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VARIATIONS IN WRITHES OF VIRTUAL KNOTS UNDER A LOCAL MOVE

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Abstract. n-writhes denoted by $J_n(K)$ are virtual knot invariants for $n \neq 0$ and are closely associated with coefficients of some polynomial invariants of virtual knots. In this work, we investigate the variations of $J_n(K)$ under arc shift move and conclude that n-writhes $J_n(K)$ vary randomly in the sense that it may change by any random integer value under one arc shift move. Also, for each $n \neq 0$ we provide an infinite family of virtual knots which can be distinguished by n-writhes $J_n(K)$, whereas odd writhe $J(K)$ fails to do so.

1. Introduction

Virtual knot theory initiated by Kauffman [7] extends the study of knots embedded in 3-dimensional sphere $\mathbb{S}^3$ to the knots embedded in thickened surfaces $S_g \times [0,1]$, where S_g denotes a surface of genus $g > 0$. A diagram of a virtual knot is a 4-planar graph whose vertices are replaced by two types of crossings, classical and virtual as shown in Figure 1. A virtual crossing is indicated by placing a small circle around the vertex and do not have over/under information like classical crossing. Equivalence between two virtual knot diagrams is defined using generalized Reidemeister moves as listed in Figure 2 which are an extension of classical Reidemeister moves. For convenience the set of generalized Reidemeister moves is denoted by GR-moves throughout this paper. A local move which along with GR-moves can be used repeatedly to convert any virtual knot diagram D into trivial knot diagram is called an unknotting operation. Finding new local moves that act as unknotting operations is an important task in virtual knot theory. Local moves like virtualization of a classical crossing, CF-move [12] shown in Figure 3, and forbidden moves [6, 10] as shown in the Figure 4 are few known unknotting operations for virtual knots.

An object associated with virtual knot diagrams which remains unchanged under GR-moves is called a virtual knot invariant. Finding new invariants

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is one of the major goals in virtual knot theory. Numerical invariants like unknotting number, odd writhe, forbidden number or polynomial invariants like index polynomial [5], affine index polynomial [9], writhe polynomial [2], and algebraic invariants like virtual knot group and virtual knot quandle are few already known virtual knot invariants. Another numerical invariant called n-writhe which we will further investigate in the paper can be defined using Gauss diagram.

Definition 1.1. A Gauss diagram $G(D)$ corresponding to a virtual knot diagram D is an oriented circle with a base point where each classical crossing is marked two times with respect to overpass and underpass. Two markings corresponding to a crossing c are then joined by a signed arrow (chord) directed from overpass to underpass (see Figure 6). The sign of the crossing, denoted by $\mathrm{sgn}(c)$, attached to each arrow is equal to the local writhe of the corresponding crossing c defined as per the convention shown in Figure 5.

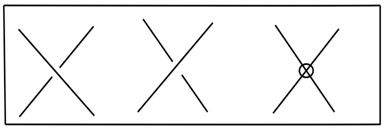


Figure 1. Classical and virtual crossings.

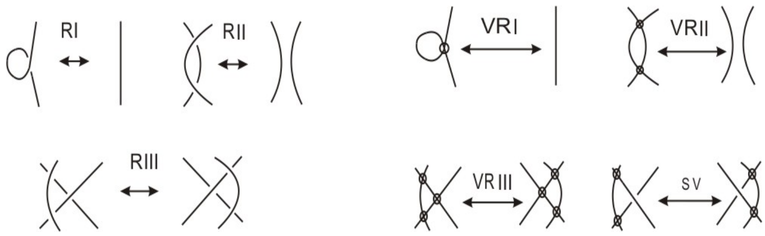
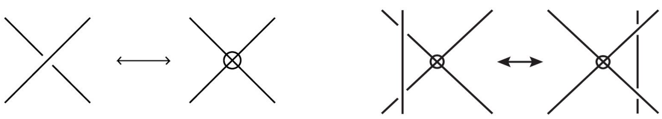


Figure 2. Generalized Reidemeister moves.



(a) Virtualization (b) CF-move

Figure 3. Virtualization and CF-move.

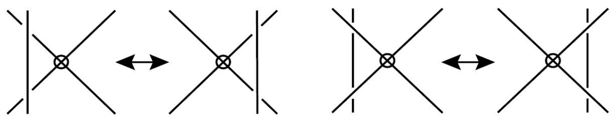


FIGURE 4. Forbidden moves.

$\mathrm{sgn}(c) = +1$ $\qquad$ $\mathrm{sgn}(c) = -1$

FIGURE 5. Crossing signs.

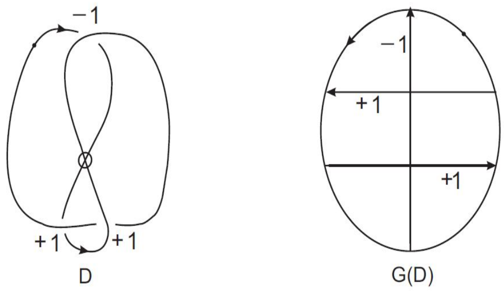


FIGURE 6. Gauss diagram corresponding to virtual knot diagram D.

