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GOLDIE EXTENDING PROPERTY ON THE CLASS OF z-CLOSED SUBMODULES

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Abstract. In this article, we define a module M to be G^z-extending if and only if for each z-closed submodule X of M there exists a direct summand D of M such that $X \cap D$ is essential in both X and D. We investigate structural properties of G^z-extending modules and locate the implications between the other extending properties. We deal with decomposition theory as well as ring and module extensions for G^z-extending modules. We obtain that if a ring is right G^z-extending, then so is its essential overring. Also it is shown that the G^z-extending property is inherited by its rational hull. Furthermore it is provided some applications including matrix rings over a right G^z-extending ring.

1. Introduction

Throughout this paper, all rings are associative with unity, R denotes such a ring, and all modules are unital right R-modules. In the spirit of [1], for a module M, think of the following relations on the set of submodules of M: (i) $X\alpha Y$ if and only if there exists $A \leq M$ such that $X \leq_e A$ and $Y \leq_e A$; (ii) $X\beta Y$ if and only if $X \cap Y \leq_e X$ and $X \cap Y \leq_e Y$. Recall that β is an equivalence relation and is equivalent to a relation defined in Goldie [9] for right ideals of a ring. It is easy to see that a module M is *extending* (or CS, or C_1) if and only if for each $X \leq M$, there exists a direct summand D of M such that $X\alpha D$ (see [1, 7, 23]). Further a module M is called *G-extending* (i.e., Goldie extending) if for each $X \leq M$, there exists a direct summand D of M such that $X\beta D$ or equivalently, for each complement $C \leq M$, there exists a direct summand D of M such that $C\beta D$ (see [1]). Clearly, every extending module is G-extending. Latter two concepts appear to be too similar to expect much difference in their application to the structure theory of rings and modules. However, there are many surprising differences as indicated in [1, 2]. Another useful generalization of CS-modules is CLS-modules [22].

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Following [22], a module M is called CLS if every z-closed submodule of M is a direct summand of M. Incidentally, a submodule N of M is called *z-closed* provided that M/N is nonsingular, i.e., $Z(M/N) = 0$. It can be seen easily that z-closed submodules are complement. Moreover, notice that z-closed submodules are the same as the φ-closed submodules in [10]. Furthermore complement submodules and z-closed submodules are coincide in a nonsingular module [22, Lemma 3] (see, also [6]).

In this paper, we study a module condition including the β relation on the set of all z-closed submodules of a module. We call a module M, *G^z-extending* if for every z-closed submodule N of M there exists a direct summand D of M such that $N\beta D$. A ring R is called *right G^z-extending* if R_R is a G^z-extending module, i.e., for every z-closed right ideal I of R there exists an idempotent element e of R such that $I\beta(eR)$. It is clear that the class of G^z-extending modules properly contains the class of G-extending modules. The notion of a G^z-extending module generalizes that of G-extending and also extending and CLS-modules by asking that only every z-closed submodule is β-related to a direct summand rather than every complement submodule.

In Section 2, we consider connections between the G^z-extending property, the C_1 condition, CLS and G-extending conditions. Moreover, we obtain basic properties as well as structural behavior of the class of G^z-extending modules. Section 3 is devoted to the decomposition theory of the G^z-extending modules. Since a direct sum of G^z-extending modules need not to be G^z-extending, we deal with when a direct sum of G^z-extending modules is also a G^z-extending. Furthermore, we investigate conditions which provide the inheritance of G^z-extending property by direct summands. In particular, we prove that if K is a projection invariant z-closed submodule of M, then $M = M_1 \oplus K$ for some $M_1 \leq M$ and both M_1, K are G^z-extending. In the last section, we focus on essential extensions of a G^z-extending module and also a G^z-extending ring. We show that if M is G^z-extending, then $\widetilde{E}(M)$ is G^z-extending where $\widetilde{E}(M)$ is the rational hull of M. Moreover, we prove that the right essential overring of a right G^z-extending ring enjoys with the G^z-extending property. Furthermore, we provide some applications including matrix rings over a right G^z-extending ring. Our results yield that being G^z-extending is not Morita invariant property.

Let R be a ring and M a right R-module. If $X \subseteq M$, then $X \leq M$, $X \leq_e M$, $Z(M)$, $Z_2(M)$, $E(M)$, $\widetilde{E}(M)$, $\mathrm{Soc}(M)$ and $\mathrm{End}(M_R)$ denote X is a submodule of M, X is an essential submodule of M, the singular submodule of M, the second singular (Goldie torsion) submodule of M, the injective hull of M, the rational hull of M, the socle of M, and the ring of endomorphisms of M, respectively. For R, $T_m(R)$ and $M_m(R)$ symbolize the ring of m-by-m upper triangular matrices over R and the ring of m-by-m full matrices over R, respectively.

Following [18], M is called a *UC-module* if every submodule of M has a unique closure in M. Recall that a submodule N of M is *projection invariant* if for every $e^2 = e \in \mathrm{End}(M_R)$, $e(N) \subseteq N$ (see [8]). A module M is said to satisfy the ${C_{11}}^z$-condition if every z-closed submodule of M has a complement which is a direct summand [11]. A ring is called *Abelian* if every idempotent is central. Other terminology and notation can be found in [4], [7], [10], [13], [23].

2. Preliminaries

Recall that the following relations on the set of submodules of M (see [1,2]).

(i) $X\alpha Y$ if and only if there exists $K \leq M$ such that $X \leq_e K$ and $Y \leq_e K$,

(ii) $X\beta Y$ if and only if $X \cap Y \leq_e X$ and $X \cap Y \leq_e Y$.

