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# HARMONICITY OF ALMOST NORDEN SUBMERSIONS BETWEEN ALMOST NORDEN MANIFOLDS

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Abstract. We define an almost Norden submersion (holomorphic and semi-Riemannian submersion) between almost Norden manifolds and show that, in most of the cases, the base manifold has the similar kind of structure as that of total manifold. We obtain necessary and sufficient conditions for almost Norden submersion to be a totally geodesic map. We also derive decomposition theorems for the total manifold of such submersions. Moreover, we study the harmonicity of almost Norden submersions between almost Norden manifolds and between Kaehler-Norden manifolds. Finally, we derive conditions for an almost Norden submersion to be a harmonic morphism.

## 1. Introduction

The notion of Riemannian submersions between Riemannian manifolds was initiated by O'Neill [20] and Gray [14]. Since a Riemannian submersion acted as an effective tool to compare geometries and structures between the manifolds and has many significant applications in differential geometry and mathematical physics therefore geometry of Riemannian submersions was further developed by many others. Then the geometry of semi-Riemannian submersions was introduced by O'Neill in his celebrated book [21]. Watson [24] defined almost Hermitian submersions between almost Hermitian submanifolds and examined the influence of a given structure defined on the total space on the determination of the corresponding structure on the fibre submanifolds and the base manifolds. Chinea [5] studied almost contact Riemannian submersions between almost contact metric manifolds. Later, almost Hermitian submersions and almost contact submersions were studied between subclasses of almost Hermitian and almost contact manifolds, for details see [10] and many references there in. Recently, Sahin [23] defined and studied various types of Riemannian submersions such as anti-invariant submersions, semi-invariant submersions, generic

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Riemannian submersions, slant submersions, semi-slant submersions and others from almost Hermitian manifolds.

Let $(M^{2m}, g, J)$ be an indefinite almost complex manifold and the semi-Riemannian metric $g$ be of neutral signature $(m, m)$. The semi-Riemannian metric $g$ is said to be a Norden metric if the almost complex structure $J$ is anti-isometry with respect to $g$, i.e., $g(JX, JY) = -g(X, Y)$ for any vector fields $X$ and $Y$ on $M$ and an almost complex manifold endowed with Norden metric is called almost Norden manifold [13]. The difference between the geometry of indefinite almost Hermitian manifolds and almost Norden manifolds arises due to the fact that in the former case, the almost complex structure $J$ is an isometry with respect to the semi-Riemannian metric $g$. Since the geometry of Riemannian and semi-Riemannian submersions has potential applications in medical imaging [18], theory of robotics [1], Kaluza-Klein theory [4], statistical analysis on manifolds [3] and many others, therefore study of Riemannian submersions is an active area of research. Moreover, as per our knowledge, the notion of Riemannian submersions on almost Norden manifolds has not been considered yet. All these facts motivate us to work on the theory of submersions between almost Norden manifolds.

In present paper, we define an almost Norden submersion (which is holomorphic and semi-Riemannian submersion) between almost Norden manifolds. It is known that Gray and Hervella [15] have obtained complete classification of almost Hermitian manifolds where Ganchev and Borisov have obtained classification of almost Norden manifolds in [13]. Using almost Norden submersions, we show that in most of the cases, the base manifold has the similar kind of structure as that of total manifold (Theorems 3.6, 3.7, 3.8 and Corollary 3.9). We obtain necessary and sufficient conditions for almost Norden submersion to be a totally geodesic map (Theorem 4.9). Then we derive decomposition theorems for the total manifold of almost Norden submersions (Theorems 4.12, 4.18). It is well known that a map between Riemannian manifolds is harmonic if it is a critical point of the energy density of that map [2]. Gudmundsson and Wood [16] obtained conditions for a holomorphic map between almost Hermitian manifolds to be a harmonic. Later, Eells and Sampson [7] showed that a holomorphic map between Kaehler manifolds is harmonic. We also obtain conditions for almost Norden submersions between almost Norden manifolds and between Kaehler-Norden manifolds to be harmonic (Corollary 4.22). We discuss horizontally weakly conformal almost Norden submersions and derive conditions for it to be a harmonic morphism (see Theorem 4.25).

## 2. Semi-Riemannian submersions

Assume that $(M^{m+n}_{r+s}, g)$ and $(B^n_s, g')$ are $(m + n)$ and $(n)$-dimensional connected semi-Riemannian manifolds of index $r + s$ and $s$, respectively, where $0 \leq r \leq m$, $0 \leq s \leq n$. A semi-Riemannian submersion [21] is a smooth surjective map

$$\pi : M \to B,$$

between semi-Riemannian manifolds $M$ and $B$ such that

A1. the fibers $\pi^{-1}(b)$, $b \in B$ are semi-Riemannian submanifolds of $M$;

A2. the differential $\pi_*$ of $\pi$ preserves scalar products of vectors normal to the fibers.

In the case of semi-Riemannian submersions, the fibers are $(\dim(M)-\dim(B))$-dimensional semi-Riemannian submanifolds of $M$ and the vectors tangent to the fibers are called vertical vectors and belong to the kernel of the linear map $\pi_*$. The distribution of vertical vectors is denoted by $\mathcal{V}$ then $\mathcal{V} = (ker\ \pi_*)$, implies $\mathcal{V}$ is integrable and fibers are its integral manifolds of maximal rank. Vectors which are normal to the fibers are called horizontal and the distribution of horizontal vectors is denoted by $\mathcal{H}$ then $\mathcal{H} = (ker\ \pi_*)^{\perp}$ which is orthogonal complement of $\mathcal{V}$ in $TM$, that is, $TM = \mathcal{V} \oplus_{ortho} \mathcal{H}$. A horizontal vector field $X$ is said to be a basic vector field on $M$ if it is $\pi$-related to a vector field $X_*$ of $B$, that is, $\pi_* X_p = X_{*\ \pi(p)}$ for every $p \in M$.

Let $\nabla^g$ be the Levi-Civita connection of $g$. Then O'Neill [20] defined tensors $\mathcal{T}$ and $\mathcal{A}$ for all vector fields $E, F$ on $M$, which characterize the geometry of Riemannian submersions, as

$$\mathcal{A}_E F = \mathcal{H}\nabla^g_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla^g_{\mathcal{H}E}\mathcal{H}F, \tag{1}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla^g_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla^g_{\mathcal{V}E}\mathcal{H}F. \tag{2}$$

Clearly $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of $M$ reversing the vertical and the horizontal distributions. Let $U, V$ be vertical vector fields and $X, Y$ be horizontal vector fields on $M$. Then using (1) and (2), we have

$$\nabla^g_U V = \mathcal{T}_U V + \mathcal{V}\nabla^g_U V, \tag{3}$$

$$\nabla^g_U X = \mathcal{H}\nabla^g_U X + \mathcal{T}_U X, \tag{4}$$

$$\nabla^g_X V = \mathcal{A}_X V + \mathcal{V}\nabla^g_X V, \tag{5}$$

$$\nabla^g_X Y = \mathcal{H}\nabla^g_X Y + \mathcal{A}_X Y. \tag{6}$$

It is easy to observe that $\mathcal{T}$ acts as second fundamental form for the fibers therefore fibers are totally geodesic if and only if $\mathcal{T}$ vanishes identically. While the tensor $\mathcal{A}$ gives the integrability of the horizontal distribution. Moreover, the tensors $\mathcal{T}$ and $\mathcal{A}$, satisfy the following relations

$$\mathcal{T}_V U = \mathcal{T}_U V, \quad \forall\ U, V \in \Gamma(\mathcal{V}), \tag{7}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y], \quad \forall\ X, Y \in \Gamma(\mathcal{H}). \tag{8}$$