In [2], Cheng and Gao assigned an integer value, called an *index value*, to each classical crossing c of a virtual knot diagram and denoted it by $\mathrm{Ind}(c)$. $\mathrm{Ind}(c)$ can be defined using Gauss diagram. For the arrow corresponding to crossing c, fix an arc on the circle in Gauss diagram starting at tail and ending at head of arrow c along anticlockwise orientation. Denote by H and T, respectively, the set of arrows whose head and tail lies on the fixed arc.

Definition 1.2. $\mathrm{Ind}(c)$ is defined as sum of signs of arrows lying in T minus the sum of signs of arrows lying in H.

For example, in Figure 7, the arc PQ along anticlockwise orientation on the circle from P to Q is the fixed arc corresponding to arrow $\overrightarrow{PQ}$. We have $H = \{c_1, c_2, c_4\}$ and $T = \{c_2, c_3, c_5\}$ therefore $\mathrm{Ind}(c) = \mathrm{sgn}(c_2) + \mathrm{sgn}(c_3) + \mathrm{sgn}(c_5) - (\mathrm{sgn}(c_1) + \mathrm{sgn}(c_2) + \mathrm{sgn}(c_4))$, i.e., $\mathrm{Ind}(c) = -2$.

In [8], Kauffman defined parity among crossings as odd/even and further used it to define a virtual knot invariant called odd writhe. Denoted by $J(D)$, the odd writhe of a virtual knot diagram D is defined as the sum of signs of all the odd crossings in D. Remark here that a crossing c is odd if and only if $\mathrm{Ind}(c)$

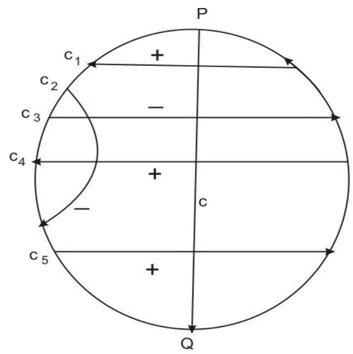


FIGURE 7. Computing Ind(c).

is an odd integer as was shown in [2]. Therefore, odd writhe $J(D)$ can also be reinterpreted as the sum of signs of all the crossings having index value an odd integer. In [14], Satoh and Taniguchi introduced the n-th writhe. For each $n \in \mathbb{Z} \setminus \{0\}$ the *n-th writhe* $J_n(D)$ of an oriented virtual knot diagram D is defined as $J_n(D) = \sum_{\text{Ind}(c)=n} \text{sgn}(c)$. It was shown that $J_n(D)$ for $n \neq 0$ is a virtual knot invariant and the odd writhe $J(D)$ is equal to the sum of all $J_n(D)$ for odd integers n. $J_n(D)$ are stronger virtual knot invariants than odd writhe $J(D)$ in the sense that any two virtual knots which can not be distinguished by $J_n(D)$ for all $n \neq 0$ are also non-distinguishable by $J(D)$. Further $J_n(D)$ distinguishes few virtual knots which are not distinguishable by $J(D)$. Such an infinite family of virtual knots are constructed in Corollary 2.6. Further importance of $J_n(D)$ lies in the fact that coefficients of few polynomial invariants like index polynomial [5], odd writhe polynomial [1] and affine index polynomial [9] can be expressed in terms of $J_n(D)$ as shown in [13,14]. A study of variations in $J_n(D)$ under a local move can give insight about the change in polynomial invariants under corresponding local move. The variations of $J_n(D)$ have been extensively studied under various local moves like Δ-move [14], forbidden moves [13] and virtualization of classical crossing [11]. A local move called *arc shift move* was defined by authors in an earlier work and the variation in odd writhe $J(K)$ under arc shift move was also studied. However, behavior of $J_n(K)$ under arc shift move is still not known. In this work we investigate the variations of $J_n(K)$ under arc shift move. In Section 2 we briefly review definition and results known for arc shift move and discuss main results of this paper. At the end in Section 3, we study the variation of affine index polynomial under arc shift move.

2. Arc shift move and variations of writhes

A local move called an arc shift move was defined for virtual knot diagrams in [3] and related Gordian complex of virtual knots was studied in [4]. An arc, say (a, b) is meant that rather going directly first to a and then b along arc (a, b), we travel first to b and then come back to a thus reversing the orientation on arc (a, b). This procedure is shown in Figure 8 for the case when both crossings of arc (a, b) are classical crossings. Some new crossings which may arise during executing an arc shift move are marked as virtual crossings.

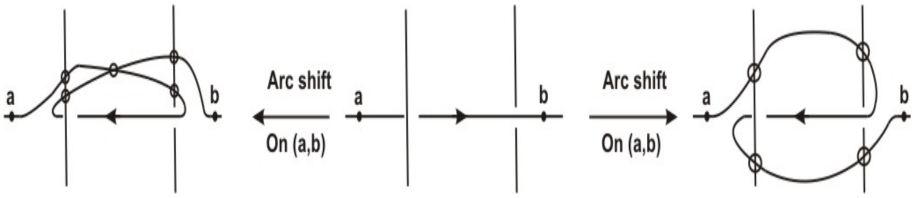


FIGURE 8. Arc shift move on arc (a, b).

