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TRACE PROPERTIES AND INTEGRAL DOMAINS, III

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Abstract. An integral domain R is an RTP domain (or has the radical trace property) (resp. an LTP domain) if $I(R:I)$ is a radical ideal for each nonzero noninvertible ideal I (resp. $I(R:I)R_P = PR_P$ for each minimal prime P of $I(R:I)$). Clearly each RTP domain is an LTP domain, but whether the two are equivalent is open except in certain special cases. In this paper, we study the descent of these notions from particular overrings of R to R itself.

1. Introduction

Throughout this article, R denotes an integral domain with quotient field K and integral closure R'. By an overring of R we mean a domain that contains R and has the same quotient field. For a nonzero fractional ideal I of R, $(R:I) = \{x \in K \mid xI \subseteq R\}$ is the dual of I and $I_v = (R:(R:I))$ is the divisorial closure of I (both "with respect to R"). The trace of an R-module B is the ideal of R generated by the set $\{\varphi(b) \mid b \in B, \varphi \in \mathrm{Hom}_R(B,R)\}$ (see, for example, [5]). We say that an ideal I of R is a *trace ideal* if it is the trace of some R-module. In such a case, I will in fact be its own trace [2, Proposition 7.2], equivalently $(R:I) = (I:I)$. Thus we may restrict our study of "trace properties" to the noninvertible ideals of R.

A domain R is said to be a *TP domain* if the trace of each noninvertible ideal is prime; alternately one may say that R has the *trace property* [5]. Every valuation domain is a TP domain (see [1]). It is quite unlikely that a domain with more than one maximal ideal is a TP domain, for one it would require that at most one maximal ideal is not invertible (see [5, Corollary 2.11]). On the other hand, such a domain may have the *radical trace property* where the trace of each nonzero noninvertible ideal is a radical ideal. Such a domain is also referred to as an *RTP domain* [6]. In the case of a Noetherian domain R, it is known that R is an RTP domain if and only if R_M is a TP domain for each maximal ideal M [6, Proposition 2.1]. Two types of domains that are related to RTP domains are TPP domains and LTP domains, introduced

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in [11] and [9], respectively. A *TPP domain* is one for which the trace of each noninvertible primary ideal is prime (in fact is its radical [11, Corollary 8]) and an *LTP domain* is one for which each trace ideal is locally prime when localized at any of its minimal primes. Evidence suggests that the three notions may be equivalent. It is known that RTP implies TPP [11, Theorem 4], and TPP implies LTP [9, Corollary 3]. Also the three are equivalent for Prüfer domains ([11, Theorem 23] and [9, Theorem 10]), one-dimensional domains ([11, Corollary 6] and [9, page 422]), and Mori domains ([11, Theorem 12] and [9, Theorem 18]).

With regard to overrings, if R is an RTP domain, then each flat overring is also an RTP domain [12, Corollary 3.17]. Our motivation is the following question (which we answer negatively in Example 2.1): If S and T are flat RTP overrings of R such that $R = S \cap T$, do we have R an RTP domain? In general, the answer to this question is "NO", so we restrict ourselves to some particular overrings more closely related to R.

Recall that a pair of valuation domains V and W with the same quotient field K are said to be independent if (0) is the only common prime ideal. Since each overring of V has the form V_P for some prime ideal P and $PV_P = P$, the following are equivalent for V and W.

(1) V and W are independent.
(2) $VW = K$.
(3) No nonzero prime ideal of $V \cap W$ survives in both V and W.

This notion was extended to pairs of domains with the same quotient field. For a pair of domains S and T with quotient field K, we say that S and T are independent if $ST = K$ and no nonzero prime ideal of $S \cap T$ survives in both S and T (see [4, Chapter 6]).

We show that if S and T are independent overrings of R such that $R = S \cap T$ and each nonzero ideal of R survives in at least one of S and T, then R is an RTP domain if and only if both S and T are RTP domains. The analogous statements hold for TPP and for LTP (Theorem 3.2).

For a different type of pair, we consider the case of when $S = (A : A)$ and $T = (B : B)$, where A and B are comaximal trace ideals of R. In this case, we have that if both S and T are RTP domains, then so is R (Theorem 2.7). Also we prove that if R has an idempotent maximal ideal M such that $(M : M)$ is an RTP domain, then so is R (Theorem 2.12).

