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MEAN VALUES OF DERIVATIVES OF L-FUNCTIONS IN EVEN CHARACTERISTIC

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Abstract. For any positive integer μ, we compute the mean value of the μ-th derivative of quadratic Dirichlet L-functions over the rational function field $\mathbb{F}_q(t)$, where q is a power of 2.

1. Introduction

In a series of papers [1,2,5], Andrade studied the mean values of derivatives of L-functions in function fields based on the use of the approximate functional equation for function field L-functions developed by Andrade and Keating in [4]. These results can be seen as a function field version of moments of derivatives of the Riemann zeta function as given by Ingham [11] which are further developed by the work of Conrey [8], Gonek [10] and Conrey, Rubinstein and Snaith [9].

For any non-square polynomial D in $\mathbb{F}_q[t]$, where q is odd, let $L(s,\chi_D)$ be the Dirichlet L-function associated to the quadratic character χ_D defined by Jacobi symbol in $\mathbb{F}_q[t]$. In [1], Andrade proved several mean values results for the derivatives of Dirichlet L-functions in function fields when the average is taken over all discriminants, i.e., over all monic polynomials of a prescribed degree in $\mathbb{F}_q[t]$. For any integer $\mu \geq 1$, he gave an exact formula for $\sum_D L^{(\mu)}(\frac{1}{2},\chi_D)$, where D runs over all non-square monic polynomials in $\mathbb{F}_q[t]$ of given degree and $L^{(\mu)}(s,\chi_D)$ is the μ-th derivative of $L(s,\chi_D)$ ([1, Theorem 1.1, Theorem 1.2]). In [2], Andrade investigated the mean values of derivatives of quadratic Dirichlet L-functions over function fields when the average is taken over monic irreducible polynomials P in $\mathbb{F}_q[t]$. He obtained asymptotic formulas for $\sum_P L'(\frac{1}{2},\chi_P)$ and $\sum_P L''(\frac{1}{2},\chi_P)$ as $\deg(P)$ goes to infinity and q fixed, where P runs over all monic irreducible polynomials of a prescribed degree in $\mathbb{F}_q[t]$. Let $\mathcal{H}_n$ denote the set of monic square-free polynomials of degree n in $\mathbb{F}_q[t]$. In [5], Andrade and Rajagopal studied the mean values of second

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derivatives of Dirichlet L-functions $L(s,\chi_D)$ at $s=\frac{1}{2}$. More precisely, they gave an asymptotic formulas for $\sum_D L''(\frac{1}{2},\chi_D)$, where D runs over $\mathcal{H}_{2g+1}$, as g goes to infinity and q fixed ([5, Theorem 2.1]). Recently, Andrade and Jung [3] extended the work of Andrade and Rajagopal to the mean values of μ-th derivatives of Dirichlet L-functions $L(s,\chi_D)$ at $s=\frac{1}{2}$. They gave asymptotic formulas for $\sum_D L^{(\mu)}(\frac{1}{2},\chi_D)$, where D runs over $\mathcal{H}_{2g+1}$ or $\mathcal{H}_{2g+2}$, as g goes to infinity and q fixed ([3, Theorem 3.1, Theorem 3.2]). The aim of this paper is to study the mean values of derivatives of Dirichlet L-functions $L(s,\chi_\mu)$ at $s=\frac{1}{2}$ in even characteristic case. We give asymptotic formulas for $\sum_{u\in\mathcal{I}_{g+1}} L^{(\mu)}(\frac{1}{2},\chi_u)$ and $\sum_{D\in\mathcal{F}_{g+1}} L^{(\mu)}(\frac{1}{2},\chi_u)$ as g goes to infinity and q fixed (Theorem 3.1, Theorem 3.2), where $\mathcal{I}_{g+1}$ and $\mathcal{F}_{g+1}$ are the sets of rational functions which play the roles of $\mathcal{H}_{2g+1}$ and $\mathcal{H}_{2g+2}$ in even characteristic, respectively. In Appendix, we also consider the case of non-maximal orders, as was done in [1] in odd characteristic case.