Now, we recall the notion of second fundamental form of a map between the Riemannian manifolds. Let $\pi : M^m \to B$ be a smooth map between Riemannian manifolds $(M, g)$ and $(B, g')$. Suppose $\mathcal{W}$ is a vector bundle of $B$. Then the pull back bundle $\pi^{-1}\mathcal{W}$ of $M$ has fibers given by $(\pi^{-1}\mathcal{W})_x = \mathcal{W}_{\pi(x)}$, $x \in M$. For a given connection $\nabla^{\mathcal{W}}$ on $\mathcal{W}$ there is a unique pull back connection $\nabla^{\pi}$ on the pull back bundle $\pi^{-1}\mathcal{W}$ of $M$ and satisfies

$$\nabla^{\pi}_X(\pi^*\sigma) = \nabla^{\mathcal{W}}_{\pi_*(X)}\sigma,$$

where $\sigma \in \Gamma(\mathcal{W})$ and $\pi^*\sigma = \sigma \circ \pi \in \Gamma(\pi^{-1}\mathcal{W})$. Moreover, it is known that the differential $\pi_*$ of $\pi$ can be viewed as a section of the bundle $T^*M \otimes \pi^{-1}TB = \mathrm{Hom}(TM, \pi^{-1}TB)$ of $M$. This bundle has a connection $\nabla$ induced from the Levi-Civita connection $\nabla^g$ of $M$ and the pull back connection $\nabla^{\pi}$. On applying the induced connection $\nabla$ to differential map $\pi_*$, we obtain the second fundamental form $\nabla\pi_*$ of $\pi$ and given by

$$(\nabla\pi_*)(X, Y) = \nabla^{\pi}_X(\pi_*(Y)) - \pi_*(\nabla^g_X Y) \tag{9}$$

for any $X, Y \in \Gamma(TM)$. It is also known that the second fundamental form is symmetric. The smooth map $\pi : M \to B$ is said to be totally geodesic if for each geodesic $\gamma$ in $M$, the image $\pi(\gamma)$ is also geodesic in $B$, equivalently, if $(\nabla\pi_*) = 0$. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame on $M$. Then the trace of second fundamental form gives us an important quantity, called the tension field of $\pi$, which is the section $\tau(\pi) \in \Gamma(\pi^{-1}TB)$ and defined by

$$\tau(\pi) = trac\ \nabla\pi_* = \sum_{i=1}^{m}(\nabla\pi_*)(e_i, e_i).$$

The map $\pi$ is said to be harmonic if its tension field $\tau(\pi)$ vanishes identically, for details see [2].

Now, we recall the notion of harmonic morphism between semi-Riemannian manifolds from [12]. A smooth map $\pi : M \to B$ between semi-Riemannian manifolds is said to be harmonic morphism if $\pi$ pull back harmonic functions (that is, local solutions of Laplace-Beltrami equation) on $B$ into harmonic functions on $M$. Analogous to Riemannian case [11], Fuglede [12] showed that a harmonic morphism is the same as a smooth map which is harmonic and horizontally weakly conformal.

A $C^1$-map $\pi : (M, g) \to (B, g')$ between semi-Riemannian manifolds is said to be horizontally weakly conformal [12] if

(i) for any $x \in M$ at which $(ker\ \pi_*)_x$ (or equivalently $(ker\ \pi_*)^{\perp}_x$) is non-degenerate and $\pi_*(x) \neq 0$, the restriction of $\pi_*(x)$ to $(ker\ \pi_*)^{\perp}_x$ is surjective and conformal in the sense that there is a (necessarily unique) real number $\lambda(x) \neq 0$ such that

$$g'_{\pi(x)}(\pi_*(X), \pi_*(Y)) = \lambda(x) g_x(X, Y), \quad \forall\ X, Y \in \Gamma(ker\ \pi_*)^{\perp}_x. \tag{10}$$

(ii) For any $x \in M$ at which $(ker\ \pi_*)_x$ is degenerate then $(ker\ \pi_*)_x^{\perp} \subset (ker\ \pi_*)_x$, that is

$$g_x(X,Y) = 0, \quad \forall\ X, Y \in \Gamma(ker\ \pi_*)_x.$$

The real number $\lambda$ is called the dilation of a horizontally weakly conformal map $\pi$, which is extended to all of $M$ by $\lambda(x) = 0$ if $\pi_*(x) = 0$ or if $(ker\ \pi_*)$ is degenerate. Here the term "weakly" refers to the possible occurrence of points $x \in M$ at which $\lambda(x) = 0$. A horizontally weakly conformal map is said to be horizontally homothetic if the gradient of its dilation $\lambda$ is vertical, i.e., $\mathcal{H}(grad\ \lambda) = 0$. If the dilation is constant and non-zero, then $\pi$ is a Riemannian submersion up to scale, i.e., it is a Riemannian submersion after a suitable homothetic change of metric on the domain or codomain.

## 3. Almost Norden submersions

Let $(M, g, J)$ be a $2m$-dimensional almost complex manifold with its almost complex structure $J$. If the metric $g$ on $M$ satisfies

$$g(JX, Y) = g(X, JY), \text{ equivalently, } g(JX, JY) = -g(X, Y) \tag{11}$$

for all vector fields $X, Y$ on $M$, then it is called a Norden metric on $M$. In this case $(J, g)$ is called the almost Norden structure on $M$ and $(M, g, J)$ is said to be an almost Norden manifold. Then metric $g$ is necessarily an indefinite metric of neutral (Kleinian) signature $(m, m)$. If $J$ is integrable, then $(M, g, J)$ is called a Norden manifold. The associated metric $\widetilde{g}$ of an almost Norden manifold $(M, g, J)$ is given by

$$\widetilde{g}(X, Y) = (g \circ J)(X, Y) = g(JX, Y) \tag{12}$$

for all vector fields $X, Y$ on $M$. Obviously, $\widetilde{g}$ is a Norden metric on $M$ and known as the twin (or dual) metric of $g$ on $M$. It should be noted that $\widetilde{g}$ is also of neutral signature $(m, m)$. Let $\nabla^g$ be the Levi-Civita connection of $g$ and a tensor field $\mathcal{F}$ of type $(0, 3)$ on $M$ is given by (for details, see [13])

$$\mathcal{F}(X, Y, Z) = (\nabla^g_X \widetilde{g})(Y, Z) = g((\nabla^g_X J)Y, Z) \tag{13}$$

for all vector fields $X, Y$ on $M$, where the tensor field $\mathcal{F}$ satisfies the following properties

$$\mathcal{F}(X, Y, Z) = \mathcal{F}(X, Z, Y), \quad \mathcal{F}(X, JY, JZ) = \mathcal{F}(X, Y, Z). \tag{14}$$

**Definition 1.** Let $(M^{2m}_m, g, J)$ and $(B^{2n}_n, g', J')$ be $2m$ and $2n$-dimensional almost Norden manifolds of index $m$ and $n$, respectively, where $n \leq m$. Assume that the surjective map $\pi : (M^{2m}_m, g, J) \to (B^{2n}_n, g', J')$ is a semi-Riemannian submersion from $M$ to $B$. Then we say the map $\pi$ is an *almost Norden submersion* between almost Norden manifolds if $\pi$ is an almost complex map (or a holomorphic map), that is

$$\pi_* \circ J = J' \circ \pi_*. \tag{15}$$

From the definition of almost Norden submersion, it is clear that the index of the vertical distribution and the horizontal distribution is $m-n$ and $n$, respectively.

**Theorem 3.1.** *Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be an almost Norden submersion between almost Norden manifolds. Then the vertical and horizontal distributions are $J$-invariant.*

*Proof.* The proof is analogous as for the almost Hermitian submersions (see Proposition 2.1 of [24]), therefore we omit the proof here. $\square$

Moreover, from Definition 1 and the last theorem, we have following assertions immediately.

