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RIGIDITY OF A RANK 1 CUSP OF PUNCTURED-SURFACE GROUPS IN HYPERBOLIC 4-SPACE

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Abstract. We prove that a punctured-torus group of hyperbolic 4-space which keeps an embedded hyperbolic 2-plane invariant has a strictly parabolic commutator. More generally, this rigidity persists for a punctured-surface group.

1. Introduction

The presence of screw parabolic isometries makes the geometry of four dimensional hyperbolic space interesting. On the boundary at infinity $\hat{\mathbb{R}}^n$ of an $(n+1)$-dimensional hyperbolic space, a parabolic isometry is Möbius conjugate to $x \mapsto Ax + a$ with $A \in \mathrm{SO}(n)$, $a \in \mathbb{R}^n \setminus \{0\}$. If $A \neq I$, then it is called *screw parabolic*; otherwise it is *strictly parabolic*. Hyperbolic 4-space is the lowest dimension hyperbolic space where screw parabolic isometries appear.

In hyperbolic 4-space $\mathbb{H}^4$, a screw parabolic isometry is not Möbius conjugate to a strictly parabolic isometry but is topologically conjugate. Furthermore, a screw parabolic isometry is not quasiconformally conjugate to a strictly parabolic isometry [10]. In fact, there are infinitely many quasiconformally distinct conjugacy classes of screw parabolic isometries [10]. This contrasts to the fact that all parabolic isometries are quasiconformally conjugate to each other and hence there is only one quasiconformal conjugacy class of parabolic isometries in $\mathbb{H}^2$ and $\mathbb{H}^3$. What does a hyperbolic 4-manifold with a screw parabolic isometry look like?

Screw parabolic isometries create examples of geometrically finite hyperbolic 4-manifolds which are not quasiconformally stable [10]. We start with a Fuchsian thrice-punctured sphere group G with a fundamental domain. Using the standard totally geodesic embedding of $\mathbb{H}^2$ in $\mathbb{H}^4$ and the Poincaré extension, we extend the action of G to $\mathbb{H}^4$. Now acting on $\mathbb{H}^4$, the group

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G has strictly parabolic isometries and keeps a hyperbolic 2-plane invariant. Adding a rotational action around the invariant plane, we can continuously deform the strictly parabolic isometries into screw parabolic isometries. In this paper, we investigate if we can apply this idea to construct other hyperbolic 4-manifolds with a screw parabolic isometry. As a result, we discover a rigidity for a strictly parabolic commutator of a punctured-surface group in contrast to the thrice-punctured sphere group.

A punctured-surface group $G_{g,1}(g \geq 1)$ is a discrete free group generated by $2g$ loxodromic isometries with the condition that the product of commutators of the generators is a parabolic isometry. We may think of $G_{g,1}$ as the image of a discrete type-preserving representation to the group of orientation preserving isometries acting on hyperbolic n-space $\mathbb{H}^n$, $\rho : \pi_1(S_{g,1}) \to \mathrm{Isom}(\mathbb{H}^n)$, where $S_{g,1}$ is a punctured hyperbolic surface of genus $g \geq 1$. The commutator condition means the loop around the puncture determines a cusp of the manifold $\mathbb{H}^n/G_{g,1}$. We prove the following rigidity.

Main Theorem. *If a punctured-surface group*

$$G_{g,1} = \langle f_1, h_1, \ldots, f_g, h_g \text{ loxodromic} \mid \Pi_{1\leq i\leq g}[f_i, h_i] \text{ parabolic}\rangle,\ g \geq 1$$

keeps a totally geodesic plane invariant in hyperbolic 4*-space, then the product of commutators* $\Pi_{1\leq i\leq g}[f_i, h_i]$ *is a strictly parabolic isometry.*

You might ask if we can deform a rank 2 maximal parabolic subgroup of a Kleinian group (that is, a discrete group of isometries acting on hyperbolic 3-space) into one containing a screw parabolic isometry in hyperbolic 4-space. The answer is negative. It is because a maximal parabolic subgroup containing an irrational screw parabolic isometry in hyperbolic 4-space can only be rank 1 [10]. An irrational screw parabolic isometry is a screw parabolic isometry which is not virtually strictly parabolic.