In 1952, Jaffard introduced the notion of a ring of Dedekind type as a commutative ring R (with identity) such that each nonzero ideal factors as a finite product of pairwise comaximal ideals such that each factor is contained in a unique maximal ideal. He showed that if R is a commutative ring with identity, then R is of Dedekind type if and only if it has finite character (each nonzero nonunit is contained in only finitely many maximal ideals) and each nonzero prime ideal is contained in a unique maximal ideal (see [8, Théorème 6]). Thus in the terminology introduced by Matlis [14], a ring is of Dedekind type if and only if it is h-local. We will make use of both ideas. If I is a nonzero ideal of an h-local domain R, then $(R : I)R_M = (R_M : IR_M)$ for each maximal ideal [3, Lemma 2.3]. A consequence is that if R is h-local, then it is an RTP domain if and only if R_M is an RTP domain for each maximal ideal M (see [13, Theorem 3.11]). Based on the work of Matlis (with regard to h-local domains) and Jaffard (with regard to the equivalent factoring property mentioned above), we will make use of the two notions to generalize [13, Theorem 3.9].

Let $\mathcal{S} = \{S_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of domains (that are not fields) with the same quotient field K such that $R := \bigcap_{\alpha\in\mathcal{A}} S_\alpha$ also has the quotient field K. Such a family is said to be a *Jaffard family* if for each nonzero ideal I of R, there is a finite nonempty subset $\operatorname{supp}_{\mathcal{S}}(I) := \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq \mathcal{A}$ such that $I_\alpha := IS_\alpha \cap R$ is equal to R for all $\alpha \in \mathcal{A}\setminus \operatorname{supp}_{\mathcal{S}}(I)$, $I_{\alpha_i} (= IS_{\alpha_i} \cap R) \subsetneq R$ for all $1 \leq i \leq n$ and $I = I_{\alpha_1} I_{\alpha_2} \cdots I_{\alpha_n}$ with $I_{\alpha_i} + I_{\alpha_j} = R$ for all $i \neq j$.

For a different approach, let $\mathcal{P} = \{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a partition of $\operatorname{Max}(R)$ and for each $\alpha \in \mathcal{A}$, let $W_\alpha = \bigcap\{R_M \mid M \in X_\alpha\}$. As above, let $\operatorname{supp}_{\mathcal{P}}(I) = \{\alpha \in \mathcal{A} \mid IW_\alpha \subsetneq W_\alpha\}$. Say that $\mathcal{P}$ is a *Matlis partition* of $\operatorname{Max}(R)$ if $|\operatorname{supp}_{\mathcal{P}}(rR)| < \infty$ for each nonzero nonunit $r \in R$ and $|\operatorname{supp}_{\mathcal{P}}(P)| = 1$ for each nonzero prime ideal P of R. Note that $\operatorname{supp}_{\mathcal{P}}(I)$ is nonempty (but finite) for each nonzero ideal I of R. Also R is h-local if and only if $\mathcal{P} = \{\{M_\alpha\} \mid M_\alpha \in \operatorname{Max}(R)\}$ is a Matlis partition of R.

In [4, Theorem 6.3.4] it was proved that each Jaffard family arises from a Matlis partition and each Matlis partition produces a Jaffard family. With regard to trace properties we show that if $R = \bigcap\{S_\alpha \mid S_\alpha \in \mathcal{S}\}$, where $\mathcal{S} = \{S_\alpha\}$ is a Jaffard family, then R is an RTP domain if and only if each S_α is an RTP domain (see Theorem 4.1). The analogous equivalences are obtained for TPP domains and for LTP domains.

2. Descent trace properties

In this section, we will study the descent of the radical trace property from special overrings of R to R itself. Our study is motivated by the following question.

Question: Suppose $R = S \cap T$ for some domains S and T with the same quotient field as R. If both S and T are RTP domains, is R an RTP domain? The answer is "No" in general" even if in the case where both S and T are flat over R as shown by the following example.