Observe that α is reflexive and symmetric, but it may not be transitive. However, β is an equivalence relation and is defined in Goldie (see [9]) for right ideals of a ring. Note that if $X, Y \leq M$ such that $X\alpha Y$, then $X\beta Y$.

Proposition 2.1. *M is CLS if and only if for each z-closed submodule X of M, there exists a direct summand D of M such that $X\alpha D$.*

Proof. The proof is routine. □

Motivated by Proposition 2.1, and Goldie's, Smith's and Akalan, Birkenmeier, Tercan's use of the β equivalence relation in Goldie [9], Smith [18], and Akalan, Birkenmeier, Tercan [1, 2], respectively, we make the following definition.

Definition. We say M is *G^z-extending* if for each z-closed submodule X of M, there exists a direct summand D of M such that $X\beta D$.

Note that M is G-extending (or Goldie extending) if and only if for each closed (or complement) submodule C of M, there is a direct summand D of M such that $C\beta D$. It is clear that the class of G^z-extending modules contains both of the classes of G-extending and CLS modules.

Next result gives equivalent conditions to G^z-extending property.

Proposition 2.2. *Let M be a module. The following conditions are equivalent.*

(i) *M is G^z-extending*;

(ii) *For each z-closed submodule Y of M, there exist $X \leq M$ and a direct summand D of M such that $X \leq_e Y$ and $X \leq_e D$*;

(iii) *For each z-closed submodule Y of M there exist a complement L of Y and a complement K of L such that $Y\beta K$ and every homomorphism $f : K \oplus L \to M$ extends to a homomorphism $g : M \to M$.*

Proof. (i)⇒(ii) Let Y be a z-closed in M. Hence there exist a direct summand D of M such that $Y\beta D$. Now take $X = Y \cap D$.

(ii)⇒(iii) From (ii), there exist D, D' such that $Y \cap D \leq_e Y$, $Y \cap D \leq_e D$ and $M = D \oplus D'$. So, take $D = K$ and $D' = L$.

(iii)⇒(i) Since Y is a complement in M from (iii) and [23, Lemma 3.97], K is a direct summand of M. Hence M is G^z-extending. □

Now we locate the G^z-extending condition with respect to several known generalizations of the extending property.

Proposition 2.3. *Let M be a module. Let us consider the following conditions*:

(i) *M is CS.*
(ii) *M is G-extending.*
(iii) *M is CLS.*
(iv) *M is G^z-extending.*
(v) *M is C_{11}^z.*

Then (i)⇒(ii)⇒(iii)⇒(iv)⇒(v). *In general, the reverse implications do not hold.*

Proof. (i)⇒(ii) and (iii)⇒(iv) are clear.

(iv)⇒(v) follows from [1, Proposition 1.6].

(ii)⇒(iii) Let N be a z-closed submodule of M. There exists a direct summand D of M such that $N \cap D \leq_e N$ and $N \cap D \leq_e D$. Since $M/N \cong M/(N \cap D)\,/\,N/(N \cap D)$, $N/(N \cap D)$ is z-closed in $M/(N \cap D)$. Hence $Z(Z(M/(N \cap D))) \leq Z(N/(N \cap D)) = N/(N \cap D) \leq Z(M/(N \cap D))$ which gives that $Z(M/(N \cap D)) = N/(N \cap D)$. Thus $D/(D \cap N) = Z(D/(D \cap N)) \leq Z(M/(N \cap D)) = N/(N \cap D)$. It follows that $D/(N \cap D) \leq N/(N \cap D)$, i.e., $N \cap D \leq D \leq N$. Since $N \cap D \leq_e N$, we have that $D = N$. Thus M is CLS.

(ii)$\not\Rightarrow$(i) Let M be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$ where p is any prime integer. Then $M_{\mathbb{Z}}$ is G-extending by [1, Corollary 3.3]. However, $M_{\mathbb{Z}}$ is not extending [19, Example 10].

(iii)$\not\Rightarrow$(ii) Let F be a field and V be a vector space over F with $\dim(V_F) = 2$. Let R be the trivial extension of F with V, i.e.,

$$R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} : f \in F,\, v \in V \right\}.$$

Then R_R is CLS because it does not have any proper z-closed submodule. Since R_R is indecomposable which is not uniform, R_R is not G-extending (see [1, Proposition 1.8]).

(iv)$\not\Rightarrow$(iii) Let K be a field of characteristic $p > 0$. Let $G = \langle x : x^p = 1 \rangle$, the cyclic group of order p. Let R denote the group algebra $K[G]$. Then R is (Quasi-)Frobenious algebra and hence self-injective Artinian ring (see [17, pages 79 and 405]). In particular, R_R is uniform and hence R is a CS-ring. Note that the augmentation ideal, $P = R(x-1)$ is the unique maximal ideal of R, and the only ideals of R are: $R > P > P^2 > \cdots > P^p = 0$. Now, let M be the R-module $R \oplus (R/P)$. By Corollary 3.4(ii), M is a G^z-extending R-module. On the other hand, $N = P \oplus \bar{0}$ is a submodule of M and $Z(M/N) = Z((R \oplus R/P)\,/\,(P \oplus \bar{0})) = Z(R/P) \oplus Z(R/P) = \bar{0} \oplus \bar{0}$. It follows that N is a z-closed submodule of M. If N were a direct summand of M, then we would have P is a direct summand of R, which gives a contradiction. Hence M_R is not a CLS-module.