Deforming a Kleinian punctured-torus group in hyperbolic 4-space can be found in [4]. However, they only considered cases involving strictly parabolic isometries. We can find some interesting results about screw parabolic isometries in [7, 15, 16].

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2. Preliminaries and notations

In this section, we will give definitions and basic facts of hyperbolic spaces and Vahlen matrices. For the basics on hyperbolic geometry, the reader is referred to [5–7, 9, 13, 14]; for Vahlen matrices, to [1–3, 8, 10–12, 17–20].

Hyperbolic $(n + 1)$-space $\mathbb{H}^{n+1}$ is the unique complete simply connected $(n+1)$-dimensional Riemannian manifold with constant sectional curvature -1. It has the natural boundary at infinity $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. Möbius transformations acting on $\hat{\mathbb{R}}^n$ are finite compositions of reflections in co-dimension 1 spheres or hyperplanes. Every orientation preserving Möbius transformation of $\hat{\mathbb{R}}^n$ extends continuously to an orientation preserving hyperbolic isometry of $\mathbb{H}^{n+1}$ and vice-versa. Therefore, we identify the group of all orientation preserving isometries of $\mathbb{H}^{n+1}$, denoted by $\mathrm{Isom}(\mathbb{H}^{n+1})$ with the group of all orientation preserving Möbius transformations of $\hat{\mathbb{R}}^n$, denoted by $\mathrm{Möb}(\hat{\mathbb{R}}^n)$. A Möbius transformation of $\hat{\mathbb{R}}^n$ can be represented as a 2×2 matrix whose entries are the Clifford numbers satisfying some conditions, called a *Vahlen matrix.* The action of the 2×2 matrix is the usual action of Möbius transformations. This is a natural generalization of the classical settings, $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$, via identifying the real numbers $\mathbb{R}$ with the Clifford algebra C_0 and the complex numbers $\mathbb{C}$ with the Clifford algebra C_1.

The Clifford algebra C_{n-1} is the associative algebra over the real numbers generated by the elements $e_1, e_2, \ldots, e_{n-1}$ subject to the relations $e_l^2 = -1$ for all $l = 1, \ldots, n-1$ and $e_l e_m = -e_m e_l$ for $l \neq m$.

An element of C_{n-1} is called a *Clifford number.* A Clifford number a is of the form $\sum a_I I$, where the sum is over all products $I = e_{v_1} e_{v_2} \cdots e_{v_p}$ with $1 \leq v_1 < v_2 < \cdots < v_p \leq n-1$ and $a_I \in \mathbb{R}$. The null product of generators is the real number 1. Here are the three involutions in the Clifford algebra:

(1) The main involution $a \mapsto a'$ is an automorphism obtained by replacing each e_i with $-e_i$. Thus, $(ab)' = a'b'$ and $(a+b)' = a' + b'$.
(2) Reversion $a \mapsto a^*$ is an anti-automorphism obtained by replacing each $e_{v_1} e_{v_2} \cdots e_{v_p}$ with $e_{v_p} e_{v_{p-1}} \cdots e_{v_1}$. Therefore, $(ab)^* = b^* a^*$ and $(a+b)^* = a^* + b^*$.
(3) Conjugation $a \mapsto \overline{a}$ is an anti-automorphism obtained by a composition. Therefore, $\overline{a} = (a')^* = (a^*)'$.

The Euclidean *norm* $|a|$ of $a = \sum a_I I \in C_{n-1}$ is given by $|a|^2 = \sum a_I^2$. A *vector* is a Clifford number of the form $x = x_0 + x_1 e_1 + \cdots + x_{n-1} e_{n-1} \in C_{n-1}$, where the x_i's are real numbers. The set of all vectors forms an n-dimensional subspace which we identify with $\mathbb{R}^n$. For any vector x, $x^* = x$ and $\overline{x} = x'$. Every non-zero vector x is invertible with $x^{-1} = \frac{\overline{x}}{|x|^2}$. Since the product of invertible elements is invertible, every product of non-zero vectors is invertible. A *Clifford group* Γ_{n-1} is a multiplicative group generated by all non-zero vectors of C_{n-1}. We note that $\Gamma_{n-1} = C_{n-1} - \{0\}$ is true for only $n = 1, 2, 3$. For example, let $a = 1 + e_1 e_2 e_3 \in C_4$, then $a \notin \Gamma_4$.