It is possible to convert any virtual knot diagram D into trivial knot diagram using arc shift moves and GR-moves thus making it an unknotting operation for virtual knots.

Theorem 2.1 ([3])**.** *Every virtual knot diagram D can be transformed into trivial knot diagram using arc shift moves and generalized Reidemeister moves.*

Minimum number of arc shift moves required to convert a diagram of virtual knot K into trivial knot is defined as arc shift number of K denoted by $A(K)$. Further, behavior of odd writhe $J(K)$ was studied under arc shift moves.

Proposition 2.2 ([3])**.** *If D and D' are two virtual knot diagrams that differ by an arc shift move, then either $J(D') = J(D)$ or $J(D') = J(D) \pm 2$.*

Variation of n-writhe $J_n(K)$ is still not known under arc shift move. In this paper, we investigate the variations in $J_n(K)$ under arc shift moves and prove in Theorem 2.3, the main result that n-writhe can change by any random integer.

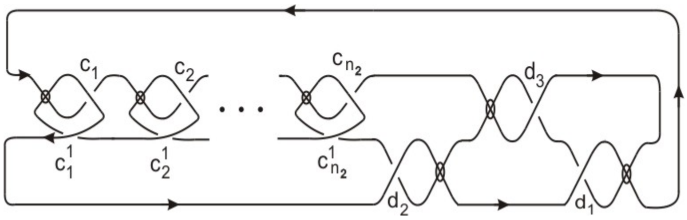
Theorem 2.3. *For any pair of integers $(n_1, n_2) \in \mathbb{Z}\setminus\{0\} \times \mathbb{Z}\setminus\{0\}$, there exists a virtual knot VK such $A(VK) = 1$ and $J_{n_1}(VK) = n_2$.*

Proof. The proof is based on construction. We construct virtual knots $VK_{n_1}^{n_2}$ for different cases of (n_1, n_2) where $(n_1, n_2) \in \mathbb{Z} \setminus \{0\} \times \mathbb{Z} \setminus \{0\}$.

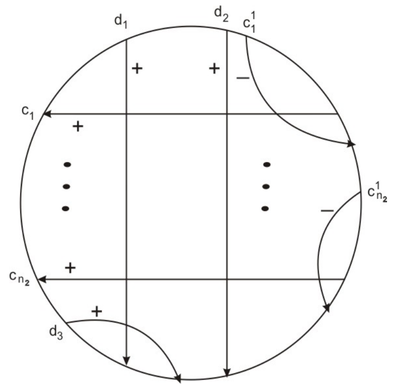
Case 1: $(1, n_2),\ n_2 > 0$.

Construct the virtual knot $VK_1^{n_2}$ as shown in Figure 9(a). We claim that this virtual knot $VK_1^{n_2}$ satisfies $A(VK_1^{n_2}) = 1$ and $J_1(VK_1^{n_2}) = n_2$.

Denoted by $G(VK_1^{n_2})$, the Gauss diagram of the virtual knot $VK_1^{n_2}$, is shown in Figure 9(b). Using Gauss diagram we compute $\mathrm{Ind}(c)$ for all the classical crossings in $VK_1^{n_2}$. We slightly abuse the notation here and use the symbol c to denote both the crossing c as well as the corresponding arrow in Gauss



(a) $VK_1^{n_2}$.



(b) $G(VK_1^{n_2})$.

FIGURE 9. Virtual knot $VK_1^{n_2}$ and Gauss diagram $G(VK_1^{n_2})$.

diagram. For the crossing c_1, we have $\mathrm{Ind}(c_1) = \mathrm{sgn}(d_1)+\mathrm{sgn}(d_2)+\mathrm{sgn}(c_1^1) = 1$. Similarly by computing $\mathrm{Ind}(c)$ for all other crossings, we have

$$\begin{aligned}
\mathrm{Ind}(c_i) &= 1 \quad \text{for } i = 1, 2, \ldots, n_2,\\
\mathrm{Ind}(c_i^1) &= -1 \quad \text{for } i = 1, 2, \ldots, n_2,\\
\mathrm{Ind}(d_1) &= -n_2 + 1,\\
\mathrm{Ind}(d_2) &= -n_2,\\
\mathrm{Ind}(d_3) &= -1.
\end{aligned} \tag{1}$$

As $n_2 > 0$, the only crossings with index value 1 are $c_1, c_2, \ldots, c_{n_2}$. Hence, $J_1(VK_1^{n_2}) = \sum_{\mathrm{Ind}(c)=1} \mathrm{sgn}(c) = \mathrm{sgn}(c_1) + \cdots + \mathrm{sgn}(c_{n_2}) = n_2$. We now show that the virtual knot $VK_1^{n_2}$ has arc shift number one by converting the diagram of $VK_1^{n_2}$ into trivial knot using one arc shift move and GR-moves. In the diagram of $VK_1^{n_2}$, apply one arc shift move at the arc (a, b) lying in the shaded region as shown in Figure 10. Resulting diagram of virtual knot L can be reduced into trivial knot by using GR-moves multiple times as can be seen in Figure 10.