2. A short background on function fields

Let $k=\mathbb{F}_q(t)$ be the rational function field with a constant field $\mathbb{F}_q$, where q is a power of 2, and $\mathbb{A}=\mathbb{F}_q[t]$. We denote by $\mathbb{A}^+$ the set of monic polynomials in $\mathbb{A}$ and by $\mathbb{P}$ the set of monic irreducible polynomials in $\mathbb{A}$. Any monic irreducible polynomial $P\in\mathbb{P}$ will be also called a prime polynomial throughout the paper. For any positive integer n, let $\mathbb{A}_n=\{f\in\mathbb{A}:\deg(f)=n\}$ and $\mathbb{A}_n^+=\mathbb{A}^+\cap\mathbb{A}_n$, $\mathbb{P}_n=\mathbb{P}\cap\mathbb{A}_n$.

The zeta function $\zeta_{\mathbb{A}}(s)$ of $\mathbb{A}$ is defined to be the following infinite series:

$$\zeta_{\mathbb{A}}(s)=\sum_{f\in\mathbb{A}^+}\frac{1}{|f|^s}=\prod_{P\in\mathbb{P}}\left(1-\frac{1}{|P|^s}\right)^{-1},\quad \mathrm{Re}(s)>1, \tag{2.1}$$

where $|f|=q^{\deg(f)}$. It is well known that $\zeta_{\mathbb{A}}(s)=\frac{1}{1-q^{1-s}}$.

In this section, we recall some basic facts on quadratic function field in even characteristic. For more details, we refer to [6, §2.2, §2.3].

2.1. Quadratic function field in even characteristic

Any separable quadratic extension K of k is of the form $K=K_u=k(x_u)$, where x_u is a zero of $X^2+X+u=0$ for some $u\in k$. Fix an element $\xi\in\mathbb{F}_q\setminus\wp(\mathbb{F}_q)$, where $\wp:k\to k$ is the additive homomorphism defined by $\wp(x)=x^2+x$. We say that $u\in k$ is normalized if it is of the form

$$u=\sum_{i=1}^{m}\sum_{j=1}^{e_i}\frac{Q_{ij}}{P_i^{2j-1}}+\sum_{\ell=1}^{n}\alpha_\ell T^{2\ell-1}+\alpha, \tag{2.2}$$

where $P_i\in\mathbb{P}$ are distinct, $Q_{ij}\in\mathbb{A}$ with $\deg(Q_{ij})<\deg(P_i)$, $Q_{ie_i}\neq 0$, $\alpha\in\{0,\xi\}$, $\alpha_\ell\in\mathbb{F}_q$ and $\alpha_n\neq 0$ for $n>0$. Let $u\in k$ be normalized one as in (2.2). The infinite prime $(1/t)$ of k splits, is inert or ramified in K_u according as $n=0$ and $\alpha=0$, $n=0$ and $\alpha=\xi$, or $n>0$. Then the field K_u is called real, inert imaginary, or ramified imaginary, respectively. The discriminant D_u of K_u is given by

$$D_u = \begin{cases} \prod_{i=1}^m P_i^{2e_i} & \text{if } n = 0, \\ \prod_{i=1}^m P_i^{2e_i} \cdot (1/t)^{2n} & \text{if } n > 0, \end{cases}$$

and the genus g_u of K_u is given by

$$g_u = \frac{1}{2} \deg(D_u) - 1. \tag{2.3}$$

For each $M \in \mathbb{A}^+$, write $r(M) = \prod_{P|M} P$ and $t(M) = M \cdot r(M)$. For $P \in \mathbb{P}$, let ν_P be the normalized valuation at P, that is, $\nu_P(M) = e$, where $P^e \| M$. Let $\mathcal{B}$ be the set of monic polynomials M such that $\nu_P(M) = 0$ or odd for any $P \in \mathbb{P}$, that is, $t(M)$ is a square, and $\mathcal{C}$ be the set of rational functions $\frac{D}{M} \in k$ such that $D \in \mathbb{A}, M \in \mathcal{B}$ and $\deg(D) < \deg(M)$. Also we let $\mathcal{E}$ be the set of rational functions $\frac{D}{M} \in \mathcal{C}$ of the form

$$\frac{D}{M} = \sum_{P|M} \sum_{i=1}^{\ell_P} \frac{A_{P,i}}{P^{2i-1}},$$

where $\deg(A_{P,i}) < \deg(P)$ for any $P \mid M$ and for all $1 \leq i \leq \ell_P = \frac{1}{2}(\nu_P(M)+1)$. Let $\mathcal{F}$ be the set of rational functions $\frac{D}{M} \in \mathcal{E}$ such that $A_{P,\ell_P} \neq 0$ for all $P \mid M$ and $\mathcal{F}' = \{u + \xi : u \in \mathcal{F}\}$. For any positive integer n, let

$$\begin{aligned} &\mathcal{B}_n = \{M \in \mathcal{B} : \deg(t(M)) = 2n\}, \quad \mathcal{C}_n = \left\{\tfrac{D}{M} \in \mathcal{C} : M \in \mathcal{B}_n\right\}, \\ &\mathcal{E}_n = \mathcal{E} \cap \mathcal{C}_n, \quad \mathcal{F}_n = \mathcal{F} \cap \mathcal{E}_n, \quad \mathcal{F}'_n = \{u + \xi : u \in \mathcal{F}_n\}. \end{aligned}$$