Example 2.1. Let $V = k(\mathrm{X},\mathrm{Y})[[\mathrm{Z}]] = k(\mathrm{X},\mathrm{Y}) + M$, $S = k(\mathrm{X})[\mathrm{Y}] + M$ and $T = k(\mathrm{Y})[\mathrm{X}] + M$. By [10, Theorem 13] S and T are Prüfer RTP domains. However $R = S \cap T = k[\mathrm{X},\mathrm{Y}] + M$ is not an RTP domain again by [10, Theorem 13] as $k[\mathrm{X},\mathrm{Y}]$ is not an RTP domain. Moreover, $S = R_{\mathcal{X}}$ and $T = R_{\mathcal{Y}}$, where $\mathcal{X} = k[\mathrm{X}] \setminus 0$ and $\mathcal{Y} = k[\mathrm{Y}] \setminus 0$. Thus S and T are flat over R.

It is well-known that if I and J are trace ideals of R, then $I \cap J$ need not be a trace ideal of R [7, Examples 5.2 and 5.3], but it is the case that $IJ(R : IJ)$ is contained in $I \cap J$. Thus if $IJ = I \cap J$, then $I \cap J$ is a trace ideal of R.

Lemma 2.2. *Let I and J be a pair of trace ideals of a domain R.*

(1) $IA(R : IA) \subseteq I$ *for each nonzero ideal A of R.*
(2) $IJ(R : IJ) \subseteq I \cap J$.
(3) *If $I \cap J = IJ$, then $I \cap J$ is a trace ideal of R.*

Proof. Let A be a nonzero ideal of R. Then $IA(R : IA) = IA((R : I) : A) = IA((I : I) : A) = IA(I : IA) \subseteq I$. Hence we have $IJ(R : IJ) \subseteq I \cap J$. Also, if $I \cap J = IJ$, then we have $(R : I \cap J) = (I \cap J : I \cap J)$ and so $I \cap J$ is a trace ideal of R. □

Recall that an ideal J of a domain R is said to be SV-stable if J is an invertible ideal of $(J : J)$.

Lemma 2.3. *Let I and J be a pair of trace ideals of a domain R such that $I \cap J = IJ$. If J is SV-stable, then IJ^{-1} is a trace ideal of J^{-1}.*

Proof. By Lemma 2.2, $I \cap J = IJ$ is a trace ideal of R. Thus $(IJ : IJ) = (R : IJ) = ((R : J) : I) = (J^{-1} : IJ^{-1})$.

For $t \in (R : IJ)$ we have $tIJ \subseteq IJ$ and therefore $tIJJ^{-1} \subseteq IJJ^{-1}$. If J is SV-stable, we may cancel the J to obtain $tIJ^{-1} \subseteq IJ^{-1}$. Thus in this case IJ^{-1} is a trace ideal of J^{-1}. □

Theorem 2.4. *Let J be a SV-stable trace ideal of R such that $(J : J)$ is an RTP domain and I a trace ideal of R.*

(1) *If I is comaximal with J, then I is a radical ideal of R.*

(2) *If $J = P$ is prime and $I \subseteq P$, then I contains both $(\sqrt{I})^2$ and $IP(R : IP)$.*

Proof. (1) Since J is a trace ideal, $(J : J) = J^{-1}$. If $I + J = R$, then we have $I \cap J = IJ$. Hence by Lemma 2.3, IJ^{-1} is a trace ideal of J^{-1}. Moreover, $(J^{-1} : IJ^{-1}) = (IJ : IJ)$. If J^{-1} is an RTP domain, then IJ^{-1} is a radical ideal of J^{-1}. To see that I is a radical ideal, let $x \in R$ be such that $x^n \in I$ for some $n \geq 1$. Then $x \in IJ^{-1}$ since IJ^{-1} is a radical ideal of J^{-1}. It follows that $xJ \subseteq IJ$. Checking locally shows that $x \in I$. Specifically, if M is a maximal ideal that does not contain I, then we certainly have $x \in R_M = IR_M$. On the other hand if N is a maximal ideal that contains I, then it does not contain J and in this case we have $x \in xR_N = xJR_N \subseteq IJR_N = IR_N$. Therefore I is a radical ideal of R.