(v)$\not\Rightarrow$(iv) Let R be the 2-by-2 upper triangular matrix ring over integers, i.e., $R = \left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right]$. Then R_R is a C_{11}-module which yields that R_R is a $C_{11}^{\,z}$-module. It is clear that $Z(R_R) = 0$ and R_R is not CS. Therefore R_R is not G^z-extending. □

Lemma 2.4. *Let M be a module. Assume* $\mathrm{End}(M_R)$ *is Abelian such that if X is a z-closed submodule of M, then $X = \sum_{i\in I} h_i(M)$, where each $h_i \in \mathrm{End}(M_R)$. Then M is $C_{11}^{\,z}$ if and only if M is CLS.*

Proof. Let M_R has $C_{11}^{\,z}$ and X be a z-closed submodule of M. Then $X = \sum_{i\in I} h_i(M)$, where each $h_i \in \mathrm{End}(M_R)$. By assumption, there exists $e^2 = e \in \mathrm{End}(M_R)$ such that eM is a complement of X in M. Let $0 \neq x \in X$. Then $x = ex + (1-e)x$. But $x = \sum_{i\in I} h_i(m_i)$, where $m_i \in M$. So $ex = e(\sum_{i\in I} h_i(m_i)) = \sum_{i\in I} h_i(em_i) \in X \cap eM = 0$. Hence $X \leq_e (1-e)M$. Since X is z-closed, $X = (1-e)M$. It follows that X is a direct summand of M. Thus M is CLS. The converse follows from Proposition 2.3. □

Proposition 2.5. *Let M be a module.*

(i) *Let M be a UC-module (e.g., M is nonsingular). Then M is G^z-extending if and only if M is CLS.*

(ii) *Let* $\mathrm{End}(M_R)$ *be Abelian such that if X is a z-closed submodule of M, then $X = \sum_{i\in I} h_i(M)$, where each $h_i \in \mathrm{End}(M_R)$. Then M is G^z-extending if and only if M is CLS.*

Proof. (i) Assume M is a G^z-extending module. Let X be any z-closed submodule of M. Then there exists a direct summand D of M such that $X\beta D$. Hence $X \cap D \leq_e X$ and $X \cap D \leq_e D$. Since X is a complement in M, by UC assumption $X = D$. So X is a direct summand of M. Hence M is a CLS-module. The converse follows from Proposition 2.3.

(ii) This part is a consequence of Proposition 2.3 and Lemma 2.4. □

Corollary 2.6. (i) *Let M be a nonsingular module. Then M is a G^z-extending module if and only if M is a G-extending module if and only if M is extending.*

(ii) *The following statements are equivalent for a nonsingular indecomposable module M:*

(1) *M is uniform.*
(2) *M is CS.*
(3) *M is G-extending.*
(4) *M is G^z-extending.*

Proof. (i) Immediate by [22, Lemma 3] (or [23, Lemma 5.58]).

(ii) The proof follows from part (i) and Proposition 2.3 [1, Proposition 1.8]. □

Proposition 2.7. *If M is G^z-extending and X is a projection invariant z-closed submodule of M, then M/X is CLS.*

Proof. Let Y/X be a z-closed submodule of M/X. Since

$$M/Y \cong (M/X)/(Y/X),$$

Y is a z-closed submodule of M. By hypothesis, there exists $d^2 = d \in \mathrm{End}(M_R)$ such that $Y\beta dM$. Since X is projection invariant in M, $X = (X \cap dM) \oplus (X \cap (1-d)M)$. Then $X \leq dM$ because $X \cap dM = X \cap (Y \cap dM) \leq_e X \cap Y = X$. Hence dM/X is a direct summand of M/X. Since $Y\beta dM$ and X is z-closed in M, by [10, Proposition 1.4], we obtain $(Y/X)\beta(dM/X)$. Therefore M/X is G^z-extending. By Proposition 2.5(i), M/X is a CLS-module. □

3. Decompositions

There are nonsingular modules $M = M_1 \oplus M_2$ in which M_1 and M_2 are CLS, but M is not CLS (e.g., let $R = \mathbb{Z}[x]$ and take $M = \mathbb{Z}[x] \oplus \mathbb{Z}[x]$). From Proposition 2.5(i), these modules M also show that the class of G^z-extending modules is not closed under direct sums. One of the main objective in this section is to determine conditions which make direct sum of G^z-extending modules is also G^z-extending. For this aim, the following kind of injectivity works well together with the G^z-extending property.

Let N, M be modules. N is said to M*-ejective* if, for each $K \leq M$ and each homomorphism $f : K \to N$, there exists a homomorphism $g : M \to N$ and a $X \leq_e K$ such that $g(x) = f(x)$ for all $x \in X$ (see [1] or [23]). Observe that if N is M-injective, then N is M-ejective. The following result generalizes [7, Lemma 7.5] and appears in [1, Theorem 2.7] (see, also [23, Theorem 2.59]). For the sake of completeness, we will mention the statement of the former result and refer to the aforementioned references for the detailed proof.

Lemma 3.1. *Let M_1 and M_2 be modules such that $M = M_1 \oplus M_2$. Then M_1 is M_2-ejective if and only if for every $K \leq M$ such that $K \cap M_1 = 0$, there exists $M_3 \leq M$ such that $M = M_1 \oplus M_3$ and $K \cap M_3 \leq_e K$.*

Proof. The proof follows by [1, Theorem 2.7] (or [23, Theorem 2.59]). □

Theorem 3.2. *Let $M = M_1 \oplus M_2$ be a direct sum of modules. Then*

(i) *If M_1 is M_2-ejective (or M_2 is M_1-ejective) and M_1 and M_2 are G^z-extending, then M is G^z-extending.*

(ii) *Assume M_1 is nonsingular. If M_1 is M_2-injective and M is G^z-extending, then M_2 is G^z-extending.*