Under the correspondence $u \mapsto K_u$, $\mathcal{F}_n$ (resp. $\mathcal{F}'_n$) corresponds to the set of all real (resp. inert imaginary) separable quadratic extensions K_u of k with genus $n-1$. For any positive integer s, let $\mathcal{G}_s$ be the set of polynomials $F(T) \in \mathbb{A}$ of the form

$$F(T) = \alpha + \sum_{i=1}^{s} \alpha_i T^{2i-1}, \quad \text{where } \alpha \in \{0, \xi\},\ \alpha_i \in \mathbb{F}_q \text{ and } \alpha_s \neq 0.$$

Let $\mathcal{F}_0 = \{0\}$. For any integers $r \geq 0$ and $s \geq 1$, let $\mathcal{I}_{(r,s)} = \{u + F : u \in \mathcal{F}_r, F \in \mathcal{G}_s\}$. For any integer $n \geq 1$, let $\mathcal{I}_n$ be the union of all $\mathcal{I}_{(r,s)}$, where (r,s) runs over all pairs of nonnegative integers such that $s > 0$ and $r + s = n$. Then, under the correspondence $u \mapsto K_u$, $\mathcal{I}_n$ corresponds to the set of all ramified imaginary separable quadratic extensions K_u of k with genus $n-1$.

We have that $\#\mathcal{B}_n = q^n$, $\#\mathcal{E}_n = q^{2n}$, $\#\mathcal{F}_n = \zeta_{\mathbb{A}}(2)^{-1} q^{2n}$ and $\#\mathcal{I}_n = 2\zeta_{\mathbb{A}}(2)^{-1} q^{2n-1}$ (see [6, Lemma 2.3]). For each $M \in \mathcal{B}$, let $\mathcal{C}_M$ be the set of rational functions $u \in \mathcal{C}$ whose denominator divides M, $\mathcal{E}_M = \mathcal{E} \cap \mathcal{C}_M$ and $\mathcal{F}_M = \mathcal{F} \cap \mathcal{C}_M$. Note that $\mathcal{E}_n$ (resp. $\mathcal{F}_n$) is the disjoint union of $\mathcal{E}_M$ (resp. $\mathcal{F}_M$) with $M \in \mathcal{B}_n$. Let $\tilde{M} = \prod_{P|M} P^{(\nu_P(M)+1)/2}$. We also note that $\#\mathcal{E}_M = |\tilde{M}|$ and $\#\mathcal{F}_M = \Phi(\tilde{M})$, where $\Phi(\tilde{M}) = \#(\mathbb{A}/\tilde{M}\mathbb{A})^\times$.

2.2. Hasse symbol and L-functions

For any $u \in k$ whose denominator is not divisible by $P \in \mathbb{P}$, the Hasse symbol $[u, P)$ with values in $\mathbb{F}_2$ is defined by

$$[u,P) = \begin{cases} 0 & \text{if } X^2 + X \equiv u \ (\mathrm{mod} P) \text{ is solvable in } \mathbb{A}, \\ 1 & \text{otherwise.} \end{cases}$$

For $N \in \mathbb{A}$ prime to the denominator of u, if $N = sgn(N) \prod_{i=1}^{s} P_i^{e_i}$, where $sgn(N)$ is the leading coefficient of N and $P_i \in \mathbb{P}$ are distinct and $e_i \geq 1$, we define $[u, N)$ to be $\sum_{i=1}^{s} e_i [u, P_i)$.

For $u \in k$ and $0 \neq N \in \mathbb{A}$, we also define the quadratic symbol:

$$\left\{ \frac{u}{N} \right\} = \begin{cases} (-1)^{[u,N)} & \text{if } N \text{ is prime to the denominator of } u, \\ 0 & \text{otherwise.} \end{cases}$$

This symbol is clearly additive in its first variable, and multiplicative in the second variable.