(2) If I is a trace ideal that is contained in P, then $(I : I) = (R : I) \supseteq (R : P) = (P : P)$ and $(R : P^2) \subseteq (R : PI) = (P^{-1} : I) = (I : PI)$. It follows that $I = IP^{-1} = IP(R : P^2) \subseteq PI(R : PI) = PI(I : PI) \subseteq I$. Since P^{-1} is an RTP domain, $I(P^{-1} : I)$ is a radical ideal of P^{-1}. Let $x, y \in \sqrt{I}$. Then $x \in P$ and $y \in I(P^{-1} : I)$. Therefore $xy \in I$ and we have $(\sqrt{I})^2 \subseteq I$. □

Corollary 2.5. *Let M be a maximal ideal of R. If M is SV-stable and $(M : M)$ is an RTP domain, then each trace ideal of R that is comaximal with M is a radical ideal of R.*

Proof. If M is SV-stable, then either it is an invertible ideal of $R = (M : M)$ or $(R : M) = (M : M) \supsetneq R$. There is nothing to prove if $R = (M : M)$ and in the other case we simply apply Theorem 2.4. □

Theorem 2.6. *Let A be an* (*integral*) *ideal of a domain R of the form $A = JE$, where J is an invertible* (*fractional*) *ideal of R and E is an idempotent* (*fractional*) *ideal of R. If $(A : A)$ is an RTP domain and I is a trace ideal of R that is comaximal with A, then I is a radical ideal of R.*

Proof. Set $A = JE$, where $JJ^{-1} = R$ and $E = E^2$. Since J is invertible, $T = (A : A) = (E : E)$. Let I be a trace ideal of R such that $R = I + A$ and write $1 = a + b$ for some $a \in A$ and $b \in I$. Let $x \in (T : IE)$. Then $xIE \subseteq T$ and so $xIE = xIE^2 \subseteq E$. Hence $xIA = xIJE \subseteq JE = A$ and so $xA \subseteq (A : I) \subseteq (R : I) = (I : I)$. Thus $xIJE = xIA \subseteq I$ and so $xIE \subseteq IJ^{-1}$. Hence $xIE \subseteq E \cap IJ^{-1} \subseteq IE$. (Notice that $E \cap IJ^{-1} \subseteq IE$. Indeed, let $y \in E \cap IJ^{-1}$. Then $y = ya + yb$. Since $ya \in IJ^{-1}A = IJ^{-1}JE = IE$ and $yb \in IE$, $y \in IE$.) Hence $(T : IE) = (IE : IE)$ and since T is an RTP domain, IE is a radical ideal of T. Now let $x \in R$ such that $x^n \in I$ for some positive integer n. For every $e \in E$, $e^n \in E$ and so $(xe)^n \in IE$. Then $xe \in IE$ since IE is a radical ideal of T. Hence $xE \subseteq IE$. Since $xa \in xA = xJE \subseteq JIE = IA \subseteq I$ and $xb \in I$, $x = xa + xb \in I$ as desired. □

Theorem 2.7. *Let A and B be trace ideals of an integral domain R. If A and B are comaximal and both $(A : A)$ and $(B : B)$ are RTP domains, then R is an RTP domain.*

Proof. Assume A and B are comaximal and both $A^{-1} = (A : A)$ and $B^{-1} = (B : B)$ are RTP domains.

Let I be a trace ideal of R. Then by Lemma 2.2, both $IA(R : IA)$ and $IB(R : IB)$ are contained in I.

Since both A^{-1} and B^{-1} are RTP domains, $I(A^{-1} : I)$ is a radical ideal of A^{-1} and $I(B^{-1} : I)$ is a radical ideal of B^{-1}.

Let $x \in \sqrt{I}$ and let $a \in A$ and $b \in B$ be such that $a + b = 1$. We have $x \in I(A^{-1} : I) = I(R : IA)$ and $x \in I(B^{-1} : I) = I(R : IB)$. Thus $xa \in IA$ and $xb \in IB$. It follows that $x = xa + xb$ is in I and therefore I is a radical ideal of R. Hence R is an RTP domain. □

The proof of the previous theorem can easily be generalized to a finite set of trace ideals whose sum is R.

Theorem 2.8. *Let $A_1, A_2, \ldots, A_n$ be trace ideals of a domain R such that $(A_i : A_i)$ is an RTP domain for each A_i. If $A_1 + A_2 + \cdots + A_n = R$, then R is an RTP domain.*

Proof. Assume $A_1 + A_2 + \cdots + A_n = R$. Then there are elements $a_1 \in A_1$, $a_2 \in A_2$, …, $a_n \in A_n$ such that $a_1 + a_2 + \cdots + a_n = 1$.