Case 2: $(2, n_2),\ n_2 > 0.$

Let $VK_2^{n_2}$ be the virtual knot shown in Figure 11(a). We claim that virtual knot $VK_2^{n_2}$ satisfies the conditions $A(VK_2^{n_2}) = 1$ and $J_2(VK_2^{n_2}) = n_2$.

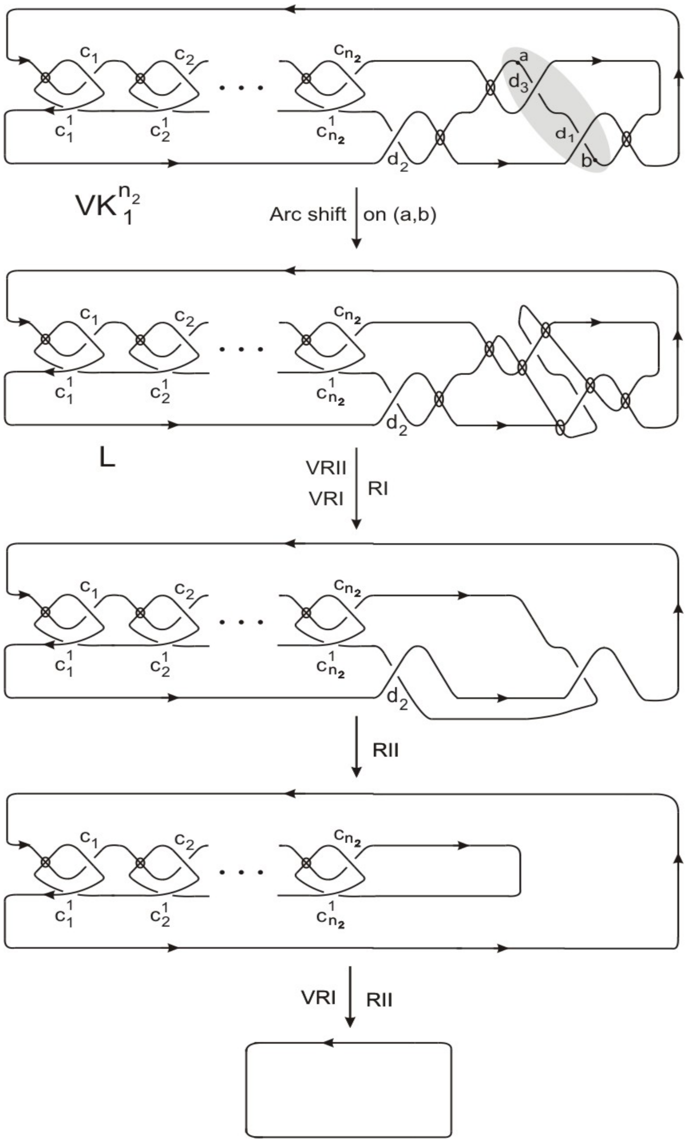
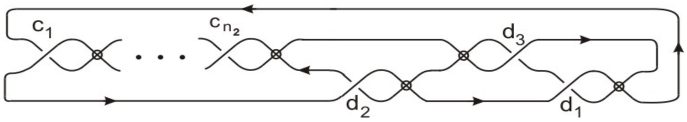
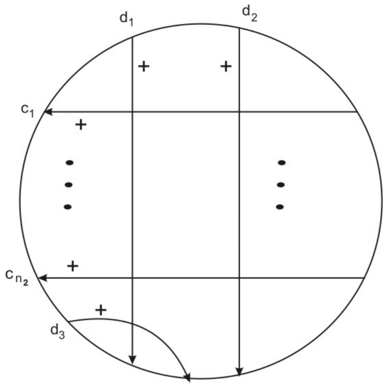


FIGURE 10. Converting $VK_1^{n_2}$ into trivial knot.

As in Case 1, we compute index values of all classical crossings using Gauss diagram $G(VK_2^{n_2})$ shown in Figure 11(b). For c_1 we have, $\mathrm{Ind}(c_1) = \mathrm{sgn}(d_1) +$



(a) $VK_2^{n_2}$.



(b) $G(VK_2^{n_2})$.

FIGURE 11. Virtual knot $VK_2^{n_2}$ and Gauss diagram $G(VK_2^{n_2})$.

$\mathrm{sgn}(d_2) = 2$. Similarly, by computations for rest of the crossings, we get

$$
\begin{aligned}
\mathrm{Ind}(c_i) &= 2 \quad \text{for } i = 1, 2, \ldots, n_2, \\
\mathrm{Ind}(d_1) &= -n_2 + 1, \\
\mathrm{Ind}(d_2) &= -n_2, \\
\mathrm{Ind}(d_3) &= -1.
\end{aligned} \tag{2}
$$

Since $n_2 > 0$, the only crossings with index values 2 are $c_1, c_2, \ldots, c_{n_2}$, hence, $J_2(VK_1^{n_2}) = \sum_{\mathrm{Ind}(c)=2} \mathrm{sgn}(c) = \mathrm{sgn}(c_1) + \cdots + \mathrm{sgn}(c_{n_2}) = n_2$.