For the field K_u, we associate a character χ_u on $\mathbb{A}^+$ which is defined by $\chi_u(f) = \{\frac{u}{f}\}$, and let $L(s, \chi_u)$ be the L-function associated to the character χ_u: for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 1$,

$$L(s, \chi_u) := \sum_{f \in \mathbb{A}^+} \frac{\chi_u(f)}{|f|^s} = \prod_{P \in \mathbb{P}} \left(1 - \frac{\chi_u(P)}{|P|^s} \right)^{-1}.$$

It is well known that $L(s, \chi_u)$ is a polynomial in q^{-s}. Letting $z = q^{-s}$, write $\mathcal{L}(z, \chi_u) = L(s, \chi_u)$. Then, $\mathcal{L}(z, \chi_u)$ is a polynomial in z of degree $2g_u + \frac{1}{2}(1 + (-1)^{\varepsilon(u)})$, where $\varepsilon(u) = 1$ if K_u is ramified imaginary and $\varepsilon(u) = 0$ otherwise. Also we have that $\mathcal{L}(z, \chi_u)$ has a "trivial" zero at $z = 1$ (resp. $z = -1$) if and only if K_u is real (resp. inert imaginary), so we can define the "completed" L-function as

$$\mathcal{L}^*(z, \chi_u) = \begin{cases} \mathcal{L}(z, \chi_u) & \text{if } K_u \text{ is ramified imaginary,} \\ (1-z)^{-1} \mathcal{L}(z, \chi_u) & \text{if } K_u \text{ is real,} \\ (1+z)^{-1} \mathcal{L}(z, \chi_u) & \text{if } K_u \text{ is inert imaginary,} \end{cases} \tag{2.4}$$

which is a polynomial of even degree $2g_u$ satisfying the functional equation

$$\mathcal{L}^*(z, \chi_u) = (qz^2)^{g_u} \mathcal{L}^* \left(\frac{1}{qz}, \chi_u \right). \tag{2.5}$$

3. Statement of results

Let μ be a positive integer. Let $L^{(\mu)}(s, \chi_u)$ be the μ-th derivative of $L(s, \chi_u)$. For any integer $n \geq 0$, let $J_\mu(n)$ be the sum of the μ-th powers of the first n positive integers, i.e., $J_\mu(n) = \sum_{\ell=1}^{n} \ell^\mu$. Faulhaber's formula tell us that $J_\mu(n)$ can be rewritten as a polynomial in n of degree $\mu + 1$ with zero constant term, that is, $J_\mu(n) = \sum_{m=1}^{\mu+1} j_\mu(m) n^m$.

Let

$$G(s)=\sum_{L\in\mathbb{A}^+}\frac{\mu(L)}{|L|^s\prod_{P|L}(1+|P|)},$$

where $\mu(L)$ is the Möbius function for polynomials. So for any integer $m\geq 0$, we have

$$\frac{G^{(m)}(s)}{(-\ln q)^m}=\sum_{L\in\mathbb{A}^+}\frac{\mu(L)\deg(L)^m}{|L|^s\prod_{P|L}(1+|P|)}. \tag{3.1}$$

We are now ready to state two of the main results of this paper. The first theorem is the mean values of derivatives of Dirichlet L-functions associated to the imaginary quadratic function field K_u with $u\in\mathcal{I}_{g+1}$.

Theorem 3.1. *Let μ be a fixed positive integer and q be a fixed power of 2. Assume that $q>2$. Then we have*

$$\begin{aligned}
&\sum_{u\in\mathcal{I}_{g+1}}\frac{L^{(\mu)}(\frac{1}{2},\chi_u)}{(\ln q)^{\mu}}\\
&=\frac{(-1)^{\mu}2^{\mu+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\left(G(1)J_{\mu}([\tfrac{g}{2}])+\frac{G^{(\mu)}(1)}{(-\ln q)^{\mu}}\right)\\
&\quad+\frac{(-2)^{\mu+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\sum_{a=1}^{\mu+1}j_{\mu}(a)\frac{G^{(a)}(1)}{(-\ln q)^{a}}\\
&\quad+\frac{2^{\mu+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\sum_{m=0}^{\mu}\binom{\mu}{m}(-g)^{\mu-m}\left(G(1)J_{m}([\tfrac{g-1}{2}])+\frac{G^{(m)}(1)}{(-\ln q)^{m}}\right)\\
&\quad-\frac{2^{\mu+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\sum_{m=0}^{\mu}\binom{\mu}{m}(-g)^{\mu-m}\sum_{a=1}^{m+1}j_{m}(a)\frac{G^{(a)}(1)}{(-\ln q)^{a}}+O(g^{\mu}2^{\frac{g}{2}}q^{\frac{3}{2}g}).
\end{aligned}$$

We remark that the main term in Theorem 3.1 is two times of the one in [3, Theorem 3.1] since under the correspond $D\mapsto k(\sqrt{D})$, $\mathcal{H}_{2g+1}\cup\gamma\mathcal{H}_{2g+1}$ corresponds to the set of all ramified imaginary separable quadratic extensions K of k with genus g in odd characteristic case, where γ is any non-square element of $\mathbb{F}_q^*$.