Let I be a trace ideal of R and let $x \in \sqrt{I}$. Since $(R : A_i) = (A_i : A_i)$ is an RTP domain, $I((R : A_i) : I) = I(R : IA_i)$ is a radical ideal of $(R : A_i)$. Hence $x \in I(R : IA_i)$. Also, by Lemma 2.2, $IA_i(A_i^{-1} : I) = IA_i(R : IA_i) \subseteq I$. Thus $x = xa_1 + xa_2 + \cdots + xa_n \in I$ and we have that I is a radical ideal of R. Therefore R is an RTP domain. □

Theorem 2.9. *Let A and B be comaximal ideals of a domain R such that A^{-1} and B^{-1} are TP domains. Then R is a TP domain if and only if each trace ideal of R is a contraction of either a prime ideal of A^{-1} or a prime ideal of B^{-1}.*

Proof. Since $A + B = R$, $R = A^{-1} \cap B^{-1}$. Let I be a trace ideal of R and set $P = I(A^{-1} : I)$ and $Q = I(B^{-1} : I)$. Clearly $PA \subseteq I$ and $QB \subseteq I$ and $I = P \cap Q$ (for if $1 = a + b$ for some $a \in A$ and $b \in B$, then for every $x \in P \cap Q$, $x = xa + xb \in PA + QB \subseteq I$). Since I is a proper ideal of R and A^{-1} and B^{-1} are TP domains, either P is a prime ideal of A^{-1} or Q is a prime ideal of B^{-1} (for if $P = A^{-1}$ and $Q = B^{-1}$, then $I = P \cap Q = A^{-1} \cap B^{-1} = R$). If $P = A^{-1}$, necessarily Q is a prime ideal of B^{-1} and $I = Q \cap R$; and if $Q = B^{-1}$, necessarily P is a prime ideal of A^{-1} and $I = P \cap R$. Assume that P is a prime ideal of A^{-1} and Q is a prime ideal of B^{-1}. Since $I = (P \cap R) \cap (Q \cap R)$, if R is a TP domain, I must be a prime ideal of R and so $P \cap R$ and $Q \cap R$ are comparable. Thus $I = P \cap R$ or $I = Q \cap R$, as desired.

The converse is trivial. □

Lemma 2.10. *If I is a trace ideal of R and B is an idempotent ideal, then B and IB are trace ideals of both R and $B^{-1} = (B : B)$.*

Proof. First note that if B is idempotent, then $BB^{-1} = B^2B^{-1} \subseteq B$. Hence $(B : B) = B^{-1} = (R : B^2) = (B^{-1} : B)$. So B is a trace ideal of both R and $(B : B)$. The proof that IB is a trace ideal of both R and $(R : B)$ is only a bit more complicated.

We have $(R : IB) = (R : IB^2) = ((R : B) : IB)$ and alternately, $(R : IB) = ((R : I) : B) = ((I : I) : B) = (I : IB) \subseteq (IB : IB^2) = (IB : IB) \subseteq (R : IB)$. Thus $(R : IB) = (IB : IB) = ((R : B) : IB)$. □

Recall that the maximal ideal of a valuation domain is either idempotent or invertible. Using this fact it is rather easy to construct a domain R that is not an RTP domain but does have a (single) trace ideal A such that $(A : A)$ is an RTP domain, where A is either idempotent or SV-stable.

Example 2.11. Let V be a valuation domain with (nonzero) maximal ideal M such that the residue field V/M is the quotient field of an almost Dedekind domain D that is not a Dedekind domain. Then the pullback of D over M is a Prüfer domain R that is not an RTP domain. But $(R : M) = (M : M) = V$ is an RTP domain (in fact a TP domain), with M either idempotent or invertible as an ideal of V – the latter obviously equivalent to M being a SV-stable trace ideal of R.

In the previous example, the ideal M is a nonmaximal prime ideal of R. What if instead we have M maximal with $(M : M)$ an RTP domain, is that enough to ensure that R is an RTP domain? In Example 2.13, we show that R need not be an RTP domain if M is an SV-stable maximal ideal such that $(M : M)$ is an RTP domain. On the other hand, we show in our next result that if M is an idempotent maximal ideal, then R is an RTP domain whenever $(M : M)$ is an RTP domain. Note that in this case, M is a radical ideal of $(M : M)$ and each minimal prime of M in $(M : M)$ is a maximal ideal of $(M : M)$ (since $((M : M) : J) = (M : MJ) = (M : M)$ for each ideal J of $(M : M)$ that contains M).