Definition. A matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is said to be a *Vahlen matrix* if the following conditions are satisfied:

(1) $a, b, c, d \in \Gamma_{n-1} \cup \{0\}$.
(2) $ad^* - bc^* = 1$.
(3) $ab^*, cd^*, c^*a, d^*b \in \mathbb{R}^n$.

A Vahlen matrix A has a multiplicative inverse $A^{-1} = \left(\begin{smallmatrix} d^* & -b^* \\ -c^* & a^* \end{smallmatrix}\right)$ which is also a Vahlen matrix. Hence, the set of all Vahlen matrices forms a group, denoted by $\mathrm{SL}(2,\Gamma_{n-1})$. Since $\Gamma_{n-1} \subset \Gamma_n$, it follows that $\mathrm{SL}(2,\Gamma_{n-1})$ is a subgroup of $\mathrm{SL}(2,\Gamma_n)$.

A Vahlen matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\Gamma_{n-1})$ induces a Möbius transformation of $\hat{\mathbb{R}}^n$ by $Ax = (ax+b)(cx+d)^{-1}$ for any vector $x = x_0+x_1e_1+\cdots+x_{n-1}e_{n-1} \in \mathbb{R}^n$, and $\infty \mapsto \infty$ if $c = 0$ and $\infty \mapsto ac^{-1}$, $-c^{-1}d \mapsto \infty$ if $c \neq 0$. Moreover, any orientation preserving Möbius transformation of $\hat{\mathbb{R}}^n$ can be presented as a Vahlen matrix. Replacing x with $x + x_ne_n \in \mathbb{R}^{n+1}$, we can automatically extend the action of A to $\hat{\mathbb{R}}^{n+1}$: $x+x_ne_n \mapsto (a(x+x_ne_n)+b)(c(x+x_ne_n)+d)^{-1}$.

The coefficient of the last generator e_n of the image is $\frac{x_n}{|cx+d|^2}$. This shows that the extension keeps the upper half space of $\mathbb{R}^{n+1}$ invariant. In fact, the group of Vahlen matrices modulo $\pm\mathrm{I}$ is isomorphic to the group of orientation preserving isometries of $\mathbb{H}^{n+1}$.

2.1. Hyperbolic 4-space and Vahlen matrices

From now on, we will only consider the case that $n = 3$ which corresponds to hyperbolic 4-space $\mathbb{H}^4$. The upper half-space model of the hyperbolic 4-space is

$$\mathbb{H}^4 = \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 \mid x_0, x_1, x_2, x_3 \in \mathbb{R},\ x_3 > 0\}.$$

The boundary at infinity is

$$\hat{\mathbb{R}^3} = \{x_0 + x_1e_1 + x_2e_2 \mid x_0, x_1, x_2 \in \mathbb{R}\} \cup \{\infty\}.$$

The isometry group $\mathrm{Isom}(\mathbb{H}^4)$ is isomorphic to $\mathrm{SL}(2,\Gamma_2)/\pm\mathrm{I}$. The Clifford algebra $C_2 = \{x_0 + x_1e_1 + x_2e_2 + x_3e_1e_2 \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\}$ is isomorphic to the quaternions $\mathbf{H} = \{x_0 + x_1i + x_2j + x_3k \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\}$ as an algebra, where we identify e_1, e_2 and e_1e_2 with i, j and k, respectively. We note that the Clifford group Γ_2 is $C_2 \setminus \{0\}$.