**Theorem 3.2.** *Let $\pi : (M_m^{2m}, g, \tilde{g}, J) \to (B_n^{2n}, g', \tilde{g}', J')$ be an almost Norden submersion between almost Norden manifolds such that $\nabla^g$ and $\nabla'$ be the Levi-Civita connections on $M$ and $B$, respectively. Let $X, Y$ be the basic vector fields on $M$ and are $\pi$-related to vector fields $X_*$ and $Y_*$ of $B$, respectively. Then*

(i) $g(X,Y) = g'(X_*, Y_*) \circ \pi$, $\tilde{g} = \tilde{g}'(X_*, Y_*) \circ \pi$.
(ii) *$JX$ is the basic vector field associated to $J'X_*$.*
(iii) *$\mathcal{H}\nabla^g_X Y$ is the basic vector field associated to $\nabla'_{X_*} Y_*$.*
(iv) *$\mathcal{H}[X,Y]$ is the basic vector field associated to $[X_*, Y_*]$.*

**Example 3.3.** Let $(\mathbb{R}^8_4, J, g)$ and $(\mathbb{R}^4_2, J', g')$ be almost Norden manifolds endowed with almost Norden structures $(J, g)$ and $(J', g')$, respectively and given by

$$g = -dx_1^2 + dx_2^2 - dx_3^2 + dx_4^2 - dx_5^2 + dx_6^2 - dx_7^2 + dx_8^2,$$
$$J(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7),$$

and

$$g' = -dy_1^2 + dy_2^2 - dy_3^2 + dy_4^2,$$
$$J'(y_1, y_2, y_3, y_4) = (y_2, -y_1, y_4, -y_3),$$

respectively. Let $\pi : \mathbb{R}^8_4 \to \mathbb{R}^4_2$ be a map defined by

$$\pi(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (x_1 + x_5, x_2 + x_6, x_3 + x_7, x_4 + x_8).$$

Then by straightforward calculations

$$(ker\ \pi_*) = span\{U_1, U_2, U_3, U_4\},$$

where

$$U_1 = \partial x_1 - \partial x_5, \quad U_2 = \partial x_2 - \partial x_6,$$
$$U_3 = \partial x_3 - \partial x_7, \quad U_4 = \partial x_4 - \partial x_8,$$

and

$$(ker\ \pi_*)^\perp = span\{Z_1, Z_2, Z_3, Z_4\},$$

where

$$Z_1 = \partial x_1 + \partial x_5, \quad Z_2 = \partial x_2 + \partial x_6, \quad Z_3 = \partial x_3 + \partial x_7, \quad Z_4 = \partial x_4 + \partial x_8.$$

It is easy to see that $JU_1 = -U_2$, $JU_3 = -U_4$, $JZ_1 = -Z_2$ and $JZ_3 = -Z_4$, therefore $(ker\ \pi_*)$ and $(ker\ \pi_*)^\perp$ are invariant with respect to $J$ and moreover

$$g(JU_i, JU_i) = -g(U_i, U_i), \quad g(JZ_i, JZ_i) = -g(Z_i, Z_i), \quad \forall\ i, j \in \{1, 2, 3, 4\}.$$

Furthermore, by direct calculations, we derive

$$\pi_* JZ_1 = -2\partial y_2 = J'\pi_* Z_1, \quad \pi_* JZ_2 = 2\partial y_1 = J'\pi_* Z_2,$$
$$\pi_* JZ_3 = -2\partial y_4 = J'\pi_* Z_3, \quad \pi_* JZ_4 = 2\partial y_3 = J'\pi_* Z_4.$$

Hence $\pi_* JZ_i = J'\pi_* Z_i$ for $i \in \{1, 2, 3, 4\}$, consequently $\pi$ is an almost Norden submersion from an almost Norden manifold $(\mathbb{R}^8_4, J, g)$ to an almost Norden manifold $(\mathbb{R}^4_2, J', g')$.

Let $(M, g, J)$ be an almost Norden manifold. Then the Nijenhuis tensor $\mathcal{N}$ of $M$ is given by

$$\mathcal{N}(X, Y) = [X, Y] + [JX, JY] - J[JX, Y] - J[X, JY],$$

which can be further expressed using the Levi-Civita connection $\nabla^g$ of $g$ as below

$$\mathcal{N}(X, Y) = (\nabla^g_X J)JY - (\nabla^g_Y J)JX + (\nabla^g_{JX} J)Y - (\nabla^g_{JY} J)X \tag{16}$$

for any $X, Y \in \Gamma(TM)$. The associated tensor $\widetilde{\mathcal{N}}$ with $\mathcal{N}$ is given by

$$\widetilde{\mathcal{N}}(X, Y) = (\nabla^g_X J)JY + (\nabla^g_Y J)JX + (\nabla^g_{JX} J)Y + (\nabla^g_{JY} J)X. \tag{17}$$

Next, let $\{e_i\}_{i=1}^{2m}$ be a basis of $T_pM$. Then 1-form associated with $\mathcal{F}$ is given by, see [13]

$$\varphi(X) = \sum_{i,j=1}^{2m} g^{ij}\mathcal{F}(e_i, e_j, X) = \sum_{i,j=1}^{2m} g^{ij} g((\nabla^g_{e_i} J)e_j, X), \quad \forall X \in T_pM,$$

where $g^{ij}$ is the inverse matrix of $g$. Furthermore, last expression can be locally defined as below

$$\varphi(X) = \sum_{i,j=1}^{2m} g^{ij} g((\nabla^g_{X_i} J)X_j, X), \quad \forall X \in T_pM,$$

where $\{X_i\}_{i=1}^{2m}$ is a local basis of $TM$.

Now, we recall an important definition of adapted local basis for an almost Norden manifold from [9] for further uses.

**Definition 2.** Let $(M, g, J)$ be an almost Norden manifold. Then for any $p \in M$, there exists a local basis $\{E_1, \ldots, E_m, F_1, \ldots, F_m\}$ of $TM$ such that

$$F_i = JE_i,\ g(E_i, E_j) = g(F_i, F_j) = 0,\ g(E_i, F_j) = \partial_{ij},\ \forall i = 1, \ldots, m, \tag{18}$$

and this local basis is called an adapted local basis to the almost Norden structure $(J, g)$.

**Lemma 3.4** ([8]). *Let $(M, g, J)$ be an almost Norden manifold. Then the 1-form $\varphi(X)$ can be locally expressed by means of an adapted local basis $\{E_1, \dots, E_m, F_1, \dots, F_m\}$ of $TM$ to $(J, g)$ as follows*:

$$\varphi(X) = \sum_{i=1}^{m} \Big\{ g((\nabla^g_{E_i} J)F_i, X) + g((\nabla^g_{F_i} J)E_i, X) \Big\}, \quad \forall X \in \Gamma(TM). \tag{19}$$

Using the tensor field $\mathcal{F}$ along with 1-form $\varphi(X)$, eight classes of almost complex manifold with Norden metric were obtained in [13] as below:

**Theorem 3.5.** *Let $(M, g, J)$ be a $2m$-dimensional almost Norden manifold. Then we have following classes of this kind of manifolds*:

(1) *The class $\mathcal{W}_0$ or Kaehlerian manifolds with Norden metric (Kaehler-Norden manifolds) characterized by the condition*

$$\mathcal{F}(X, Y, Z) = 0.$$

(2) *The class $\mathcal{W}_1$ or conformally Kaehlerian manifolds with Norden metric characterized by the condition*

$$\begin{aligned}\mathcal{F}(X, Y, Z) = \frac{1}{2m}\{&g(X, Y)\varphi(Z) + g(X, Z)\varphi(Y) + g(X, JY)\varphi(JZ) \\ &+ g(X, JZ)\varphi(JY)\}\end{aligned}$$