Proof. (i) Let Y be a z-closed submodule of M. If $Y \cap M_1 = 0$, then by Lemma 3.1, there exists $M_3 \leq M$ such that $M = M_1 \oplus M_3$ and $Y \cap M_3 \leq_e Y$. Since $M_3 \cong M_2$, then M_3 is G^z-extending. Now $M_3/(Y \cap M_3) \cong (Y + M_3)/Y \leq M/Y$ yields that $Y \cap M_3$ is a z-closed submodule of M_3. By Proposition 2.2, there exists $X \leq_e Y \cap M_3$ and D, a direct summand of M_3, such that $X \leq_e D$. Since

M_3 is a direct summand of M, D is a direct summand of M. Now assume $Y \cap M_1 \neq 0$. Then there exists $K \leq Y$ such that $(Y \cap M_1) \oplus K \leq_e Y$. Since $K \cap M_1 = 0$, by Lemma 3.1, there exists $M_4 \leq M$ such that $M = M_1 \oplus M_4$ and $K \cap M_4 \leq_e K$. Since $M_4 \cong M_2$, M_4 is G^z-extending. It can be seen that $Y \cap M_1$ and $Y \cap M_4$ are z-closed submodules of M_1 and M_4, respectively. Thus there exist $X_1 \leq_e Y \cap M_1$, $X_2 \leq_e Y \cap M_4$, a direct summand D_1 of M_1 such that $X_1 \leq_e D_1$, and D_2 a direct summand of M_4 such that $X_2 \leq_e D_2$. Since $X_2 \leq_e Y \cap M_4$, $X_2 \cap K \leq_e K \cap Y \cap M_4 = K \cap M_4$. It follows that $X_2 \cap K \leq_e K$. So we have that $X_1 \oplus (X_2 \cap K) \leq_e (Y \cap M_1) \oplus K = Y$. Since $X_1 \oplus (X_2 \cap K) \leq X_1 \oplus X_2 \leq Y$, $X_1 \oplus X_2 \leq_e Y$. Observe that $X_1 \oplus X_2 \leq_e D_1 \oplus D_2$ and $D_1 \oplus D_2$ is a direct summand of $M_1 \oplus M_4 = M$. Therefore, M is G^z-extending.

(ii) Let Y be a z-closed submodule of M_2. Since $M/Y = (M_1 \oplus M_2)/(0 \oplus Y) \cong M_1 \oplus (M_2/Y)$, by assumption, Y is a z-closed submodule of M. By Proposition 2.2, there exist $X \leq_e Y$ and $D \leq M$ such that $X \leq_e D$ and D is a direct summand of M. Then $D \cap M_1 = 0$. By [7, Lemma 7.5], there exists $M' \leq M$ such that $M = M_1 \oplus M'$ and $D \leq M'$. There exists an isomorphism $\alpha : M' \to M_2$ such that $\alpha(x) = x$ for all $x \in X$ (i.e., $\alpha = \pi_2|_{M'}$). Let $D_2 = \alpha(D)$. Then $X \leq_e D_2$ and D_2 is a direct summand of M_2. Hence M_2 is G^z-extending. □

Corollary 3.3. *Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ be a finite direct sum. If M_i is M_j-ejective for all $j > i$ and each M_i is G^z-extending, then M is G^z-extending.*

Proof. An induction argument similar to that in [1, Corollary 3.2] and Theorem 3.2 yields the result. □

Corollary 3.4. *Let $M = M_1 \oplus M_2$. Then*

(i) *If M_1 is injective and M_2 is G^z-extending, then M is G^z-extending.*
(ii) *If M_1 is G^z-extending and M_2 is semisimple, then M is G^z-extending.*

Proof. Since in parts (i) and (ii) M_1 is M_2-injective, the proof follows from Theorem 3.2. □

Example 3.5. (i) Let M_1 and M_2 be Abelian groups (i.e., $\mathbb{Z}$-modules) with M_1 divisible and $M_2 = \mathbb{Z}/\mathbb{Z}p^n$, where p is a prime and n is a positive integer. Corollary 3.3 yields that $M = M_1 \oplus M_2$ is G^z-extending. Recall that M is not extending when $M_1 = \mathbb{Q}$ (see [19]).

(ii) Let M_1 be a G^z-extending module with a finite composition series $0 = X_0 \leq X_1 \leq \cdots \leq X_m = M_1$. Let $M_2 = X_m/X_{m-1} \oplus \cdots \oplus X_1/X_0$. Then $M = M_1 \oplus M_2$ is G^z-extending by Corollary 3.4(ii). However, M is not extending, in general (see [7, Corollary 7.4]).

Proposition 3.6. *Let M be a G^z-extending module N a z-closed submodule of M.*

(i) *If for each* $e^2 = e \in \mathrm{End}(M_R)$*, there exists* $f^2 = f \in \mathrm{End}(N_R)$ *such that* $N \cap eM \leq_e fN$*, then* N *is* G^z*-extending.*

(ii) *If for each* $e^2 = e \in \mathrm{End}(M_R)$*, there exists* $f^2 = f \in \mathrm{End}(M_R)$ *such that* $eM\beta fM$ *and* $fN \subseteq N$*, then* N *is* G^z*-extending. In particular, all projection invariant* z*-closed submodules of* M *are* G^z*-extending.*

Proof. (i) Let Y be a z-closed submodule of N. Hence Y is a z-closed submodule of M. By Proposition 2.2, there exist $X \leq_e Y$ and $e^2 = e \in \mathrm{End}(M_R)$ such that $X \leq_e eM$. Then $X \leq_e eM \cap N \leq_e fN$ for some $f^2 = f \in \mathrm{End}(N_R)$. Thus N is G^z-extending.