The second theorem is the mean values of derivatives of Dirichlet L-functions associated to the real quadratic function field K_u with $u\in\mathcal{F}_{g+1}$.

Theorem 3.2. *Let μ be a fixed positive integer and q be a fixed power of 2. Assume that $q>2$. Then we have*

$$\begin{aligned}
&\sum_{u\in\mathcal{F}_{g+1}}\frac{L^{(\mu)}(\frac{1}{2},\chi_u)}{(-\ln q)^{\mu}}\\
&=\frac{2^{\mu}q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\left(G(1)J_{\mu}([\tfrac{g}{2}])+\frac{G^{(\mu)}(1)}{(-\ln q)^{\mu}}\right)-\frac{2^{\mu}q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\sum_{a=1}^{\mu+1}j_{\mu}(a)\frac{G^{(a)}(1)}{(-\ln q)^{a}}
\end{aligned}$$

$$- G(1)(g+1)^{\mu}q^{2g+[\frac{g}{2}]-\frac{g}{2}+\frac{3}{2}} - G(1)q^{2g+[\frac{g-1}{2}]-\frac{g-1}{2}+\frac{3}{2}}\sum_{m=0}^{\mu}\binom{\mu}{m}\frac{g^{\mu-m}\delta^{(m)}(\frac{1}{2})}{(-\ln q)^m}$$
$$+\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\sum_{a+b+c=\mu}\frac{\mu!}{a!b!c!}\frac{(2g)^a(-2)^c\delta^{(b)}(\frac{1}{2})}{(-\ln q)^b}\left(G(1)J_c([\tfrac{g-1}{2}])+\frac{G^{(c)}(1)}{(-\ln q)^c}\right)$$
$$-\frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\sum_{a+b+c=\mu}\frac{\mu!}{a!b!c!}\frac{(2g)^a(-2)^c\delta^{(b)}(\frac{1}{2})}{(-\ln q)^b}\sum_{a=1}^{c+1}j_c(a)\frac{G^{(a)}(1)}{(-\ln q)^a}+O(g^{\mu}2^{\frac{g}{2}}q^{\frac{3g}{2}}),$$

where $\delta(s)=\frac{1-q^{-s}}{1-q^{s-1}}$.

4. Main tools

In this section we present a few auxiliary results that will be used in the proof of the main theorems.

Lemma 4.1. *For any* $f\in\mathbb{A}_n^+$ *with* $n\le g$*, which is not a perfect square, we have*

$$\sum_{u\in\mathcal{I}_{g+1}}\left\{\frac{u}{f}\right\}\ll 2^{\frac{n}{2}}gq^g.$$

Proof. This is Proposition 3.20 in [6]. □

Lemma 4.2. *Let* n *be a positive integer. For any* $f\in\mathbb{A}_d^+$*, which is not a perfect square, we have*

$$\sum_{u\in\mathcal{F}_n}\left\{\frac{u}{f}\right\}\ll 2^{\frac{d}{2}}q^n.$$

Proof. This is Proposition 3.15 in [6]. □

Lemma 4.3. *Let* $L\in\mathbb{A}^+$*. Given any* $\epsilon>0$*, we have*

$$\sum_{\substack{f\in\mathbb{A}_n^+\\(f,L)=1}}\Phi(f)=\frac{q^{2n}}{\zeta_{\mathbb{A}}(2)}\prod_{P|L}(1+|P|^{-1})^{-1}+O\left(q^{(1+\epsilon)n}\right).$$

Proof. This is Lemma 3.3 in [6]. □

Applying Lemma 4.3 with $\epsilon=\frac{1}{2}$, we have the following corollary.