Theorem 2.12. *If M is an idempotent maximal ideal of a domain R such that $(M : M)$ is an RTP domain, then R is an RTP domain.*

Proof. Let I be a trace ideal of R. If $I + M = R$, by Theorem 2.6, I is a radical ideal of R. Assume that $I \subseteq M$. Let $x \in \sqrt{I}$ be such that $x^2 \in I$. Since $I \subseteq M$, $\sqrt{I} \subseteq M$ so $x \in M$ and therefore $x^3 \in IM = \sqrt{IM}$ (Lemma 2.10) which puts $x \in IM$ and therefore $x \in I$ and we have that I is a radical ideal of R. Hence R is an RTP domain. □

Example 2.13. An example of a domain R with a SV-stable trace maximal ideal M such that $(M : M)$ is an RTP domain but R is not an RTP domain. Let k be a field, X an indeterminate over k and set $R = k[[\mathrm{x}^2, \mathrm{x}^5]]$. Clearly R is a one-dimensional local Noetherian domain with maximal ideal $M = (\mathrm{x}^2, \mathrm{x}^5)$ and $M^{-1} = k[[\mathrm{x}^2, \mathrm{x}^3]]$. Thus R is not an RTP domain but M^{-1} is an RTP domain. Also it is clear that M is SV-stable since $M = \mathrm{x}^2 k[[\mathrm{x}^2, \mathrm{x}^3]]$.

3. Independent pairs

Recall from above that a pair of domains S and T with the same quotient field K are *independent* if $ST = K$ and no nonzero prime ideal of $S \cap T$ survives in both S and T. In the event $S \cap T$ also has the quotient field K, then all that one needs to check is that no nonzero prime ideal of $S \cap T$ survives in both S and T. In fact, a slightly weaker condition suffices.

As noted earlier, if R is an RTP domain, then each flat overring of R is an RTP domain [12, Corollary 3.17]. The analogous implication holds for LTP domains [12, Corollary 3.20]. Next we show that RTP can also be replaced by TPP.

Theorem 3.1. *If R is a TPP domain, then each flat overring of R is also a TPP domain.*

Proof. Assume R is a TPP domain and let S be a flat overring of R. Then R is also an LTP domain [9, Corollary 3] and thus we at least have that S is an LTP domain [12, Corollary 3.20].

Let Q' be a nonzero noninvertible primary ideal of S with $P' := \sqrt{Q'}$. Then since S is flat over R, $Q := Q' \cap R$ is a P-primary ideal of R, where

$P = \sqrt{Q} = P' \cap R$ with $Q' = QS$ and $P' = PS$. Thus $Q(R:Q)S \subseteq Q'(S:Q')$ and so Q is not an invertible ideal of R. Since R is a TPP domain we have $Q(R:Q) = P$ [11, Corollary 8] and thus we at least have that $Q'(S:Q') \supseteq Q(R:Q)S = PS = P'$. As Q' is not invertible, $Q'(S:Q') = P'$ if P' is a maximal ideal of S.

In the event that P' is not a maximal ideal of S', we at least have $Q'(S_{N'} : QS_{N'}) \supseteq P'S_{N'}$ for each maximal ideal N' of S'.

Let N' be a maximal ideal of S that properly contains P'. Then $N := N' \cap R$ is a prime ideal of R that contains P with $N' = NS$ and $S_{N'} = S_{NS} = R_N$. Moreover, R_N is a TPP domain. As PR_N is a nonmaximal prime ideal of R_N, $QR_N(R_N : QR_N) = PR_N$ [11, Theorem 6 and Corollary 8]. But we also have $QR_N = Q'S_{N'}$ and $PR_N = P'S_{N'}$. Hence $P'S_{N'} \subseteq Q'(S:Q')S_{N'} \subseteq Q'(S_{N'} : Q'S_{N'}) = P'S_{N'}$ and therefore $Q'(S:Q') = P'$. Thus S is a TPP domain. □

Theorem 3.2. *Let R be a domain with a pair of proper independent overrings S and T such that $R = S \cap T$ and each nonzero ideal of R survives in at least one of S and T.*

(1) *R is an RTP domain if and only if both S and T are RTP domains.*
(2) *R is a TPP domain if and only if both S and T are TPP domains.*
(3) *R is an LTP domain if and only if both S and T are LTP domains.*

Proof. Since both S and T are flat overrings of R, if R is an RTP domain, then both S and T are RTP [12, Corollary 3.17]. For TPP, both S and T are TPP domains if R is a TPP domain (Theorem 3.1). Also, if R is an LTP domain, then both S and T are LTP domains [12, Corollary 3.20].