(ii) Let Y be a z-closed submodule of N. Then Y is a z-closed submodule of M. Then there exists $e^2 = e \in \mathrm{End}(M_R)$ such that $Y\beta eM$. Hence $Y\beta fM$. Since $fN \subseteq N$, N is G^z-extending. □

Theorem 3.7. *Let* K *be a projection invariant* z*-closed submodule of* M*. If* M *is* G^z*-extending, then there exists* $M_1 \leq M$ *such that* $M = M_1 \oplus K$ *and* M_1 *and* K *are* G^z*-extending.*

Proof. There exists $e^2 = e \in \mathrm{End}(M_R)$ such that $K\beta eM$. But $K = eK \oplus (1-e)K$, $eK = K \cap eM$, and $(1-e)K = K \cap (1-e)M$ because K is projection invariant. Since $K\beta eM$, then $eK \leq_e eM$ and $eK \leq_e K$. Hence $K \cap (1-e)M = 0$. So $K = eK \leq_e eM$. Since K is z-closed in M, $eM/K = Z(eM/K) \leq Z(M/K) = 0$. It follows that $eM = K$. Let $M_1 = (1-e)M$. Therefore $M = M_1 \oplus K$. Observe that, by Proposition 3.6(ii), K is G^z-extending.

Now, let us show that M_1 is G^z-extending. Since $M/K \cong M_1$, M_1 is a CLS-module, by Proposition 2.7. Then Proposition 2.3 yields that M_1 is G^z-extending. □

As a direct consequence of Theorem 3.7, we have the following corollary.

Corollary 3.8. *If* M *is* G^z*-extending and* $M = \underset{i\in I}{\oplus} M_i$*, where each* M_i *is projection invariant* z*-closed submodule of* M*, then* M_i *is* G^z*-extending.*

Recall that the decomposition $M = A \oplus B$ is said to be *exchangeable* if for any direct summand X, there exist $A' \leq A$ and $B' \leq B$ such that $M = X \oplus A' \oplus B'$ (see [14, Definition 4]). It is known that if M is G-extending and the decomposition $M = A \oplus B$ is exchangeable, then A is also G-extending (see [12, Lemma 2.3]). Next we show that a similar result to the aforementioned fact holds for G^z-extending modules.

Proposition 3.9. *If* M *is* G^z*-extending and the decomposition* $M = M_1 \oplus M_2$ *is exchangeable, then* M_1 *is also* G^z*-extending. In particular, if* M *is* G^z*-extending with the finite internal exchange property, then so is any direct summand of* M*.*

Proof. Let X be z-closed in M_1. By $M/(X \oplus M_2) \cong M_1/X$, $X \oplus M_2$ is z-closed in M. Since $M = M_1 \oplus M_2$ is exchangeable and G^z-extending, there exist a decomposition $M = D \oplus M_1' \oplus M_2'$ and an essential submodule Y of D such that $Y \leq_e X \oplus M_2$ and $M_i' \leq M_i$ $(i = 1, 2)$. Since $M_2' = 0$, $M_1 = M_1' \oplus (D \cap M_1)$. Now $Y \leq_e D$ and $Y \leq_e X \oplus M_2$ yield that $Y \cap M_1 \leq_e D \cap M_1$ and $Y \cap M_1 \leq_e (X \oplus M_2) \cap M_1 = X$. It follows that M_1 is G^z-extending. □

Observe that if M_1 and M_2 are projection invariant, then $M = M_1 \oplus M_2$ is exchangeable. The converse is not true in general. Using the latter result, Corollary 3.8 can also be obtained without any further proof.

Our next a few results provide applications of Theorems 3.2, 3.7 in terms of left exact preradicals. To this end, for left exact preradicals see [20]. First, we need to have the following basic lemma.

Lemma 3.10 ([1, Proposition 2.2])**.** *Let ρ be a left exact preradical and $M = M_1 \oplus M_2$, where $\rho(M) \leq_e M_2$. Then M_1 is M_2-ejective.*

Corollary 3.11. *If $M = M_1 \oplus M_2$, where M_1 and M_2 are G^z-extending and $\rho(M) \leq_e M_2$ for some left exact preradical ρ, then M is G^z-extending.*

Proof. By Lemma 3.10, M_1 is M_2-ejective. Now, by Theorem 3.2(i), M is G^z-extending. □

Corollary 3.12. *Let ρ be the radical for a stable hereditary torsion theory (e.g., $\rho = Z_2$). Then a module M is G^z-extending if and only if $M = M_1 \oplus M_2$, where M_1 and M_2 are G^z-extending, and $\rho(M) = M_2$.*

Proof. Since $\rho(M)$ is a projection invariant z-closed submodule, $M = M_1 \oplus \rho(M)$ for some $M_1 \leq M$. Then both $\rho(M)$ and M_2 are G^z-extending by Theorem 3.7. The converse follows from Corollary 3.11. □

Observe that M in Example 3.5(i) can be decomposed as $M = M_1 \oplus M_2$, but $M_1 = A \oplus B$ where A is torsion-free and B is torsion. Then $M = A \oplus B \oplus M_2$ and $Z_2(M) = B \oplus M_2$. Hence Example 3.5(i) provides a nonextending module M which illustrates Corollary 3.11.

Corollary 3.13. *Let M be a G^z-extending module. Then*

(i) *If D is a direct summand of M such that $Z(D) = 0$, then D is CLS.*
(ii) *Every direct summand of M is G^z-extending if and only if every direct summand of $Z_2(M)$ is G^z-extending.*

Proof. (i) There exists $C \leq M$ such that $M = C \oplus D$. Then $Z_2(M) = Z_2(C) = C \cap Z_2(M)$. By Corollary 3.12, $C = Z_2(M) \oplus B$ for some $B \leq C$, and $M = Z_2(M) \oplus B \oplus D$, where $B \oplus D$ is CLS by Proposition 2.5(i). [22, Lemma 7] yields that D is CLS.