*for all vector fields $X, Y, Z$ on $M$.*

(3) *The class $\mathcal{W}_2$ or special complex manifolds with Norden metric characterized by the condition*

$$\mathcal{F}(X, Y, JZ) + \mathcal{F}(Y, Z, JX) + \mathcal{F}(Z, X, JY) = 0, \quad \varphi = 0,$$

*or equivalently $\mathcal{N} = 0$, $\varphi = 0$.*

(4) *The class $\mathcal{W}_3$ or quasi-Kaehlerian manifolds with Norden metric characterized by the condition*

$$\mathcal{F}(X, Y, Z) + \mathcal{F}(Y, Z, X) + \mathcal{F}(Z, X, Y) = 0,$$

*or equivalently $\widetilde{\mathcal{N}} = 0$.*

(5) *The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ or complex manifolds with Norden metric characterized by the condition*

$$\mathcal{N} = 0.$$

(6) *The class $\mathcal{W}_2 \oplus \mathcal{W}_3$ or semi-Kaehlerian with Norden metric characterized by the condition*

$$\varphi = 0.$$

(7) *The class $\mathcal{W}_1 \oplus \mathcal{W}_3$ characterized by the condition*

$$\mathfrak{S}_{XYZ}\mathcal{F}(X, Y, Z) = \frac{1}{m}\mathfrak{S}_{XYZ}\Big\{g(X, Y)\varphi(Z) + g(X, JY)\varphi(JZ)\Big\},$$

*where $\mathfrak{S}$ denotes the cyclic sum by $X, Y, Z$.*

(8) *The class $\mathcal{W}$ or the whole class of almost Norden manifolds.*

**Theorem 3.6.** *Let $\pi : (M, g, \tilde{g}, J) \to (B, g', \tilde{g}', J')$ be an almost Norden submersion. If $M$ is a Kaehler-Norden manifold, then $B$ is also a Kaehler-Norden manifold.*

*Proof.* Let $X, Y, Z$ be basic vector fields on $M$ and are $\pi$-related to vector fields $X_*, Y_*, Z_*$ on $B$, respectively. Then on using (13) and Theorem 3.2, it follows that

$$\begin{aligned} \mathcal{F}(X,Y,Z) &= g(\nabla^g_X JY - J\nabla^g_X Y, Z) \\ &= g(\mathcal{H}\nabla^g_X JY - \mathcal{H}J\nabla^g_X Y, Z) \\ &= g'(\nabla'_{X_*} J'Y_* - J'\nabla'_{X_*} Y_*, Z_*) \circ \pi \\ &= g'((\nabla'_{X_*} J')Y_*, Z_*) \circ \pi \\ &= \mathcal{F}'(X_*, Y_*, Z_*) \circ \pi \\ &= \pi^* \mathcal{F}'(X_*, Y_*, Z_*). \end{aligned} \tag{20}$$

Let $M$ be a Kaehler-Norden manifold, that is, $\mathcal{F}(X, Y, Z) = 0$. Then from the last expression, we have $\pi^*\mathcal{F}'(X_*, Y_*, Z_*) = 0$. Since $\pi$ is a semi-Riemannian submersion, then $\pi^*$ is a linear isometry, therefore $\mathcal{F}'(X_*, Y_*, Z_*) = 0$ and consequently $B$, is a Kaehler-Norden manifold. □

**Theorem 3.7.** *Let $\pi : (M, g, J) \to (B, g', J')$ be an almost Norden submersion. If $M$ is a quasi-Kaehlerian manifold or a complex manifold with Norden metric, then $B$ is also a quasi-Kaehlerian manifold or a complex manifold with Norden metric, respectively.*

*Proof.* It is known from Theorem 3.2 that the basic vector fields on $M$, $\pi$-related to $J'X_*$, $\nabla'_{X_*} J'Y_*$ and $J'\nabla'_{X_*} Y_*$ for any vector fields $X_*, Y_*$ on $B$ are $JX$, $\mathcal{H}\nabla^g_X JY$ and $\mathcal{H}J\nabla^g_X Y$, respectively, for any basic vector fields $X$, $Y$ on $M$, $\pi$-related to vector fields $X_*$, $Y_*$ on $B$, respectively. Therefore from (16) and (17), it follows immediately that the basic vector fields on $M$, $\pi$-related to the Nijenhuis tensors $\mathcal{N}'(X_*, Y_*)$ and $\widetilde{\mathcal{N}}'(X_*, Y_*)$ on $B$ are $\mathcal{H}\mathcal{N}(X, Y)$ and $\mathcal{H}\widetilde{\mathcal{N}}(X, Y)$, respectively. Hence from Theorem 3.5, if $M$ is a quasi-Kaehlerian manifold or a complex manifolds with Norden metric, then $B$ is also of the same class, respectively. □

**Theorem 3.8.** *Let $\pi : (M^{2m}_m, g, J) \to (B^{2n}_n, g', J')$ be an almost Norden submersion such that $M$ is a semi-Kaehlerian manifold or a special complex manifold with Norden metric. Then $B$ is also a semi-Kaehlerian manifold or a special complex manifold with Norden metric if and only if*

$$\mathcal{T}_{U_i} V_i = \frac{J}{2}(\mathcal{T}_{U_i} U_i - \mathcal{T}_{V_i} V_i), \tag{21}$$

*where $\{U_1, \ldots, U_{m-n}, V_1 = JU_1, \ldots, V_{m-n} = JU_{m-n}\}$ is an adapted local basis of $(ker\ \pi_*)$.*

*Proof.* Assume that $\{U_1, \ldots, U_{m-n}, V_1 = JU_1, \ldots, V_{m-n} = JU_{m-n}\}$ and $\{X_1, \ldots, X_n, Y_1 = JX_1, \ldots, Y_n = JX_n\}$ are adapted local basis of $(ker\ \pi_*)$ and $(ker\ \pi_*)^\perp$, respectively. Then using (19), we have

$$\begin{aligned}\varphi(Z) = & \sum_{i=1}^{m-n} \Big\{ g((\nabla^g_{U_i} J)V_i, Z) + g((\nabla^g_{V_i} J)U_i, Z) \Big\} \\ & + \sum_{j=1}^{n} \Big\{ g((\nabla^g_{X_j} J)Y_j, Z) + g((\nabla^g_{Y_j} J)X_j, Z) \Big\}\end{aligned}$$

for any $Z \in \Gamma(ker\ \pi_*)^\perp$. The above expression further can be written as

$$\begin{aligned}\varphi(Z) = & - \sum_{i=1}^{m-n} g(\nabla^g_{U_i} U_i + J\nabla^g_{U_i} V_i - \nabla^g_{V_i} V_i + J\nabla^g_{V_i} U_i, Z) \\ & - \sum_{j=1}^{n} g(\nabla^g_{X_j} X_j + J\nabla^g_{X_j} Y_j - \nabla^g_{Y_j} Y_j + J\nabla^g_{Y_j} X_j, Z).\end{aligned}$$