As in Case 1, we show that virtual knot $VK_2^{n_2}$ can be transformed into trivial knot using one arc shift move. It is easy to observe that a part of virtual knot $VK_2^{n_2}$ lying on right to the dashed line in Figure 12 is identical to that of the virtual knot $VK_1^{n_2}$ from Figure 9(a). Therefore, $VK_2^{n_2}$ can be reduced into the virtual knot diagram L shown in Figure 13 by applying one arc shift move. It is easy to see that L is equivalent to trivial knot via RI, VRI moves and hence $A(VK_2^{n_2}) = 1$ follows.

Case 3: $(n_1, n_2),\ n_1 \geq 3$ and $n_2 > 0$.

This generalized virtual knot diagram denoted by $VK_{n_1}^{n_2}$ is constructed as shown in Figure 14(a). We claim that this virtual knot $VK_{n_1}^{n_2}$ satisfies $A(VK_{n_1}^{n_2}) = 1$ and $J_{n_1}(VK_{n_1}^{n_2}) = n_2$.

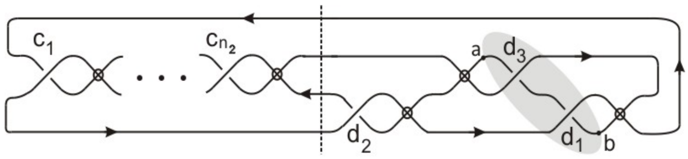


FIGURE 12. $VK_2^{n_2}$.

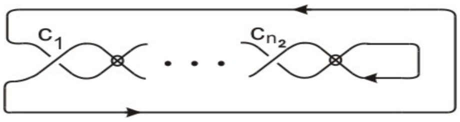
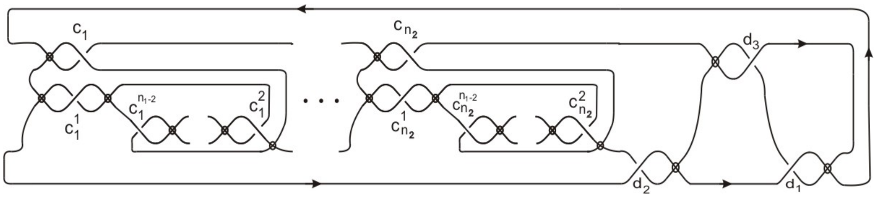
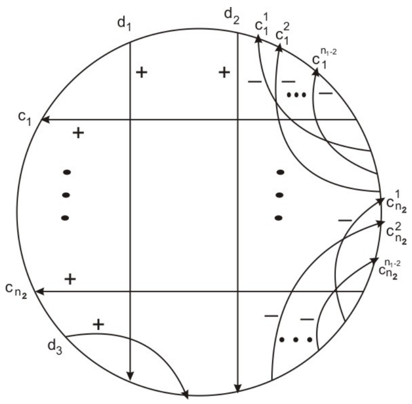


FIGURE 13. L.



(a) $VK_{n_1}^{n_2}$.



(b) $G(VK_{n_1}^{n_2})$.

FIGURE 14. Virtual knot $VK_{n_1}^{n_2}$ and Gauss diagram $G(VK_{n_1}^{n_2})$.

As in the earlier cases, we compute $\mathrm{Ind}(c)$ using Gauss diagram $G(VK_{n_1}^{n_2})$ shown in Figure 14(b).

For crossing c_1, we have

$$\mathrm{Ind}(c_1) = \mathrm{sgn}(d_1) + \mathrm{sgn}(d_2) - \sum_{i=1,\dots,n_1-2} \mathrm{sgn}(c_1^i) = 2 + (n_1 - 2) = n_1.$$

Similarly, we have

$$\begin{aligned}
&\mathrm{Ind}(c_i) = n_1 \quad \text{for } i = 1, 2, \dots, n_2,\\
&\mathrm{Ind}(c_i^1) = n_1 - 2 \quad \text{for } i = 1, 2, \dots, n_2,\\
&\mathrm{Ind}(c_i^j) = 0 \quad \text{for } i = 1, 2, \dots, n_2 \text{ and } j = 2, \dots, n_1 - 2,\\
&\mathrm{Ind}(d_1) = -n_2 + 1,\\
&\mathrm{Ind}(d_2) = -n_2,\\
&\mathrm{Ind}(d_3) = -1.
\end{aligned} \tag{3}$$

As $n_2 > 0$, the only crossings of index n_1 are $c_1, \dots, c_{n_2}$, hence,

$$J_{n_1}(VK_{n_1}^{n_2}) = \sum_{\mathrm{Ind}(c)=n_1} \mathrm{sgn}(c) = \mathrm{sgn}(c_1) + \cdots + \mathrm{sgn}(c_{n_2}) = n_2.$$

Analogous to earlier two cases, apply one arc shift move on the arc (a, b) lying on right hand side of dashed line in $VK_{n_1}^{n_2}$ (see Figure 15).