(ii) ($\Rightarrow$) Since M is G^z-extending, Corollary 3.12 yields that $Z_2(M)$ is a G^z-extending direct summand of M. Since every direct summand of $Z_2(M)$ is a direct summand of M, by assumption every direct summand of $Z_2(M)$ is G^z-extending.

($\Leftarrow$) Let D be a direct summand of M. There exists $C \leq M$ such that $M = C \oplus D$. Then $Z_2(M) = Z_2(D) \oplus Z_2(C)$. Hence $D = Z_2(D) \oplus D_1$ for some $D_1 \leq D$. By part (i), D_1 is CLS. By hypothesis, $Z_2(D)$ is G^z-extending. From Corollary 3.12, D is G^z-extending. □

The following properties are well known and work well as companion conditions with the extending condition and its generalizations (see [23]). A module M is said to have: (1) the C_2 *property* if $X \leq M$ is isomorphic to a direct summand of M, then X is a direct summand of M; (2) the C_3 *property* if whenever M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M; (3) the *summand intersection property*, SIP, if whenever M_1 and M_2 are direct summands of M, then $M_1 \cap M_2$ is a direct summand of M. As in the G-extending case, it is unknown to the authors whether or not the G^z-extending property is inherited by direct summands. The following result and its corollary bring affirmative answers in some special cases for the aforementioned problem.

Theorem 3.14. *Let M be a G^z-extending module. If M has SIP or satisfies the C_3 condition, then any z-closed direct summand of M is G^z-extending.*

Proof. Let $M = N \oplus N'$ for some submodules N, N' of M where N is z-closed in M. Using Proposition 3.6(i), where N is taken to be z-closed in M and applying the SIP gives that N is a G^z-extending module.

Now assume that M satisfies the C_3 condition. Let $\pi : M \to N$ be the canonical projection. Let K be any z-closed submodule of N. Since N is z-closed in M, K is z-closed in M. By hypothesis, there exists a direct summand L of M such that $K \cap L \leq_e K$ and $K \cap L \leq_e L$. Since M satisfies C_3 condition, $N' \oplus L$ is a direct summand of M. It can be seen that $N' \oplus L = N' \oplus \pi(L)$ (see [23, Lemma 2.71]). Hence $\pi(L)$ is a direct summand of N. For any $0 \neq y \in \pi(L)$, $y = \pi(x)$ for some $0 \neq x \in L$. There exists an $r \in R$ such that $0 \neq xr \in K \cap L$. So $xr = k = x_1$, where $k \in K$ and $x_1 \in L$. Now $0 \neq xr = \pi(x)r = k = \pi(x_1) \in K \cap \pi(L)$. It follows that $K \cap \pi(L) \leq_e \pi(L)$. It is clear that $\pi(L) = N \cap (N' \oplus \pi(L)) = N \cap (N' \oplus L)$. Hence $K \cap \pi(L) = K \cap (N' \oplus L) \leq_e K$. Thus N is G^z-extending. □

Corollary 3.15. *Let M be a G^z-extending module. If M satisfies the C_2 condition, then any z-closed direct summand of M is G^z-extending.*

Proof. Since C_2 condition implies the C_3 condition, the proof follows by Theorem 3.14. □

4. Extensions

It is easy to see that if M is a G-extending or C_{11}-module with uniform dimension 2, then M is a direct sum of uniform submodules. However, if M is a G^z-extending module with uniform dimension 2, M need not to be a direct sum of uniform submodules (see, Proposition 2.3(iii)$\not\Rightarrow$(ii)). In this section, we investigate G^z-extending essential extensions of a module or ring. We show that if a ring is right G^z-extending, then so is its essential overring. Moreover, it is shown that the G^z-extending property is inherited by its rational hull. Furthermore, we provide some applications including matrix rings over a right G^z-extending ring.

Let us begin with the following useful result which provides relative injectivity on certain direct summands of a G-extending module (or nonsingular G^z-extending module, by Corollary 2.6). First note that there are examples which show that if $M_1 = \mathrm{Soc}(M_1)$ and $\mathrm{Soc}(M_2) = 0$, then $M = M_1 \oplus M_2$ need not to be G-extending, in general (see [23, page 185]).

Proposition 4.1. *Let R be any ring, M_1 a semisimple right R-module, and M_2 a right R-module with zero socle such that $M = M_1 \oplus M_2$ is a G-extending UC-module. Then M_1 is M_2-injective.*

Proof. Obviously, $M_1 = \mathrm{Soc}(M)$. Let N be any submodule of M_2, and let $\varphi : N \to M_1$ be a homomorphism. Let $L = \{x - \varphi(x) : x \in N\}$. Then L is a submodule of M and $L \cap M_1 = 0$. There exist submodules K, K' of M such that $M = K \oplus K'$, $K \cap L \leq_e L$ and $K \cap L \leq_e K$. It is clear that K is a closure of $K \cap L$ in M. By hypothesis, $L \leq K$. Since $K \cap L \cap M_1 = L \cap M_1 = 0$, $K \cap L \cap \mathrm{Soc}(M) = \mathrm{Soc}(L) = 0$. It follows that $\mathrm{Soc}(K) = K \cap M_1 = 0$. Hence $M_1 = \mathrm{Soc}(M) \subseteq K'$. Thus $K' = M_1 \oplus (K' \cap M_2)$ and $M = K \oplus M_1 \oplus (K' \cap M_2)$. Let $\pi : M \to M_1$ denote the canonical projection with kernel $K \oplus (K' \cap M_2)$. Let θ be the restriction of π to M_2. Then $\theta : M_2 \to M_1$. Let x be any element of N. Since $x = (x - \varphi(x)) + \varphi(x)$, $\theta(x) = \varphi(x)$. It follows that M_1 is M_2-injective. $\square$