For the converse in the RTP and LTP cases we can start with a nonzero trace ideal I of R. By [4, Theorem 6.2.3], $I = IS \cap IT$. Also, by [4, Theorem 6.2.2], $IS = I(R:I)S = I(S:IS)$ and $IT = I(R:I)T = I(T:IT)$. Hence IS is a trace ideal of S and IT is a trace ideal of T.

We start with LTP, then consider RTP and finish with TPP.

By [4, Theorem 6.2.3], if P is a minimal prime of I, then P survives in exactly one of S and T. If $PS \neq S$, then PS is a prime ideal of S such that $R_P = S_{PS}$. If S is an LTP domain, then $IR_P = IS_{PS} = PS_{SP} = PR_P$. If, instead, $PT \neq T$ and T is an LTP domain, then $IR_P = IT_{PT} = PT_{PT} = PR_P$. Hence R is an LTP domain if both S and T are LTP domains.

If both S and T are RTP domains, then IS is a radical ideal of S and IT is a radical ideal of T. Since $I = IS \cap IT$, I is a radical ideal of R. Thus R is an RTP domain when both S and T are RTP domains.

Finally in the case both S and T are TPP domains, suppose N is a nonzero prime ideal of R and Q is an N-primary ideal of R that is not invertible. As in the LTP case, either NS is a prime ideal of S with $NT = T = QT$ or NT is a prime ideal of T with $NS = S = QS$. By [4, Theorem 6.2.2], $Q(R:Q)S = Q(S:QS)$ and $Q(R:Q)T = Q(T:QT)$. Moreover, $Q(R:Q) = Q(S:QS) \cap Q(T:QT)$ by [4, Theorem 6.2.3].

If NS is a prime ideal of S, then QS is a NS-primary ideal with $\sqrt{QS} = NS$. Since $S \cap T = R$ and $QT = T$ in this case, QS is not invertible as an ideal of S. Thus $Q(S : QS) = NS$ [11, Corollary 8] and it follows that $Q(R : Q) = NS \cap NT = N$.

Similarly, $Q(T : Q) = NT$ when NT is a prime ideal of T and we again have $Q(R : Q) = N$. Thus R is a TPP domain. □

4. Jaffard families and Matlis partitions

Recall that a domain R is said to be of *Dedekind type* if each nonzero ideal factors as a finite product of pairwise comaximal ideals with each factor in a unique maximal ideal [8]. Jaffard proved that R is of Dedekind type if and only if it has finite character and each nonzero prime ideal is contained in a unique maximal [8, Théorème 6]. Thus R is of Dedekind type if and only if it is h-local.

Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of domains (that are not fields) with the same quotient field K such that $R := \bigcap_{\alpha \in \mathcal{A}} S_\alpha$ also has the quotient field K. Such a family is said to be a *Jaffard family* if for each nonzero ideal I of R, there is a finite nonempty subset $\operatorname{supp}_{\mathcal{S}}(I) := \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathcal{A}$ such that $I_\alpha := IS_\alpha \cap R = R$ for all $\alpha \in \mathcal{A} \setminus \operatorname{supp}_{\mathcal{S}}(I)$, $I_{\alpha_i} (= IS_{\alpha_i} \cap R) \subsetneq R$ for all $1 \leq i \leq n$ and $I = I_{\alpha_1} I_{\alpha_2} \cdots I_{\alpha_n}$ with $I_{\alpha_i} + I_{\alpha_j} = R$ for all $i \neq j$. Since $R = \bigcap_{\alpha \in \mathcal{A}} S_\alpha$, the factoring property implies $I = \bigcap_{\alpha \in \mathcal{A}} IS_\alpha$.