We denote the subgroup of all unit elements of Γ_2 by Γ_2^{unit}:

$$\Gamma_2^{\mathrm{unit}} = \{x \in \Gamma_2 \mid |x| = 1\}.$$

For $x = x_0 + x_1e_1 + x_2e_2 + x_3e_1e_2 \in \Gamma_2^{\mathrm{unit}}$, i.e., $\sum x_i^2 = 1$, we can rewrite

$$x_0 + x_1e_1 + x_2e_2 + x_3e_1e_2 = x_0 + \sqrt{1 - x_0^2}\,\frac{x_3 - x_2e_1 + x_1e_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\,e_1e_2.$$

It follows that a unit element of Γ_2 can be written as $\cos\theta + \sin\theta ve_1e_2$, where $\theta \in [0,\pi)$, $v \in \mathbb{R}^3$ is a unit vector. Hence, every non-zero element $\lambda \in \Gamma_2$ can be written as $\lambda = |\lambda|\,(\cos\theta + \sin\theta ve_1e_2)$, where $\theta \in [0,\pi)$, $v \in \mathbb{R}^3$ is a unit vector.

We denote the line generated by a vector $v \in \mathbb{R}^3$ by $\langle v\rangle$. In particular, $\mathbb{R} = \langle 1\rangle$ and $\widehat{\mathbb{R}} = \langle 1\rangle \cup \{\infty\}$ in $\hat{\mathbb{R}}^3$.

Let f be a loxodromic isometry which fixes 0 and ∞. Then f is of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{*-1} \end{smallmatrix}\right)$ for a non-unit Clifford number $\lambda = |\lambda|\,(\cos\theta + \sin\theta ve_1e_2) \in \Gamma_2$, where $\theta \in [0,\pi)$, $v \in \mathbb{R}^3$ is a unit vector. On the boundary at infinity $\hat{\mathbb{R}}^3$, the line $\langle v\rangle$ passing through the two fixed points 0 and ∞ is kept invariant by the action of f. For $\theta \neq 0$, $\langle v \rangle$ is a unique invariant line among all the lines passing through the two fixed points 0 and ∞ of the action of f. If $\theta = 0$, f is also called *hyperbolic* and every line passing through the two fixed points 0 and ∞ is kept invariant by the action of f.

A parabolic isometry which fixes ∞, rotates around $\langle 1 \rangle$ by 2θ and sends 0 to 1 is of the form $\left(\begin{smallmatrix} \alpha & \alpha \\ 0 & \alpha \end{smallmatrix}\right)$, where $\alpha = \cos\theta + \sin\theta e_1 e_2 \in \Gamma_2$, $0 \leq \theta < \pi$. If $\theta = 0$, then it is strictly parabolic; otherwise it is screw parabolic.

Theorem 2.1 ([8])**.** (1) *For a unit element* $\alpha = \cos\theta + \sin\theta v e_1 e_2 \in \Gamma_2^{\mathrm{unit}}$ *satisfying* $v \in \mathbb{R}^3$ *and* $|v| = 1$, $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha' \end{smallmatrix}\right) \in \mathrm{SL}(2, \Gamma_2)$ *is a rotation around* $\langle v \rangle$ *by* 2θ *in* $\mathbb{R}^3$.

(2) *An isometry* $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{*-1} \end{pmatrix} \in \mathrm{SL}(2, \Gamma_2)$ *is loxodromic if and only if* $|\lambda| \neq 1$.

(3) *An isometry* $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda' \end{pmatrix} \in \mathrm{SL}(2, \Gamma_2)$ *with* $|\lambda| = 1$ *is*

strictly parabolic if $\lambda \in \mathbb{R}$,

screw parabolic if $\mu \notin \mathbb{R}^3$,

elliptic otherwise.

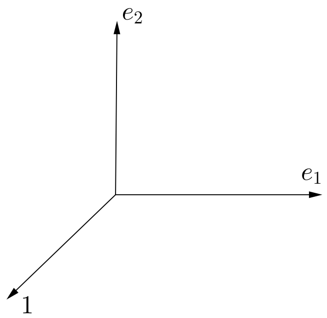

FIGURE 1. The boundary at infinity $\hat{\mathbb{R}}^3$ of hyperbolic 4-space

3. Rigidity

Lemma 3.1. *An elliptic isometry* $R \in \mathrm{M\ddot{o}b}(\hat{\mathbb{R}}^3)$ *which is a rotation around* $\langle 1 \rangle$ *commutes with an element of* $\mathrm{SL}(2, \mathbb{R})$.