Since $U_i, V_i \in \Gamma(ker\ \pi_*)$ and $X_j, Y_j \in \Gamma(ker\ \pi_*)^\perp$ therefore, on using (3) and (6), we get

$$\begin{aligned}\varphi(Z) = & - \sum_{i=1}^{m-n} g(\mathcal{T}_{U_i} U_i + J\mathcal{T}_{U_i} V_i - \mathcal{T}_{V_i} V_i + J\mathcal{T}_{V_i} U_i, Z) \\ & - \sum_{j=1}^{n} g(\mathcal{H}\nabla^g_{X_j} X_j + \mathcal{H}J\nabla^g_{X_j} Y_j - \mathcal{H}\nabla^g_{Y_j} Y_j + \mathcal{H}J\nabla^g_{Y_j} X_j, Z),\end{aligned}$$

and further using (7) and Theorem 3.2, we derive

$$\begin{aligned}\varphi(Z) = & - \sum_{i=1}^{m-n} g(\mathcal{T}_{U_i} U_i + J\mathcal{T}_{U_i} V_i - \mathcal{T}_{V_i} V_i + J\mathcal{T}_{U_i} V_i, Z) \\ & - \sum_{j=1}^{n} g'(\nabla'_{X_{*j}} X_{*j} + J'\nabla'_{X_{*j}} Y_{*j} - \nabla'_{Y_{*j}} Y_{*j} + J'\nabla'_{Y_{*j}} X_{*j}, Z_*) \circ \pi \\ = & - \sum_{i=1}^{m-n} g(\mathcal{T}_{U_i} U_i - \mathcal{T}_{V_i} V_i + 2J\mathcal{T}_{U_i} V_i, Z) + \varphi'(Z_*). \qquad (22)\end{aligned}$$

Hence the proof is complete from the above expression and using the proof of Theorem 3.7. □

**Corollary 3.9.** *Assume $\pi : (M^{2m}_m, g, J) \to (B^{2n}_n, g', J')$ is an almost Norden submersion such that the adapted local basis $\{U_1, \ldots, U_{m-n}, V_1 = JU_1, \ldots, V_{m-n} = JU_{m-n}\}$ of $(ker\ \pi_*)$ satisfies (21). If $M$ is a conformally Kaehlerian manifold with Norden metric or of class $\mathcal{W}_1 \oplus \mathcal{W}_3$, then $B$ is a conformally Kaehlerian manifold with Norden metric or of class $\mathcal{W}_1 \oplus \mathcal{W}_3$, respectively.*

*Proof.* Proof follows directly using (20) and (22). □

## 4. Totally geodesic map, decomposition theorems and harmonicity

An almost Norden manifold $(M, g, J)$ is said to be a Kaehler-Norden manifold [17], if $\nabla^g J = 0$, where $\nabla^g$ is the Levi-Civita connection of $g$. Hence from (13), it is clear that $M$ is a Kaehler-Norden manifold if and only if $\mathcal{F}(E, F, G) = 0$ for any $E, F, G \in \Gamma(TM)$.

**Theorem 4.1.** *Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be an almost Norden submersion from a Kaehler-Norden manifold $M$. Let $X, Y$ be horizontal vector fields and $V$ be a vertical vector field on $M$. Then*

(i) $g(\mathcal{A}_X V, Y) = -g(\mathcal{A}_X Y, V)$ *and similarly* $g(\mathcal{A}_X V, JY) = -g(\mathcal{A}_X Y, JV)$.
(ii) $g(\mathcal{A}_{JX} Y, V) = g(\mathcal{A}_X JY, V)$.
(iii) $\mathcal{A}_X JX = 0$.
(iv) $\mathcal{A}_X JY = -\mathcal{A}_Y JX$.
(v) $\mathcal{A}_X JY = J\mathcal{A}_X Y$.

*Proof.* Let $\nabla^g$ be the Levi-Civita connection of $g$. Then, using (5) and (6), we get

$$g(\mathcal{A}_X V, Y) = g(\nabla^g_X V, Y) = -g(V, \nabla^g_X Y) = -g(V, \mathcal{A}_X Y).$$

Now, for any horizontal vector field $X$ and a vertical vector field $V$, we have $\pi_*[X, V] = [\pi_*(X), 0] = 0$, implies $[X, V]$ is always vertical. Since $\nabla^g$ is a torsion free connection on $M$, the vertical and horizontal distributions are $J$-invariant (Theorem 3.1), then using (6), we obtain

$$g(\mathcal{A}_{JX} Y, V) = g(\nabla^g_{JX} Y, V) = -g(Y, \nabla^g_{JX} V) = -g(Y, \nabla^g_V JX).$$

Since $M$ is a Kaehler-Norden manifold, therefore above expression becomes

$$g(\mathcal{A}_{JX} Y, V) = -g(JY, \nabla^g_V X) = -g(JY, \nabla^g_X V) = g(\nabla^g_X JY, V) = g(\mathcal{A}_X JY, V).$$

Hence the assertion (ii) is complete. Further, assertion (ii) can be written as $g(\mathcal{A}_{JX} Y - \mathcal{A}_X JY, V) = 0$, then using (8), we get $2g(\mathcal{A}_X JX, V) = 0$ and then the non-degeneracy of the vertical distribution implies $\mathcal{A}_X JX = 0$. Finally, the assertion (iv) follows directly from (iii) by standard polarization technique. Assertion (v) is trivial using the Kaehlerian property of $M$ with (6). □

We can derive the following relations analogous to relations in the above theorem.

**Theorem 4.2.** *Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be an almost Norden submersion from a Kaehler-Norden manifold $M$. Let $X$ be a horizontal vector field and $U, V$ be vertical vector fields on $M$. Then*

(i) $g(\mathcal{T}_U V, X) = -g(\mathcal{T}_U X, V)$.
(ii) $g(\mathcal{T}_U X, JV) = -g(JX, \mathcal{T}_U V)$.
(iii) $g(\mathcal{T}_{JU} V, X) = g(\mathcal{T}_U JV, X)$.
(iv) $\mathcal{T}_U JV = J\mathcal{T}_U V$.

**Theorem 4.3.** *Let $\pi : (M_m^{2m}, g, \tilde{g}, J) \to (B_n^{2n}, g', \tilde{g}', J')$ be an almost Norden submersion from a Kaehler-Norden manifold $M$. Let $X, Y$ be horizontal vector fields and $U, V$ be vertical vector fields on $M$. Then*

$$\mathcal{A}_{JX}V = J\mathcal{A}_X V = \mathcal{A}_X JV. \tag{23}$$

*Proof.* Since $M$ is a Kaehler-Norden manifold, then using (4), we have

$$g(\mathcal{T}_V JX, U) = g(\nabla^g_V X, JU) = g(J\mathcal{T}_V X, U),$$

then non-degeneracy of the vertical distribution implies

$$\mathcal{T}_V JX = J\mathcal{T}_V X. \tag{24}$$

Now, from (4) we have

$$\nabla^g_V X = \mathcal{H}(\nabla^g_V X) + \mathcal{T}_V X = \mathcal{H}(\nabla^g_X V) + \mathcal{T}_V X = \mathcal{A}_X V + \mathcal{T}_V X. \tag{25}$$

Then using (24) and (25), we obtain

$$(\nabla^g_V J)X = \mathcal{A}_{JX}V - J\mathcal{A}_X V,$$

since $M$ is a Kaehler-Norden manifold, we obtain

$$\mathcal{A}_{JX}V = J\mathcal{A}_X V. \tag{26}$$

Again, as $M$ is a Kaehler-Norden manifold then from (13), we have

$$0 = (\nabla^g_{JX}\widetilde{g})(Y, JV) = g((\nabla^g_{JX}J)Y, JV),$$

further using (5), (6), (26), (i) and (ii) of Theorem 4.1, it implies

$$\begin{aligned} 0 &= g(\nabla^g_{JX}JY, JV) + g(\nabla^g_{JX}Y, V) \\ &= g(\mathcal{A}_{JX}JY, JV) - g(Y, \mathcal{A}_{JX}V) \\ &= -g(\mathcal{A}_X Y, JV) - g(Y, \mathcal{A}_{JX}V) \\ &= g(Y, \mathcal{A}_X JV) - g(Y, \mathcal{A}_{JX}V). \end{aligned}$$

Then the non-degeneracy of the vertical distribution gives

$$\mathcal{A}_X JV = \mathcal{A}_{JX}V. \tag{27}$$

Hence from (26) and (27), our assertion follows. □

It is known that the vertical distribution ($ker\ \pi_*$) is always integrable and in general, horizontal distribution $(ker\ \pi_*)^\perp$ is not integrable. In the following theorem, we obtain a necessary and sufficient condition for the horizontal distribution $(ker\ \pi_*)^\perp$ to be integrable.