It follows that $VK_{n_1}^{n_2}$ gets deformed into virtual knot diagram L_1. We can further reduce L_1 into virtual knot L_2 by applying one $VRII$ and one RII move at crossings c_1, c_1^1. Further, by repeated applications of $VRIII, SV$ moves at the crossings $c_1^2, \dots, c_1^{n_1-2}$, we can convert L_2 into virtual knot L_3. Notice that applying VRI, RI moves at crossings $c_1^2, \dots, c_1^{n_1-2}$ in L_3 results in virtual knot L_4 which is identical to L_1 except no crossings in the block that contained crossing c_1 earlier. This whole procedure is shown in Figure 15. Thus, by repeating the same procedure $n_2 - 3$ times in the diagram L_4 we can convert it into trivial knot. Therefore, $A(VK_{n_1}^{n_2}) = 1$.

We prove Theorem 2.3 for remaining cases by using the behavior of n-writhe under taking the mirror image and reversing orientation of a virtual knot diagram D. Let D^R be the *reverse* of D, obtained from D by reversing the orientation and let D^* be the *mirror image* of D, obtained by switching all the classical crossings in D. Then, we have $J_{\pm n}(D^*) = -J_{\mp n}(D)$ and $J_{\pm n}(D^R) = J_{\mp n}(D)$.

Case 4: $(n_1, n_2),\ n_1 < 0, n_2 > 0.$

Let $VK_{|n_1|}^{n_2}$ be the virtual knot corresponding to the case $|n_1|, n_2 > 0$ as discussed in previous cases. Assume that L denotes the virtual knot ${VK_{|n_1|}^{n_2}}^R$ then we have,

$$\begin{aligned}
J_{n_1}(L) &= J_{n_1}({VK_{|n_1|}^{n_2}}^R)\\
&= J_{-n_1}(VK_{|n_1|}^{n_2})\\
&= J_{|n_1|}(VK_{|n_1|}^{n_2}) \text{ as } |n_1| = -n_1 \text{ for } n_1 < 0\\
&= n_2.
\end{aligned} \tag{4}$$

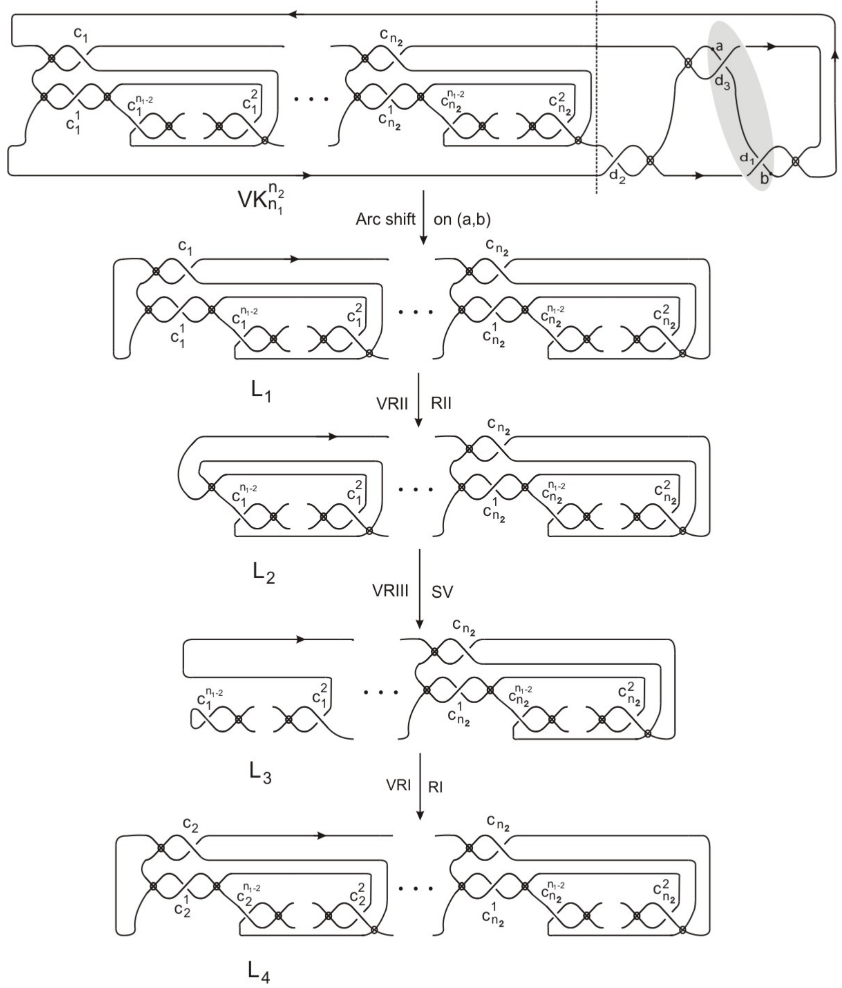


FIGURE 15. $VK_{n_1}^{n_2}$ converted into trivial knot.