Theorem 4.1. *Let R be a domain and let $\mathcal{S} = \{S_\alpha\}$ be a Jaffard family such that $R = \bigcap S_\alpha$.*

(1) *R is an RTP domain if and only if each S_α is an RTP domain.*
(2) *R is a TPP domain if and only if each S_α is a TPP domain.*
(3) *R is an LTP domain if and only if each S_α is an LTP domain.*

Proof. By [4, Theorem 6.3.1], each S_α is R-flat. Thus S_α is an RTP domain when R is an RTP domain [12, Corollary 3.17], S_α is a TPP domain when R is a TPP domain (Theorem 3.1), and S_α is an LTP domain when R is an LTP domain [12, Corollary 3.20].

Let I be a nonzero noninvertible ideal of R and for each α, let $J_\alpha = I(S_\alpha : IS_\alpha)$. By [4, Theorem 6.3.1], $I(R : I)S_\alpha = J_\alpha$ and therefore $I(R : I) = \bigcap J_\alpha$. If S_α is an RTP domain, then J_α is a radical ideal of S_α (perhaps equal to S_α). Thus $I(R : I)$ is a radical ideal of R if each S_α is an RTP domain.

For TPP, start with a nonzero noninvertible primary ideal Q of R and let $P := \sqrt{Q}$. By [4, Theorem 6.3.1], $\operatorname{supp}_{\mathcal{S}}(P) = \{\beta\}$ for some $S_\beta \in \mathcal{S}$. Then PS_β is a prime ideal of S_β and QS_β is a primary ideal of S_β with $\sqrt{QS_\beta} = PS_\beta$.

Since Q is not invertible as an ideal of R, $Q(R : Q)$ is contained in a maximal ideal M. As M contains P, it must be that MS_β is a maximal ideal of S_β. Moreover, we have $MS_\beta \supseteq Q(R : Q)S_\beta = Q(S_\beta : QS_\beta)$. Hence QS_β is not an invertible ideal of S_β. If S_β is an TPP domain, then we have $Q(R : Q)S_\beta = Q(S_\beta : QS_\beta) = PS_\beta$ [11, Corollary 8]. As both Q and P blow up in all other

S_αs, we have $Q(R : Q) = P$ as they are equal when extended to each S_α ([4, Theorem 6.3.1]). Thus R is a TPP if each S_α is a TPP domain.

Finally we show that R is an LTP domain if each S_α is an LTP domain. For this we may start with a trace ideal I of R with minimal prime P. As in the TPP case, P is survives in a unique S_β. Moreover, $R_P = (S_\beta)_{PS_\beta}$ and $PR_P = P(S_\beta)_{PS_\beta}$. Since $I(R : I) = I$, we also have $IS_\beta = I(R : I)S_\beta = I(S_\beta : IS_\beta)$. Hence IS_β is a trace ideal of S_β. Also PS_β is a minimal prime of IS_β and thus $IR_P = I(S_\beta)_{PS_\beta} = P(S_\beta)_{PS_\beta} = PR_P$. Therefore R is an LTP domain. □

Recall that a domain R is said to be h-local if each nonzero nonunit is contained in only finitely many maximal ideals and each nonzero prime ideal is contained in a unique maximal ideal. A domain for which each nonzero nonunit is contained in only finitely many maximal ideals is said to have finite character. Notice that R is h-local if and only if $\mathcal{P} = \{\{M_\alpha\} \mid M_\alpha \in \mathrm{Max}(R)\}$ is a Matlis partition of R.

Corollary 4.2 (cf. [13, Theorem 3.9])**.** *Let R be an h-local domain. Then R is an RTP domain if and only if R_M is an RTP domain for each maximal ideal M.*

In the case R is one-dimensional, then it is h-local if and only if it has a finite character. Moreover, a local one-dimensional RTP domain is a TP domain. In contrast, [11, Example 35] presents an example of local two-dimensional RTP domain that is not a TP domain.

Corollary 4.3. *If R is a one-dimensional domain with finite character, then R is an RTP domain if and only if R_M is a TP domain for each maximal ideal M.*

Proof. If R is one-dimensional, then it is h-local if and only if it has a finite character. Also for each maximal ideal M, the only nonzero prime of R_M is MR_M. Hence R_M is a TP domain if and only if it is an RTP domain. The result follows from Corollary 4.2. □

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