**Theorem 4.4.** *Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be an almost Norden submersion between almost Norden manifolds. Then the horizontal distribution $(ker\ \pi_*)^\perp$ is integrable if and only if*

$$g'(\pi_* X, (\nabla\pi_*)(Y, V)) = g'(\pi_* Y, (\nabla\pi_*)(X, V)) \tag{28}$$

*for any $X, Y \in \Gamma(ker\ \pi_*)^\perp$ and $V \in \Gamma(ker\ \pi_*)$.*

*Proof.* Let $X, Y \in \Gamma(ker\ \pi_*)^\perp$ and $V \in \Gamma(ker\ \pi_*)$. Then, using the fact the $\nabla^g$ is a Levi-Civita connection on $M$, we get

$$g([X,Y],V) = g(\nabla^g_X Y, V) - g(\nabla^g_Y X, V) = -g(Y, \nabla^g_X V) + g(X, \nabla^g_Y V).$$

Since $\pi$ is a semi-Riemannian submersion, then last expression can be written as

$$g([X,Y],V) = -g'(\pi_* Y, \pi_*(\nabla^g_X V)) + g'(\pi_* X, \pi_*(\nabla^g_Y V)),$$

further using (9), we obtain

$$g([X,Y],V) = g'(\pi_* Y, (\nabla \pi_*)(X,V)) - g'(\pi_* X, (\nabla \pi_*)(Y,V)),$$

this completes the proof. □

**Corollary 4.5.** *Let $\pi : (M^{2m}_m, g, J) \to (B^{2n}_n, g', J')$ be an almost Norden submersion between almost Norden manifolds. If submersion $\pi$ is totally geodesic, that is, $(\nabla \pi_*) = 0$, then the horizontal distribution $(ker\ \pi_*)^\perp$ is always integrable.*

Analogous to the proof of Theorem 4.4, we can prove the following results immediately.

**Theorem 4.6.** *Let $\pi : (M^{2m}_m, g, J) \to (B^{2n}_n, g', J')$ be an almost Norden submersion between almost Norden manifolds. Then the horizontal distribution $(ker\ \pi_*)^\perp$ defines totally geodesic foliations on $M$ if and only if*

$$(\nabla \pi_*)(X,V) = 0 \tag{29}$$

*for any $X \in \Gamma(ker\ \pi_*)^\perp$ and $V \in \Gamma(ker\ \pi_*)$.*

**Theorem 4.7.** *Let $\pi : (M^{2m}_m, g, J) \to (B^{2n}_n, g', J')$ be an almost Norden submersion between almost Norden manifolds. Then the vertical distribution $(ker\ \pi_*)$ defines totally geodesic foliations on $M$ if and only if*

$$(\nabla \pi_*)(U,V) = 0 \tag{30}$$

*for any $U, V \in \Gamma(ker\ \pi_*)$.*

A smooth map $\pi : (M_1, g_1, J_1) \to (M_2, g_2, J_2)$ between almost Hermitian manifolds is called *pluriharmonic* if its second fundamental form $\nabla \pi_*$ satisfies

$$(\nabla \pi_*)(X,Y) + (\nabla \pi_*)(J_1 X, J_1 Y) = 0 \tag{31}$$

for any $X, Y \in \Gamma(TM_1)$ [19].

**Theorem 4.8.** *Let $\pi : (M^{2m}_m, g, J) \to (B^{2n}_n, g', J')$ be an almost Norden submersion such that $M$ is a Kaehler-Norden manifold. Then $\pi$ is a pluriharmonic map.*

*Proof.* Let $X, Y \in \Gamma(TM)$. Then using (9), we have

$$(\nabla \pi_*)(X, JY) = \nabla^\pi_X(\pi_*(JY)) - \pi_*(\nabla^g_X JY).$$

Since $\pi$ is an almost Norden submersion, that is, $\pi_* \circ J = J' \circ \pi_*$ and $M$ is a Kaehler-Norden manifold, then using Theorem 3.6, $B$ is also a Kaehler-Norden manifold therefore we get

$$\begin{aligned}(\nabla\pi_*)(X, JY) &= \nabla^{\pi}_X J'(\pi_* Y) - \pi_*(J(\nabla^g_X Y))\\ &= \nabla'_{X_*} J'(\pi_* Y) - J'(\pi_*(\nabla^g_X Y))\\ &= J'((\nabla\pi_*)(X, Y)).\end{aligned}$$

Using last relation with the symmetric property of the second fundamental form of $\pi$, we further obtain

$$(\nabla\pi_*)(JX, JY) = -(\nabla\pi_*)(X, Y),$$

and hence $\pi$ is a pluriharmonic map. □

**Theorem 4.9.** *Let $\pi : (M^{2m}_m, g, J) \to (B^{2n}_n, g', J')$ be an almost Norden submersion from an almost Norden manifold $M$. Then $\pi$ is a totally geodesic map if and only if*

$$\mathcal{A}_X V = 0, \quad \mathcal{T}_U V = 0 \tag{32}$$

*for any $X, Y \in \Gamma(ker\ \pi_*)^{\perp}$ and $U, V \in \Gamma(ker\ \pi_*)$. Moreover, if $\pi$ is a totally geodesic map, then the fibers are totally geodesic.*

*Proof.* From the definition of totally geodesic map, it is obvious that the almost Norden submersion $\pi$ is a totally geodesic map if and only if

$$(\nabla\pi_*)(X, Y) = 0,\ (\nabla\pi_*)(X, V) = 0,\ (\nabla\pi_*)(U, V) = 0$$

for any $X, Y \in \Gamma(ker\ \pi_*)^{\perp}$ and $U, V \in \Gamma(ker\ \pi_*)$. Using Theorem 3.2 and (9), it is obvious that the second fundamental form $\nabla\pi_*$ of the almost Norden submersion $\pi$ satisfies

$$(\nabla\pi_*)(X, Y) = 0, \quad X, Y \in \Gamma(ker\ \pi_*)^{\perp}. \tag{33}$$

For $X \in \Gamma(ker\ \pi_*)^{\perp}$ and $V \in \Gamma(ker\ \pi_*)$, using (5) and (9), we have

$$(\nabla\pi_*)(X, V) = -\pi_*(\nabla^g_X V) = -\pi_*(\mathcal{A}_X V). \tag{34}$$

Also, for $U, V \in \Gamma(ker\ \pi_*)$, using (3) and (9), we have

$$(\nabla\pi_*)(U, V) = -\pi_*(\nabla^g_U V) = -\pi_*(\mathcal{T}_U V). \tag{35}$$

Hence the proof is complete from (33)–(35). □

**Corollary 4.10.** *Let $\pi : (M^{2m}_m, g, J) \to (B^{2n}_n, g', J')$ be an almost Norden submersion from a Kaehler-Norden manifold $M$. Let $\pi$ be a totally geodesic map. Then from* (i) *of Theorem 4.1, we get*

$$g(\mathcal{A}_X Y, V) = -g(\mathcal{A}_X V, Y) = 0,$$

*then the non-degeneracy of the vertical distribution implies $\mathcal{A}_X Y = 0$ for any $X, Y \in \Gamma(ker\ \pi_*)^{\perp}$. Thus, if the almost Norden submersion $\pi$ from a Kaehler-Norden manifold to an almost Norden manifold is totally geodesic, then the fibers of $\pi$ are totally geodesic and the horizontal distribution is integrable.*

Now, we recall an important theorem for product structures from [22].