Proof. A rotation R around $\langle 1 \rangle$ can be written as $R = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)$, where $\alpha = \cos\theta + \sin\theta e_1 e_2 \in \Gamma_2^{\mathrm{unit}}$, $\theta \in (0, \pi)$. For an element $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$, $RA = \left(\begin{smallmatrix} a\alpha & b\alpha \\ c\alpha & d\alpha \end{smallmatrix}\right) = AR$. $\square$

Lemma 3.2. *Let $f \in \text{Möb}(\hat{\mathbb{R}}^3)$ be a loxodromic isometry which keeps $\hat{\mathbb{R}}$ invariant. Then f can be written as $\phi^{-1}RF\phi$, where $\phi \in \text{SL}(2,\mathbb{R})$, R is a rotation around $\langle 1\rangle$ and $F \in \text{SL}(2,\mathbb{R})$ is hyperbolic.*

Proof. Since f keeps the line $\hat{\mathbb{R}}$ invariant, the two fixed points of f, called a and b, are also contained in $\hat{\mathbb{R}}$. Let $\phi \in \text{SL}(2,\mathbb{R})$ be an isometry which sends a and b to 0 and ∞, respectively. Then $\phi f \phi^{-1}$ fixes 0 and ∞ and hence it can be written as $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda^{*-1}\end{smallmatrix}\right)$ for a non-unit Clifford number $\lambda = r(\cos\theta + \sin\theta v e_1 e_2) \in \Gamma_2$, where $r > 0$, $\theta \in [0,\pi)$, $v \in \mathbb{R}^3$ is a unit vector. Since $\phi f \phi^{-1}$ also keeps $\mathbb{R}$ invariant, it follows that $\theta = 0$ or $v = \pm 1$. That is

$$\phi f \phi^{-1} = \begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\begin{pmatrix} r & 0\\ 0 & \frac{1}{r}\end{pmatrix},$$

where $\alpha = \cos\theta + \sin\theta e_1 e_2$. Thus, we proved the lemma. □

For $g \geq 1$, a *punctured-surface group* $G_{g,1}$ of hyperbolic 4-space is a discrete free group generated by $2g$ loxodromic isometries with the condition that the product of commutators of generators is a parabolic isometry:

$$G_{g,1} = \langle f_1, h_1, \ldots, f_g, h_g \text{ loxodromic} \mid \Pi_{1\leq i\leq g}[f_i,h_i] \text{ parabolic}\rangle < \text{Isom}(\mathbb{H}^4).$$

We may think of $G_{g,1}$ as the image of a discrete type-preserving representation $\rho : \pi_1(S_{g,1}) \to \text{Isom}(\mathbb{H}^4)$, where $S_{g,1}$ is a punctured hyperbolic surface of genus g. We call such a group a *punctured-torus group*.

First, we prove a rigidity for a punctured-torus group $G_{1,1}$.

Theorem 3.3. *If a punctured-torus group $G_{1,1}$ keeps a totally geodesic plane P invariant, then the commutator is a strictly parabolic isometry.*

Proof. Without loss of generality, we may assume that the boundary at infinity of P is $\partial P = \widehat{\mathbb{R}}$ and f fixes 0 and ∞. Since f and h keep $\hat{\mathbb{R}}$ invariant, applying Lemma 3.2, it follows that $f = R_1 F$ and $h = \phi^{-1}R_2 H\phi$, where R_1, R_2 are rotations around $\mathbb{R}$ and $\phi, F, H \in \text{SL}(2,\mathbb{R})$.

$$\begin{aligned}[f,h] &= [R_1F, \phi^{-1}R_2H\phi]\\ &= R_1F\Big(\phi^{-1}R_2H\phi\Big)\Big(R_1F\Big)^{-1}\Big(\phi^{-1}R_2H\phi\Big)^{-1}\\ &= [F, \phi^{-1}H\phi].\end{aligned}$$

In the above computation, we use Lemma 3.1. Since F and $\phi^{-1}H\phi$ are in $\text{SL}(2,\mathbb{R})$, the commutator also belongs to $\text{SL}(2,\mathbb{R})$ and hence $[f,h]$ is a strictly parabolic isometry. □

Now, we generalize the rigidity to a punctured-surface group $G_{g,1}$ with $g \geq 2$.