Corollary 4.2. (i) *Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$, where each M_i is uniform. If $E(M_i) \ncong E(M_j)$ for all $i \neq j$, then M is G^z-extending.*

(ii) *Let S be a simple module and M_1, $M_2 \leq E(S)$. If there exists a homomorphism $h : M_2 \to S$ such that $h(S) \neq 0$, then $M = M_1 \oplus M_2$ is G^z-extending.*

Proof. (i) From [1, Corollary 4.11], M is G-extending. Thus Proposition 2.3 gives that M is CLS and hence it is G^z-extending.

(ii) By [1, Corollary 4.14], M_1 is M_2-ejective and so it is G-extending. Now, by Proposition 2.3, M is G^z-extending. $\square$

Example 4.3. (i) Let M be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$ and let T be the polynomial ring $\mathbb{Z}[x]$. Then $M_{\mathbb{Z}}$ is included in Corollary 4.2(i). On the other hand, it is well known that T^2 is not a G^z-extending T-module [23] (or [7]). Hence we obtain that the condition $E(M_i) \ncong E(M_j)$ for all $i \neq j$, is not superfluous in Corollary 4.2(i).

(ii) Let K be a field and $R = K[x, y]$, the commutative local Frobenious K-algebra (see [1, Example 4.15]) defined by the relations $xy = x^2 - y^2 = 0$. Then R_R is a uniform injective module with simple submodule Kx^2. Let $M_2 = xR = \{k_1x + k_2x^2 : k_i \in K\}$, and let h be the R-homomorphism,

$h : xR \to Kx^2$, defined by $h(k_1x + k_2x^2) = k_2x^2$. Then $h(Kx^2) \neq 0$. Thus, by Corollary 4.2(ii), $M = M_1 \oplus xR$ is G^z-extending for any $M_1 \leq R_R$.

In the shed light of second part of Example 4.3(i), we have the following fact on G^z-extending modules with Goldie dimension 2.

Lemma 4.4. *If M_R is a G^z-extending module with uniform dimension* 2*, then every direct summand of M is G^z-extending.*

Proof. Let K be any direct summand of M. If $K = M$, then by hypothesis K is G^z-extending. Assume that $K \neq M$. Then K is uniform. So K is G^z-extending. □

It is natural to think of whether G^z-extending property is closed under essential extensions of a module or not. However, the next example provides a negative answer, i.e., there exists a G^z-extending module which has an essential extension that does not enjoy the G^z-extending property.

Example 4.5. Let F be any field and $R = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$. It is straightforward to see that $\mathrm{Soc}(R_R) \leq_e R_R$. Obviously $\mathrm{Soc}(R)$ is a G^z-extending right R-module. However, it is well known that R_R is not G^z-extending (see [21, Theorem 3.4]).

In contrast to essential extensions of a module which satisfies G^z-extending condition, we have the following useful result. First recall that S is a *right essential overring* of a ring R if S is an overring of R such that R_R is essential in S_R (see, for example [4, 23]).

Theorem 4.6. *Let S be a right essential overring of R (i.e., $R_R \leq_e S_R$). If R_R is G^z-extending, then S_R and S_S are G^z-extending.*

Proof. Let Y_R be any z-closed submodule of S_R. It is easy to see that $X = Y \cap R$ is a z-closed submodule of R_R. By Proposition 2.2, there exist $K_R \leq R_R$ and $e^2 = e \in R$ such that $K_R \leq_e X_R$ and $K_R \leq_e eR_R$. Observe that $K_R \leq_e Y_R$. Now, let us show that $K_R \leq_e eS_R$. Let $0 \neq es \in eS$. There exists $r_1 \in R$ such that $0 \neq esr_1 \in R$. Hence $0 \neq esr_1 \in eR$, so there exists $r_2 \in R$ such that $0 \neq esr_1r_2 \in K$. Thus $K_R \leq_e eS_R$. By Proposition 2.2, S_R is G^z-extending. A proof similar to the above shows that $KS_S \leq_e Y_S$ and $KS_S \leq_e eS_S$. Therefore S_S is G^z-extending. □

Corollary 4.7. *Let $T = T_m(R)$ and $M = M_m(R)$. If T_T is G^z-extending, then M_T and M_M are G^z-extending.*

Proof. This result is a consequence of Theorem 4.6, and the fact that M_T is a rational extension of T_T. □

In [15, 16], Osofsky raised the following question: if $E(R)$ has a ring multiplication which extends its right R-module scalar multiplication, must $E(R_R)$ be right self-injective? In [5], examples were constructed giving a negative answer to this question. The next result shows that for such a ring R, $E(R)$ must be at least right G^z-extending.

Corollary 4.8. *Let R_R be G^z-extending. If $E(R)$ has a ring multiplication which extends its right R-module scalar multiplication, then $E(R)_{E(R)}$ is G^z-extending.*

Next theorem shows that if a module satisfies G^z-extending condition, then so its rational hull. Recall that for a module M, the rational hull of M is defined as the following submodule of $E(M)$:

$$\widetilde{E}(M) = \{x \in E(M) : h(M) = 0 \Rightarrow h(x) = 0 \text{ for all } h \in \mathrm{End}(E(M))\}.$$

Notice that $E(M) \cong \widetilde{E}(M)$ whenever M is nonsingular (see [13]).

