frac{c_\rho\gamma+C_\rho}{c_\rho(\alpha_p^p-\gamma C_p^p)}\right)^{1/p}C_p$$

holds with probability at least

$$1-A\exp\bigg(-B\frac{\gamma^2nc_\rho^2}{12C_\rho+2\gamma c_\rho}\bigg),$$

where A and B are as in Lemma 4.3.

Proof. For every $c\in\ell^p(\Lambda\cap[-N,N]^d)$ and $g=\sum_{\lambda\in\Lambda\cap[-N,N]^d}c(\lambda)\phi_\lambda$. By Lemma 4.4, for random sequence $\{x_j\}_{j=1}^n$, the sampling inequality

$$nc_\rho\bigg(\|g\|^p_{L^p(C_K)}-\gamma\|g\|^p_{L^p(\mathbb{R}^d)}\bigg)\le\sum_{j=1}^n|g(x_j)|^p\le n\big(c_\rho\gamma+C_\rho\big)\|g\|^p_{L^p(\mathbb{R}^d)}\tag{45}$$

holds for all $c\in\ell^p(\Lambda\cap[-N,N]^d)$ with probability at least

$$1-A\exp\bigg(-B\frac{\gamma^2nc_\rho^2}{12C_\rho+2\gamma c_\rho}\bigg).$$

We know from Lemma 2.1 that

$$c_p\|c\|_{\ell^p}\le\|g\|_{L^p(\mathbb{R}^d)}\le C_p\|c\|_{\ell^p}.\tag{46}$$

Furthermore, it can be easily verified from (43) that

$$\sum_{j=1}^n|g(x_j)|^p=\|Uc\|^p_{\ell^p}.\tag{47}$$

Then, combining (44)-(47) we obtain

$$nc_\rho(\alpha_p^p-\gamma C_p^p)\le\frac{\|Uc\|^p_{\ell^p}}{\|c\|^p_{\ell^p}}\le n\big(c_\rho\gamma+C_\rho\big)C_p^p.$$

This together with (42) leads to the desired result. □

In the following theorem, we present a reconstruction algorithm for finite dimensional subspace $V_N^p(\Phi,\Lambda)$.

Theorem 5.2. *Suppose that random variables* $X=\{x_j:j\in\mathbb{N}\}$, *density function* ρ *and the generators* Φ *are as in Theorem 5.1. Let U be as in* (43), $\Psi(x)=\big(\phi_\lambda(x)\big)^T_{\lambda\in\Lambda\cap[-N,N]^d}$ *and* $(S_j(x))^T_{1\le j\le n}=U(U^TU)^{-1}\Psi$. *Then for any* $0<\gamma<\left(\frac{\alpha_p}{C_p}\right)^p$, *the reconstruction formula*

$$f(x)=\sum_{j=1}^n f(x_j)S_j(x)\tag{48}$$

holds for all $f \in V_N^p(\Phi, \Lambda)$ *with probability at least*

$$1 - A\exp\left(-B\frac{\gamma^2 n c_\rho^2}{12C_\rho + 2\gamma c_\rho}\right),$$

where A *and* B *are as in Lemma 4.3.*

Proof. For $f = \sum_{\lambda \in \Lambda \cap [-N,N]^d} c(\lambda)\phi_\lambda \in V_N^p(\Phi, \Lambda)$, we try to solve the system of linear equations

$$f(x_j) = \sum_{\lambda \in \Lambda \cap [-N,N]^d} c(\lambda)\phi_\lambda(x_j), \quad 1 \le j \le n \tag{49}$$

for the coefficients $\{c(\lambda)\}_{\lambda \in \Lambda \cap [-N,N]^d}$. The system (49) of linear equations can be rewritten as

$$Uc = f|_{\{x_j : j = 1,2,\ldots,n\}}. \tag{50}$$

It follows from Lemma 4.4 that

$$\begin{aligned}\sum_{j=1}^n |f(x_j)|^p &\ge n c_\rho\Big(\alpha_p^p \|c\|_{\ell^p}^p - \gamma\|f\|_{L^p(\mathbb{R}^d)}^p\Big)\\ &\ge n c_\rho\Big(\alpha_p^p - \gamma C_p^p\Big)\|c\|_{\ell^p}^p\end{aligned}$$

holds with probability at least

$$1 - A\exp\left(-B\frac{\gamma^2 n c_\rho^2}{12C_\rho + 2\gamma c_\rho}\right).$$