Case 5: $(n_1, n_2),\ n_1 > 0, n_2 < 0.$

Let L be the virtual knot $(VK_{n_1}^{|n_2|^R})^*$. Then we get,

$$\begin{aligned} J_{n_1}(L) &= J_{n_1}((VK_{n_1}^{|n_2|^R})^*) \\ &= -J_{-n_1}(VK_{n_1}^{|n_2|^R}) \\ &= -J_{n_1}(VK_{n_1}^{|n_2|}) \\ &= -|n_2| \\ &= n_2 \text{ as } n_2 < 0. \end{aligned} \tag{5}$$

Case 6: $(n_1, n_2),\ n_1 < 0, n_2 < 0.$

Denote by L the virtual knot $VK_{|n_1|}^{|n_2|^*}$ so that

$$\begin{aligned} J_{n_1}(L) &= J_{n_1}(VK_{|n_1|}^{|n_2|^*}) \\ &= -J_{-n_1}(VK_{|n_1|}^{|n_2|}) \\ &= -J_{|n_1|}(VK_{|n_1|}^{|n_2|}),\ \text{as } n_1 < 0 \\ &= -|n_2| \\ &= n_2 \text{ as } n_2 < 0. \end{aligned} \tag{6}$$

For each of the virtual knot L considered in Cases 4, 5 and 6, arc shift number $A(L) = 1$, which follows similarly as in Cases 1, 2 and 3. This completes the proof of Theorem 2.3. □

Whereas it follows from Theorem 2.3 that n-writhe can increase and decrease by any non zero integer under arc shift move, there is also a possibility that it remains unaltered as shown in next theorem. In fact, there exist infinite family of such virtual knots as we prove in Theorem 2.4.

Theorem 2.4. *There exists an infinite family of virtual knots* $\{K_m\}_{m\geq 1}$ *such that* $A(K_m) = 1$ *and* $J_n(K_m) = 0$ *for any* $n \in \mathbb{Z} \setminus \{0\}$.

Proof. The proof is based on construction, we construct virtual knots $\{K_m\}_{m\geq 1}$, such that, $A(K_m) = 1$ and $J_n(K_m) = 0$.

Case 1: $n = 1.$

Let K_m for $m \geq 1$ be the virtual knot shown in Figure 16.

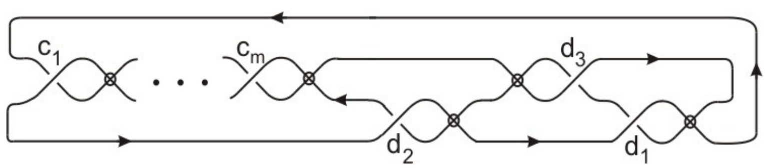


FIGURE 16. Virtual knot K_m.

For crossings in K_m, we have

$$\text{Ind}(c_i) = 2 \quad \text{for} \quad i = 1, 2, \ldots, m.$$

$$\text{Ind}(d_1) = -m + 1,\ \text{Ind}(d_2) = -m \quad \text{and} \quad \text{Ind}(d_3) = -1.$$

Note that, 2 is the only positive index value among all crossings of K_m. It follows that, there is no crossing in K_m having index value 1 and hence $J_1(K_m) = 0$ for $m \geq 1$. Observe that K_m is identical to the virtual knot $VK_2^{n_2}$ from Figure 11(a) for $n_2 = m$, thus, $A(K_m) = 1$.

Case 2: $n \geq 2.$

Consider the virtual knot K_m for $m \geq 1$ as shown in Figure 17.

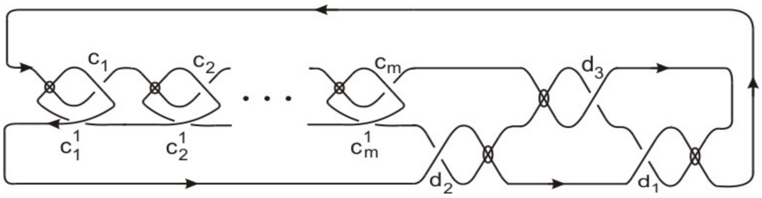


FIGURE 17. Virtual knot K_m.

For crossings in K_m, we have

$$\mathrm{Ind}(c_i) = 1,\ \mathrm{Ind}(c_i^1) = -1 \ \text{ for } \ i = 1, 2, \ldots, m.$$

$$\mathrm{Ind}(d_1) = -m + 1,\ \mathrm{Ind}(d_2) = -m \ \text{ and } \ \mathrm{Ind}(d_3) = -1.$$

Therefore, for $n \geq 2$, there is no crossing in K_m with index value equal to n and hence $J_n(K_m) = 0$ for $m \geq 1$. Note that K_m is identical to the virtual knot $VK_1^{n_2}$ from Figure 9(a) for $n_2 = m$, thus, $A(K_m) = 1$. Other cases follows by reversing the orientation on virtual knots considered in Cases 1 and 2. ☐

We have a corollary that immediately follows from Theorem 2.3.