**Theorem 4.11.** *Let $g$ be a Riemannian metric tensor on the manifold $\overline{M} = M \times N$ and assume that the canonical foliations $\mathcal{D}_M$ and $\mathcal{D}_N$ intersect perpendicularly everywhere. Then $g$ is the metric tensor of*

(i) *a double-twisted product $M \times_{(f,g)} N$ if and only if $\mathcal{D}_M$ and $\mathcal{D}_N$ are totally umbilical foliations,*

(ii) *a twisted product $M \times_f N$ if and only if $\mathcal{D}_M$ is a totally geodesic foliation and $\mathcal{D}_N$ is a totally umbilical foliation,*

(iii) *a warped product $M \times_f N$ if and only if $\mathcal{D}_M$ is a totally geodesic foliation and $\mathcal{D}_N$ is a spherical foliation, i.e., it is umbilical and its mean curvature vector field is parallel, and*

(iv) *a usual product of Riemannian manifolds if and only if $\mathcal{D}_M$ and $\mathcal{D}_N$ are totally geodesic foliations.*

Hence from Theorems 4.6, 4.7, 4.9 and 4.11, we have following result directly.

**Theorem 4.12.** *Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be an almost Norden submersion between almost Norden manifolds. Then $M$ is a locally product manifold if and only if $\pi$ is a totally geodesic map.*

**Theorem 4.13.** *Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be an almost Norden submersion between almost Norden manifolds such that $M_{(ker\ \pi_*)^\perp}$ is a totally umbilical integral manifold of the distribution $(ker\ \pi_*)^\perp$. Then $M_{(ker\ \pi_*)^\perp}$ is totally geodesic.*

*Proof.* Let $h$ be the second fundamental form of $M_{(ker\ \pi_*)^\perp}$. Then $h(X,Y) = g(X,Y)\widetilde{H}$ for any $X, Y \in \Gamma(ker\ \pi_*)^\perp$, where $\widetilde{H}$ is the mean curvature of $M_{(ker\ \pi_*)^\perp}$. Let $U \in \Gamma(ker\ \pi_*)$. Then

$$g(\nabla_X^g X, U) = g(h(X,X), U) = g(X,X)g(\widetilde{H}, U).$$

On the other hand, using (6) and (8), we have

$$g(\nabla_X^g X, U) = g(\mathcal{A}_X X, U) = 0.$$

Hence $g(X,X)g(\widetilde{H}, U) = 0$ then the non-degeneracy of $g$ on $(ker\ \pi_*)$ and $(ker\ \pi_*)^\perp$ implies that $\widetilde{H} = 0$. Hence the assertion is complete. □

**Corollary 4.14.** *There does not exist an almost Norden submersion $\pi$ between almost Norden manifolds $(M_m^{2m}, g, J)$ and $(B_n^{2n}, g', J')$ such that $M$ is a twisted product of the form $M_{(ker\ \pi_*)} \times_f M_{(ker\ \pi_*)^\perp}$, where $M_{(ker\ \pi_*)^\perp}$ and $M_{(ker\ \pi_*)}$ are integral manifolds of the distributions $(ker\ \pi_*)^\perp$ and $(ker\ \pi_*)$, respectively.*

**Corollary 4.15.** *There does not exist an almost Norden submersion $\pi$ between almost Norden manifolds $(M_m^{2m}, g, J)$ and $(B_n^{2n}, g', J')$ such that $M$ is a double-twisted product of the form $M_{(ker\ \pi_*)} \times_{(f,g)} M_{(ker\ \pi_*)^\perp}$ or of the form $M_{(ker\ \pi_*)^\perp} \times_{(f,g)} M_{(ker\ \pi_*)}$, where $M_{(ker\ \pi_*)^\perp}$ and $M_{(ker\ \pi_*)}$ are integral manifolds of the distributions $(ker\ \pi_*)^\perp$ and $(ker\ \pi_*)$, respectively.*

**Corollary 4.16.** *There does not exist an almost Norden submersion $\pi$ between almost Norden manifolds $(M_m^{2m}, g, J)$ and $(B_n^{2n}, g', J')$ such that $M$ is a warped product of the form $M_{(ker\ \pi_*)} \times M_{(ker\ \pi_*)^\perp}$, where $M_{(ker\ \pi_*)^\perp}$ and $M_{(ker\ \pi_*)}$ are integral manifolds of the distributions $(ker\ \pi_*)^\perp$ and $(ker\ \pi_*)$, respectively.*

Now, we recall that a Riemannian submersion $\pi : (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds is called a Riemannian submersion with totally umbilical fibers if

$$\mathcal{T}_U V = g_1(U, V)H \tag{36}$$

for $U, V \in \Gamma(ker\ \pi_*)$, where $H$ is the mean curvature vector of the fibers.

**Theorem 4.17.** *Let $\pi$ be an almost Norden submersion with totally umbilical fibers from a Kaehler-Norden manifold $(M_m^{2m}, g, J)$ to an almost Norden manifold $(B_n^{2n}, g', J')$. Then fibers are totally geodesic.*

*Proof.* For $U \in \Gamma(ker\ \pi_*)$, using (3) and the Kaehlerian property of $M$, we get

$$J\mathcal{T}_{JU}U + J(\mathcal{V}\nabla^g_{JU}U) = \mathcal{T}_{JU}JU + \mathcal{V}\nabla^g_{JU}JU.$$

Further, using (36) and (11), we obtain

$$g(JU, U)JH + J(\mathcal{V}\nabla^g_{JU}U) = -g(U, U)H + \mathcal{V}\nabla^g_{JU}JU. \tag{37}$$

On taking the scalar product of both sides of (37) with $X \in \Gamma(ker\ \pi_*)^\perp$ and using Theorem 3.1, we get

$$g(JU, U)g(JH, X) = -g(U, U)g(H, X). \tag{38}$$

Similarly, on taking the scalar product of both sides of (37) with

$$JX \in \Gamma(ker\ \pi_*)^\perp,$$

we get

$$g(JU, U)g(H, X) = g(U, U)g(H, JX). \tag{39}$$

Since the vertical distribution is non-degenerate, then on solving (38) and (39), it follows that

$$g(H, JX)^2 + g(H, X)^2 = 0,$$

then non-degeneracy of the horizontal distribution implies $H = 0$. Hence, the proof is complete. □

**Theorem 4.18.** *Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be an almost Norden submersion between almost Norden manifolds. Then $M$ is a locally twisted product manifold of the form $M_{(ker\ \pi_*)^\perp} \times_f M_{(ker\ \pi_*)}$ if and only if*

$$\mathcal{T}_V X = -g(X, \mathcal{T}_V V)\|V\|^{-2}V, \text{ and } \mathcal{A}_X V = 0$$

*for any $X \in \Gamma(ker\ \pi_*)^\perp$ and $V \in \Gamma(ker\ \pi_*)$, where $M_{(ker\ \pi_*)^\perp}$ and $M_{(ker\ \pi_*)}$ are integral manifolds of the distributions $(ker\ \pi_*)^\perp$ and $(ker\ \pi_*)$, respectively.*

*Proof.* For $X \in \Gamma((ker\ \pi_*)^{\perp})$ and $U, V \in \Gamma(ker\ \pi_*)$, using (4), we get

$$g(\nabla^g_V U, X) = -g(U, \nabla^g_V X) = -g(U, \mathcal{T}_V X). \tag{40}$$

This implies that $(ker\ \pi_*)$ is totally umbilical if and only if

$$\mathcal{T}_V X = -X(\mu)V, \tag{41}$$

where $\mu$ is some function on $M$. On taking scalar product with $V$ to both sides of (41) and then using (40), we obtain

$$-X(\mu)\|V\|^2 = g(\mathcal{T}_V X, V) = -g(X, \mathcal{T}_V V),$$

this further implies that

$$X(\mu) = g(X, \mathcal{T}_V V)\|V\|^{-2}. \tag{42}$$

Hence from (41) and (42), $(ker\ \pi_*)$ is totally umbilical if and only if

$$\mathcal{T}_V X = -g(X, \mathcal{T}_V V)\|V\|^{-2} V.$$

Thus the proof is complete using the last expression, Theorem 4.6 and (34). $\square$

In [16], Gudmundsson and Wood obtained expression for the tension field of a holomorphic map between an almost Hermitian manifold and a quasi-Kaehler manifold. Then, Chinea [6] derived expression for the tension field of a holomorphic map between almost Hermitian manifolds. In the next theorem, we obtain expression for the tension field of an almost Norden submersion between almost Norden manifolds by using adapted local basis.