Theorem 3.4. *Let $G_{g,1}$ be a punctured-surface group with genus $g \geq 2$. If $G_{g,1}$ keeps a totally geodesic plane P invariant, then the product of commutators $\Pi_{1\leq i\leq g}[f_i,h_i]$ is a strictly parabolic isometry.*

Proof. Without loss of generality, we may assume that the boundary at infinity of P is $\partial P = \widehat{\mathbb{R}}$. Let $i = 1, \ldots, g$. Since loxodromic f_i and h_i keep $\widehat{\mathbb{R}}$ invariant on the boundary at infinity $\hat{\mathbb{R}}^3$, applying Lemma 3.2, f_i and h_i can be written as $f_i = \phi_i^{-1} R_i F_i \phi_i$ and $h_i = \psi_i^{-1} K_i H_i \psi_i$, where $\phi_i, \psi_i \in \mathrm{SL}(2, \mathbb{R})$, R_i and K_i are rotations around $\mathbb{R}$, F_i and $H_i \in \mathrm{SL}(2, \mathbb{R})$.

$$\begin{aligned}[f_i, h_i] &= \phi_i^{-1} R_i F_i \phi_i \big(\psi_i^{-1} K_i H_i \psi_i\big)\big(\phi_i^{-1} R_i F_i \phi_i\big)^{-1}\big(\psi_i^{-1} K_i H_i \psi_i\big)^{-1} \\ &= [\phi_i^{-1} F_i \phi_i, \psi_i^{-1} H_i \psi_i] \in \mathrm{SL}(2, \mathbb{R}).\end{aligned}$$

In the above computation, we use Lemma 3.1. Thus, $\Pi_{1 \le i \le g}[f_i, h_i]$ can only be strictly parabolic. □

Due to the rigidity of Theorem 3.3, we see that for a Fuchsian punctured-torus group (that is a punctured-torus group of isometries acting on hyperbolic 2-space), adding a rotational action around the invariant plane does not deform the strictly parabolic commutator into a screw parabolic commutator. Then we might ask if we can apply the idea to a non-Fuchsian punctured-torus group. In below, we carry out the idea in a way, but we are only able to have a strictly parabolic commutator. However, we have a 2-dimensional parameter family of genuine punctured-torus groups acting on $\mathbb{H}^4$ as a result. Before we start to construct the 2-dimensional parameter family, we note that if a group $G < \mathrm{Isom}(\mathbb{H}^4)$ keeps a 3-dimensional totally geodesic subspace invariant, then it cannot contain a screw parabolic isometry.

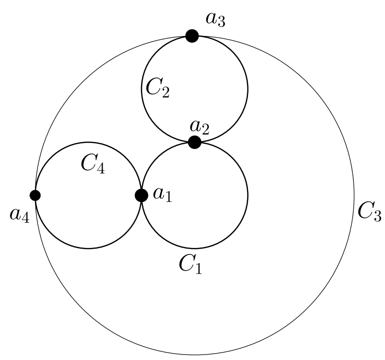

FIGURE 2. A fundamental domain of a punctured-torus group

Now we will construct a 2-dimensional parameter family of genuine punctured-torus groups acting on $\mathbb{H}^4$. We start with a Kleinian punctured-torus group $G = \langle F, H \rangle < \mathrm{Isom}(\mathbb{H}^3)$ whose fundamental domain is D on $\hat{\mathbb{R}}^2$ as follows (see Figure 2).

- D is a domain bounded by four circles C_1, C_2, C_3 and C_4.

- C_1 and C_3 are concentric.
- C_2 is tangent to C_1 and C_3 at a_2 and a_3, respectively.
- C_4 is tangent to C_1 and C_3 at a_1 and a_4, respectively.
- F is a hyperbolic isometry which maps C_1 to C_3, $a_1 \mapsto a_4$ and $a_2 \mapsto a_3$.
- H is a hyperbolic isometry which maps C_2 to C_4, $a_2 \mapsto a_1$ and $a_3 \mapsto a_4$.