Theorem 4.9. *If M is G^z-extending, then $\widetilde{E}(M)$ is G^z-extending.*

Proof. Let K be a z-closed submodule of $\widetilde{E}(M)$. Then $X = K \cap M$ is a z-closed submodule of M. By hypothesis, there exist $Y \leq M_R$ and $e^2 = e \in \mathrm{End}(M_R)$ such that $Y \leq_e X$ and $Y \leq_e eM$. Observe that $Y \leq_e K$. By [13], there exists $f \in \mathrm{End}(\widetilde{E}(M))$ such that $f|_M = e$. Since $E(M)$ is injective, there exists $\bar{e} \in \mathrm{End}(E(M))$ such that $\bar{e}|_{\widetilde{E}(M)} = f$. Let $m \in M$. Then $[\bar{e} - \bar{e}^2](m) = [e - e^2](m) = 0$. From the definition of $\widetilde{E}(M)$, $[\bar{e} - \bar{e}^2](y) = 0$ for all $y \in \widetilde{E}(M)$. Hence $f = f^2$. By $eM \leq_e f(\widetilde{E}(M))$, $Y \leq_e f(\widetilde{E}(M))$. It follows that $\widetilde{E}(M)$ is G^z-extending. □

Proposition 4.10. *Let M be a right R-module and let N be a submodule of M, where $R = ReR$ for some idempotent e in R and $S = eRe$. Then*

(i) *$N \leq_e M_R$ if and only if $Ne \leq_e (Me)_S$.*
(ii) *N is z-closed in M_R if and only if Ne is z-closed in $(Me)_S$.*

Proof. (i) It follows from [23, Proposition 2.77(i)].

(ii) Assume that $Z(Me/Ne)_S = \bar{0}$. Let $m + N \in Z(M/N)_R$. Let $r \in R$. Therefore, $(m + N)re = mre + N \in Z(M/N)_R$. There exists $F \leq_e R_R$ such that $(mre + N)F = \bar{0}$. Now, it is clear that $eR \cap F \leq_e eR$. By (i), $(eR \cap F)e \leq_e (eRe)_S = S_S$. But $mre + Ne \in Me/Ne$ and $(eR \cap F)e \leq Fe \leq F$. Thus, $(mre + Ne)(eR \cap F)e \subseteq (mre + N)(eR \cap F)e = \bar{0}$ yields that $(mre + Ne)(eR \cap F)e = \bar{0}$. Then $mre + Ne = \bar{0}$ because $(Me/Ne)_S$ is nonsingular. Hence $(m + Ne)Re = \bar{0}$. Therefore $(m + Ne)ReR = \bar{0}$, so that $(m + Ne)R = \bar{0}$, i.e., $m + Ne = \bar{0}$. It follows that $m \in Ne \subseteq N$, and hence $m \in N$. So $m + N = \bar{0}$. Now, suppose that $Z(M/N)_R = \bar{0}$. Let $me + Ne \in Z(Me/Ne)_S$. Then $(me + Ne)G = \bar{0}$ for some $G \leq_e S_S$. By (i), $GR \leq_e (eR)_R$. Thus $GR \oplus (1-e)R \leq_e R_R$. Since $(me+N)[GR \oplus (1-e)R] = \bar{0}$, $me + N \in Z(M/N)_R$. Hence $me + N = \bar{0}$ which implies that $me + Ne = \bar{0}$. So $Z(Me/Ne)_S = \bar{0}$. □

Theorem 4.11. *Let M be a right R-module, where $R = ReR$ for some idempotent e in R, and let $S = eRe$. Then*

(i) *The right R-module M is G^z-extending if and only if the right S-module Me is G^z-extending.*
(ii) *R_R is G^z-extending if and only if the right S-module Re is G^z-extending.*

Proof. The proof follows from Proposition 4.10 and [23, Proposition 2.77(iii)]. □

Corollary 4.12. *Let R be any ring. Then $M_m(R)$ is G^z-extending if and only if the free right R-module R^m is G^z-extending.*

Proof. It is clear that $M_m(R) = M_m(R)eM_m(R)$, where e is the matrix unit with 1 in the (1,1)th position and zero elsewhere. The result now follows from Theorem 4.11. □

Without any further proof Theorem 4.11 and Corollary 4.12 hold true whenever G^z-extending condition replaced with CLS in their statements.

Example 4.13. Let R be the polynomial ring $\mathbb{Z}[x]$. Then $M_2(R) = M_2(\mathbb{Z}[x]) \cong M_2(\mathbb{Z})[x]$. Note that $M_2(\mathbb{Z})$ is a right G^z-extending ring by Corollary 4.12. However, by Example 4.3 $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$ is not a G^z-extending $\mathbb{Z}[x]$-module. Now, Corollary 4.12 yields that $M_2(\mathbb{Z}[x])$ is not G^z-extending.

Observe that Example 4.13 makes it clear that over a right G^z-extending ring neither the ring of polynomials nor the full matrix ring need to be a right G^z-extending, in general. Thus, being G^z-extending is not a Morita invariant property.

Corollary 4.14. *If $T_m(R)$ is G^z-extending, then the free right R-module R^m is G^z-extending.*

Proof. It follows by [3, Corollary 1.8(ii)], Corollaries 4.7 and 4.12. □

It is not known so far whether direct summands of a G-extending module enjoy with the property (see [1,23]). Like the former case the authors desire to obtain whether the G^z-extending property is inherited by its direct summands or not? We think of it is legitimate to deal with the following open problem which is actually based on the aforementioned cases.

Open problem. In Theorem 3.2(ii) whether the assumption $Z(M_1) = 0$ is superfluous or not?

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