**Theorem 4.19.** *Let $\pi : (M^{2m}_m, g, J) \to (B^{2n}_n, g', J')$ be an almost Norden submersion between almost Norden manifolds. Then the tension field $\tau(\pi)$ of $\pi$ is given by*

$$\tau(\pi) = -\pi_*(trac\ \mathcal{T}). \tag{43}$$

*Proof.* Assume that $\{U_1, \ldots, U_{m-n}, V_1 = JU_1, \ldots, V_{m-n} = JU_{m-n}\}$ and $\{X_1, \ldots, X_n, Y_1 = JX_1, \ldots, Y_n = JX_n\}$ are adapted local bases of $(ker\ \pi_*)$ and $(ker\ \pi_*)^{\perp}$, respectively and satisfies (18), where $X_j$ are basic vector fields on $M$, which are $\pi$-related to vector fields $X_{*j}$ of $B$, respectively. Then the tension field $\tau(\pi)$ of $\pi$ is given by

$$\begin{aligned}\tau(\pi) = &\sum_{i=1}^{m-n}((\nabla\pi_*)(U_i, V_i) + (\nabla\pi_*)(V_i, U_i))\\ &+\sum_{j=1}^{n}((\nabla\pi_*)(X_j, Y_j) + (\nabla\pi_*)(Y_j, X_j)),\end{aligned} \tag{44}$$

further using (33) and (3), we obtain

$$\tau(\pi) = -\sum_{i=1}^{m-n}(\pi_*(\nabla^g_{U_i} V_i) + \pi_*(\nabla^g_{V_i} U_i))$$

$$= -\sum_{i=1}^{m-n} (\pi_*(\mathcal{T}_{U_i}V_i + \mathcal{T}_{V_i}U_i))$$
$$= -\pi_*\Big\{\sum_{i=1}^{m-n} (\mathcal{T}_{U_i}V_i + \mathcal{T}_{V_i}U_i)\Big\}$$
$$= -\pi_*(trac\ \mathcal{T}).$$

Hence the proof is complete. □

Hence from the last theorem, we have following important assertion immediately.

**Theorem 4.20.** *Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be an almost Norden submersion. Then $\pi$ is a harmonic map if and only if $\pi$ has minimal fibers.*

Let $(M_m^{2m}, g, J)$ be a Kaehler-Norden manifold. Then using (iv) of Theorem 4.2 in (43), we derive

$$\tau(\pi) = -2J' \sum_{i=1}^{m-n} \pi_*(\mathcal{T}_{U_i}U_i).$$

Hence, we have following important assertion directly.

**Theorem 4.21.** *Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be an almost Norden submersion from a Kaehler-Norden manifold. Then $\pi$ is a harmonic map if and only if $\mathcal{T}_{U_i}U_i = 0$ for adapted local basis $\{U_1, \ldots, U_{m-n}, V_1 = JU_1, \ldots, V_{m-n} = JU_{m-n}\}$ of $(ker\ \pi_*)$.*

Using Theorem 4.17, we deduce the following result.

**Corollary 4.22.** *Let $\pi$ be an almost Norden submersion with totally umbilical fibers from a Kaehler-Norden manifold $(M_m^{2m}, g, J)$ to an almost Norden manifold $(B_n^{2n}, g', J')$. Then $\pi$ is a harmonic map.*

Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be an almost Norden submersion from an almost Norden manifold $M$. If $\pi$ is horizontally weakly conformal, then we say it as a horizontally weakly conformal almost Norden submersion, with dilation $\lambda$. Now, we recall an important result from [2, p. 119], for further use.

**Theorem 4.23.** *Suppose $\pi : M \to B$ is a horizontally conformal submersion between Riemannian manifolds. Then*

$$(\nabla\pi_*)(X, Y) = X(\ln\lambda)\pi_*(Y) + Y(\ln\lambda)\pi_*(X) - g(X, Y)\pi_*(grad\ \ln\lambda) \tag{45}$$

*for any horizontal vector fields $X$ and $Y$.*

It is also known that if $\{E_i\}_{i=1}^n$ is an orthonormal frame on a semi-Riemannian manifold $(N, h)$ of dimension $n$, then any $X \in \Gamma(TN)$ can be written as

$$X = \sum_{i=1}^{n} \epsilon_i h(X, E_i)E_i, \tag{46}$$

where $\epsilon_i$ is the signature of $\{E_i\}$.

**Theorem 4.24.** *Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be a horizontally weakly conformal almost Norden submersion, with dilation $\lambda$, from an almost Norden manifold $M$. Then the tension field of $\pi$ is given by*

$$\tau(\pi) = -\pi_*(trac\ \mathcal{T}) - 2(n-1)\pi_*(grad\ \ln\lambda). \tag{47}$$

*Proof.* Using the symmetric property of the second fundamental form of $\pi$ in (44), we have

$$\tau(\pi) = \sum_{i=1}^{m-n}((\nabla\pi_*)(U_i, V_i) + (\nabla\pi_*)(V_i, U_i)) + 2\sum_{j=1}^{n}(\nabla\pi_*)(X_j, Y_j).$$

Further, follow the proof of Theorem 4.19 and (45) in the last expression, we derive

$$\begin{aligned}\tau(\pi) = &-\pi_*(trac\ \mathcal{T}) + 2\pi_*\Big\{\sum_{j=1}^{n}\big(X_j(\ln\lambda)Y_j + Y_j(\ln\lambda)X_j \\ &- g(X_j, Y_j)\mathcal{H}(grad\ \ln\lambda)\big)\Big\},\end{aligned}$$

using the relation $X(\ln\lambda) = g(grad\ \ln\lambda, X)$ in the last expression, we have

$$\begin{aligned}\tau(\pi) = &-\pi_*(trac\ \mathcal{T}) + 2\pi_*\Big\{\sum_{j=1}^{n}\big(g(grad\ \ln\lambda, X_j)Y_j + g(grad\ \ln\lambda, Y_j)X_j\big)\Big\} \\ &- 2n\pi_*(grad\ \ln\lambda).\end{aligned} \tag{48}$$

For a horizontal vector field $X$, using adapted local basis $\{X_1, \ldots, X_n, Y_1 = JX_1, \ldots, Y_n = JX_n\}$ of $(ker\ \pi_*)^\perp$ in (46), we can write

$$X = \sum_{j=1}^{n} g(X, X_j)Y_j + g(X, Y_j)X_j. \tag{49}$$

Then on using (49) in (48), we obtain

$$\tau(\pi) = -\pi_*(trac\ \mathcal{T}) - 2(n-1)\pi_*(grad\ \ln\lambda).$$

Hence the result is complete. □

Thus using Theorem 4.20, we have following result.

**Theorem 4.25.** *Let $\pi : (M_m^{2m}, g, J) \to (B_n^{2n}, g', J')$ be a non-constant horizontally weakly conformal almost Norden submersion, with dilation $\lambda$, from an almost Norden manifold $M$. Then any two of the following assertions imply the third:*

(i) *$\pi$ is harmonic (and so a harmonic morphism).*
(ii) *$\pi$ is horizontally homothetic.*
(iii) *$\pi$ has minimal fibers.*

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