Then $U^T U$ is invertible, which implies that

$$c = (U^T U)^{-1} U^T f|_{\{x_j : j = 1,2,\ldots,n\}}. \tag{51}$$

Then we can obtain the desired reconstruction formula (48). □

References

[1] R. F. Bass and K. Gröchenig, *Random sampling of multivariate trigonometric polynomials*, SIAM J. Math. Anal. **36** (2004/05), no. 3, 773–795. https://doi.org/10.1137/S0036141003432316

[2] R. F. Bass and K. Gröchenig, *Random sampling of bandlimited functions*, Israel J. Math. **177** (2010), 1–28. https://doi.org/10.1007/s11856-010-0036-7

[3] R. F. Bass and K. Gröchenig, *Relevant sampling of band-limited functions*, Illinois J. Math. **57** (2013), no. 1, 43–58. http://projecteuclid.org/euclid.ijm/1403534485

[4] G. Bennett, *Probability inequalities for the sum of independent random variable*, J. Amer. Statist. Assoc. **57** (1962), no. 297, 33–45.

[5] S. H. Chan, T. Zickler, and Y. M. Lu, *Monte Carlo non-local means: random sampling for large-scale image filtering*, IEEE Trans. Image Process. **23** (2014), no. 8, 3711–3725. https://doi.org/10.1109/TIP.2014.2327813

[6] F. Cucker and D.-X. Zhou, *Learning theory: an approximation theory viewpoint*, Cambridge Monographs on Applied and Computational Mathematics, 24, Cambridge University Press, Cambridge, 2007. https://doi.org/10.1017/CBO9780511618796

[7] Y. C. Eldar, *Compressed sensing of analog signals in shift-invariant spaces*, IEEE Trans. Signal Process. **57** (2009), no. 8, 2986–2997. https://doi.org/10.1109/TSP.2009.2020750

[8] H. Führ and J. Xian, *Relevant sampling in finitely generated shift-invariant spaces*, J. Approx. Theory **240** (2019), 1–15. https://doi.org/10.1016/j.jat.2018.09.009

[9] Y. Li, J. Wen, and J. Xian, *Reconstruction from convolution random sampling in local shift invariant spaces*, Inverse Problems **35** (2019), no. 12, 125008, 15 pp. https://doi.org/10.1088/1361-6420/ab40f7

[10] Y. Lu and J. Xian, *Non-uniform random sampling and reconstruction in signal spaces with finite rate of innovation*, Acta Appl. Math. **169** (2020), 247–277. https://doi.org/10.1007/s10440-019-00298-6

[11] S. Mulleti and C. S. Seelamantula, *Ellipse fitting using the finite rate of innovation sampling principle*, IEEE Trans. Image Process. **25** (2016), no. 3, 1451–1464. https://doi.org/10.1109/TIP.2015.2511580

[12] S. Rudresh and C. S. Seelamantula, *Finite-rate-of-innovation-sampling-based super-resolution radar imaging*, IEEE Trans. Signal Process. **65** (2017), no. 19, 5021–5033. https://doi.org/10.1109/TSP.2017.2721917

[13] S. Smale and D.-X. Zhou, *Online learning with Markov sampling*, Anal. Appl. (Singap.) **7** (2009), no. 1, 87–113. https://doi.org/10.1142/S0219530509001293

[14] Q. Sun, *Nonuniform average sampling and reconstruction of signals with finite rate of innovation*, SIAM J. Math. Anal. **38** (2006/07), no. 5, 1389–1422. https://doi.org/10.1137/05063444X

[15] Q. Sun, *Frames in spaces with finite rate of innovation*, Adv. Comput. Math. **28** (2008), no. 4, 301–329. https://doi.org/10.1007/s10444-006-9021-4

[16] G. A. Velasco, *Relevant sampling of the short-time Fourier transform of time-frequency localized functions*, arXiv:1707.09634v1, (2017).

[17] M. Vetterli, P. Marziliano, and T. Blu, *Sampling signals with finite rate of innovation*, IEEE Trans. Signal Process. **50** (2002), no. 6, 1417–1428. https://doi.org/10.1109/TSP.2002.1003065

[18] J. Yang, *Random sampling and reconstruction in multiply generated shift-invariant spaces*, Anal. Appl. (Singap.) **17** (2019), no. 2, 323–347. https://doi.org/10.1142/S0219530518500185

[19] J. Yang and X. Tao, *Random sampling and approximation of signals with bounded derivatives*, J. Inequal. Appl. **2019** (2019), Paper No. 107, 14 pp. https://doi.org/10.1186/s13660-019-2059-x

[20] J. Yang and W. Wei, *Random sampling in shift invariant spaces*, J. Math. Anal. Appl. **398** (2013), no. 1, 26–34. https://doi.org/10.1016/j.jmaa.2012.08.030

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