Note that the commutator $[F, H]$ is strictly parabolic with a fixed point a_4. Let L_1 be a common orthogonal line of C_1 and C_3, which passes a_1, a_4, and L_2 a common orthogonal circle of C_2 and C_4, which passes through a_3, a_4 (Figure 3). We will use L_1 and L_2 to deform the group G.

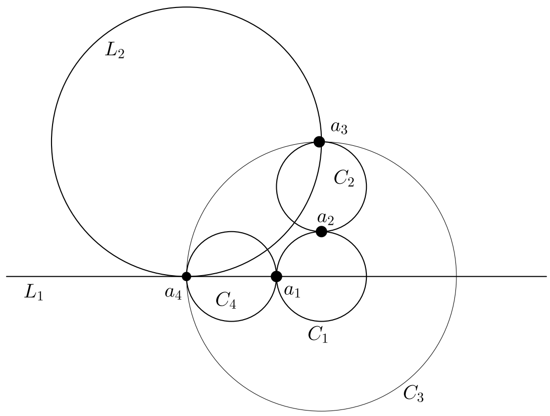

FIGURE 3. Deforming a punctured-torus group

Now, we embed $\mathbb{H}^3$ to $\mathbb{H}^4$ as $P = \{x_0 + x_1e_1 + x_3e_3 \in \mathbb{H}^4 \mid x_0, x_1, x_3 \in \mathbb{R}, x_3 > 0\}$ so that $\partial P = \langle 1, e_1\rangle \cup \{\infty\} \subseteq \hat{\mathbb{R}}^3$. Without loss of generality, we may assume $a_4 = 0$, $L_1 = \mathbb{R}$ and $L_2 \subseteq \langle 1, e_1\rangle$ is a circle whose center is $\frac{1}{2}e_1$ and radius $\frac{1}{2}$.

Let $S_i \subseteq \mathbb{R}^3$ $(i = 1, 2, 3, 4)$ be a sphere which has C_i as a great circle and $D' \subseteq \mathbb{R}^3$ the domain bounded by S_1, S_2, S_3 and S_4. Then D' is a fundamental domain for the action of G on $\partial\mathbb{H}^4$. The corresponding side pairing is $F(S_1) = S_3$ and $H(S_2) = S_4$.

Let R_i $(i = 1, 2)$ be a non-trivial rotation around L_i by θ_i:

$$R_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}, \; R_2 = \begin{pmatrix} \alpha_2 & 0 \\ 2\sin\theta_2 e_2 & \alpha_2 \end{pmatrix} \in \mathrm{SL}(2, \Gamma_2),$$

where $\alpha_i = \cos\theta_i + \sin\theta_i e_1e_2 \in \Gamma_2^{\mathrm{unit}}$ and $\theta_i \in (0, \frac{\pi}{2})$.

We take $F' = R_2F$ and $H' = R_1H$. Then F' and H' are loxodromic and still map S_1 and S_2 to S_3 and S_4, respectively. Using Lemma 3.1, R_1 commutes with F. Since the hyperbolic isometry H keeps L_2 invariant, R_2 commutes with H. Since the commutator $[F, H]$ is strictly parabolic fixing 0, it is of the form $\left(\begin{smallmatrix} 1 & 0 \\ v & 1 \end{smallmatrix}\right)$ for a non-zero vector $v \in \mathbb{R}^3$. Thus,

$$\begin{aligned} [F', H'] &= (R_2 F)(R_1 H)(F^{-1} R_2^{-1})(H^{-1} R_1^{-1}) \\ &= R_2 R_1 [F, H] R_2^{-1} R_1^{-1} \\ &= \begin{pmatrix} \alpha & 0 \\ \square & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} \alpha^* & 0 \\ \square & \alpha^* \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \square & 1 \end{pmatrix}, \end{aligned} \tag{1}$$

where $\alpha = \cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2) e_1 e_2$. Therefore, $[F', H']$ is a strictly parabolic isometry and $G' = \langle F', H' \rangle$ is a punctured-torus group which is not Kleinian, nor Fuchsian. Since the generators F' and H' have rotation-parameters θ_1 and θ_2, respectively, what we have here is a 2-dimensional parameter family of genuine punctured-torus groups acting on hyperbolic 4-space.

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