Corollary 2.5. *If $S = \{VK \mid A(VK) = 1\}$, then $S_n = \{J_n(VK) \mid VK \in S\}$ is unbounded for each $n \in \mathbb{Z} \setminus \{0\}$.*

Proof. For $n \neq 0$ and any arbitrary $m \in \mathbb{Z} \setminus \{0\}$ we constructed VK_n^m from Theorem 2.3 for which $A(VK_n^m) = 1$ and $J_n(VK_n^m) = m$. Since m is arbitrary, $\{J_n(VK_n^m) \mid m \in \mathbb{Z} \setminus \{0\}\}$ is unbounded for each $n \neq 0$. As $\{J_n(VK_n^m) \mid m \in \mathbb{Z} \setminus \{0\}\} \subseteq S_n = \{J_n(VK) \mid VK \in S\}$, S_n is unbounded for each $n \neq 0$. ☐

As another corollary of Theorem 2.3, we show that for each $n \neq 0$, there exists an infinite family of virtual knots that can be distinguished using n-writhe J_n whereas odd writhe $J(K)$ fails to do so.

Corollary 2.6. *For every $n \in \mathbb{Z} \setminus \{0\}$, there exists an infinite family of virtual knots $\{K_n^i\}_{i \geq 1}$ having identical odd writhe J but distinct n-writhe J_n.*

Proof. For a fixed non zero integer n, let K_n^j denote the virtual knot VK_n^j from Theorem 2.3 for $j = 1, 2, \ldots$. From the conditions satisfied in Theorem 2.3, we have $J_n(K_n^j) = j$ for $j \geq 1$ and thus J_n distinguishes them. Since arc shift number $A(K_n^j) = 1$, from Proposition 2.2 it follows that $J(K_n^j) = 0, 2$ or -2 for $j \geq 1$. By pigeonhole principle there must exist an infinite subfamily $K_n^{j_1}, K_n^{j_2}, \ldots$, having odd writhe as one of 0, 2 or -2. ☐

3. Affine index polynomial for the virtual knots $VK_{n_1}^{n_2}$

In this section we compute *affine index polynomial* for virtual knots $VK_{n_1}^{n_2}$ and observe that coefficients and degree of *affine index polynomial* shows unbounded behavior under arc shift move. For convenience of reader we briefly review the definition of *affine index polynomial*. Let D be an oriented virtual knot diagram and $C(D)$ denote the set of all classical crossings in D. By an *arc* we mean an edge between two consecutive classical crossings along the orientation. Note that the notion of *arc* here is slightly different from the way *arc* was defined while discussing *arc shift move*. Now assign an integer value to each arc in D in such a way that the labeling around each crossing point of D follows the rule as shown in Figure 18.

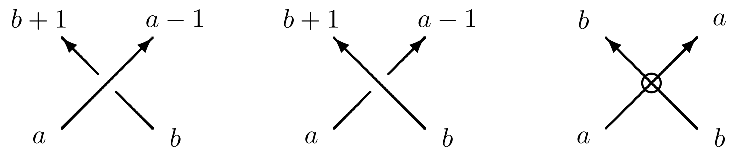


FIGURE 18. Labeling of arcs.

After labeling assign a weight $W_D(c)$ to each classical crossing c defined in [9] as

$$W_D(c) = \mathrm{sgn}(c)(a - b - 1).$$

Then the Kauffman's *affine index polynomial* [9] of virtual knot diagram D is defined as

$$P_D(t) = \sum_{c \in C(D)} \mathrm{sgn}(c)(t^{W_D(c)} - 1), \tag{7}$$

where the summation runs over the set $C(D)$ of classical crossings of D. It was proved [2, Theorem 3.6] that

$$\mathrm{Ind}(c) = W_D(c) = \mathrm{sgn}(c)(a - b - 1). \tag{8}$$

Therefore it follows that coefficients of t^n in $P_D(t)$ are given by n-writhe $J_n(D)$ and hence *affine index polynomial* for D can be rewritten as

$$P_D(t) = \sum_{n \in \mathbb{Z}} J_n(D)\,(t^n - 1)\,.$$

Using the definition of n-writhe, i.e., $J_n(D) = \sum_{\mathrm{Ind}(c)=n} \mathrm{sgn}(c)$, we compute all possible n-writhes for the virtual knots $VK_{n_1}^{n_2}$ when $n_1 \geq 3$ as listed in Table 1.

TABLE 1. Calculations of n-writhe J_n for $VK_{n_1}^{n_2}$, $n_1 \geq 3$.

n	$J_n(VK_{n_1}^{n_2})$
$-n_2$	$+1$
$-n_2 + 1$	$+1$
-1	$+1$
0	$-n_2(n_1 - 3)$
$n_1 - 2$	$-n_2$
n_1	n_2

Remark 3.1. For $VK_{n_1}^{n_2}$, by plugging n-writhes into the formula, we get

$$P_{VK_{n_1}^{n_2}}(t) = t^{-n_2} + t^{-n_2+1} + t^{-1} - 3 - n_2 t^{n_1-2} + n_2 t^{n_1}.$$

Since $A(VK_{n_1}^{n_2}) = 1$, $VK_{n_1}^{n_2}$ can be converted into trivial knot K_0 using one arc shift move. For trivial knot we have affine index polynomial $P_{K_0}(t) = 0$. In $P_{VK_{n_1}^{n_2}}(t)$, as $n_1 \geq 3$ and $n_2 > 0$ are arbitrary integers, it follows that both the coefficients and degree of affine index polynomial vary unboundedly under arc shift move.

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