Corollary 4.4. *We have*

$$\sum_{L\in\mathbb{A}_\ell^+}\sum_{\substack{f\in\mathbb{A}_n^+\\(f,L)=1}}\Phi(f)=\frac{q^{2n+\ell}}{\zeta_{\mathbb{A}}(2)}\sum_{D\in\mathbb{A}_{\le\ell}^+}\frac{\mu(D)}{|D|\prod_{P|D}(1+|P|)}+O\left(q^{\frac{3n}{2}+\ell}\right). \tag{4.1}$$

Lemma 4.5. *Let $m \geq 0$ be an integer. Then we have*

$$\sum_{\substack{L\in\mathbb{A}^+\\ \deg(L)>[\frac{g}{2}]}} \frac{\mu(L)\deg(L)^m}{|L|\prod_{P|L}(1+|P|)} = O(g^m q^{-\frac{g}{2}}).$$

Proof. This is Lemma 3.4 in [5]. □

Lemma 4.6. *For $h \in \{g-1, g\}$ and $m \in \{0, 1, \ldots, \mu\}$, we have*

$$\begin{aligned}
&\sum_{\ell=0}^{[\frac{h}{2}]}(2\ell)^m q^{-\ell} \sum_{L\in\mathbb{A}_\ell^+}\sum_{s=1}^{g}\sum_{\substack{M\in\mathcal{B}_s\\(M,L)=1}}\sum_{u\in\mathcal{I}_M} 1\\
&= \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\left(G(1)J_m([\tfrac{h}{2}]) + \frac{G^{(m)}(1)}{(-\ln q)^m}\right)\\
&\quad - \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\sum_{a=1}^{m+1} j_m(a)\frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^m q^{\frac{3g}{2}}).
\end{aligned}$$

Proof. Put

$$\mathcal{M}_{h,m}(\mu) = \sum_{\ell=0}^{[\frac{h}{2}]}(2\ell)^m q^{-\ell} \sum_{L\in\mathbb{A}_\ell^+}\sum_{s=1}^{g}\sum_{\substack{M\in\mathcal{B}_s\\(M,L)=1}}\sum_{u\in\mathcal{I}_M} 1.$$

Since $\#\mathcal{I}_M = \frac{2}{\zeta_{\mathbb{A}}(2)}q^{g+1-s}\Phi(\tilde{M})$, we have

$$\mathcal{M}_{h,m}(\mu) = \frac{2^{m+1}q^{g+1}}{\zeta_{\mathbb{A}}(2)}\sum_{\ell=0}^{[\frac{h}{2}]}\ell^m q^{-\ell}\sum_{s=1}^{g} q^{-s}\sum_{L\in\mathbb{A}_\ell^+}\sum_{\substack{\tilde{M}\in\mathbb{A}_s^+\\(\tilde{M},L)=1}}\Phi(\tilde{M}).$$

Then, by using (4.1), we can get

$$\begin{aligned}
\mathcal{M}_{h,m}(\mu) &= \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\sum_{\ell=0}^{[\frac{h}{2}]}\ell^m \sum_{D\in\mathbb{A}_{\leq\ell}^+}\frac{\mu(D)}{|D|\prod_{P|D}(1+|P|)} + O(g^m q^{\frac{3g}{2}})\\
&= \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\sum_{D\in\mathbb{A}_{\leq[\frac{h}{2}]}^+}\frac{\mu(D)}{|D|\prod_{P|D}(1+|P|)}\sum_{\deg(D)\leq\ell\leq[\frac{h}{2}]}\ell^m + O(g^m q^{\frac{3g}{2}}). \qquad (4.2)
\end{aligned}$$

For integer $m \geq 0$, recall that $J_m(n) = \sum_{\ell=1}^{n}\ell^m$, which is a polynomial in n of degree $m+1$ with zero constant term. Write $J_m(n) = \sum_{a=1}^{m+1} j_m(a)n^a$. Then we have

$$\sum_{\deg(L)\leq\ell\leq[\frac{h}{2}]}\ell^m = J_m([\tfrac{h}{2}]) + \deg(L)^m - \sum_{a=1}^{m+1} j_m(a)\deg(L)^a. \qquad (4.3)$$

Inserting (4.3) into (4.2), we have

$$\begin{aligned}\mathcal{M}_{h,m}(\mu) = {} & \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)} J_m([\tfrac{h}{2}]) \sum_{D\in\mathbb{A}^+_{\le[\frac{h}{2}]}} \frac{\mu(D)}{|D|\prod_{P|D}(1+|P|)} \\ & + \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \sum_{D\in\mathbb{A}^+_{\le[\frac{h}{2}]}} \frac{\mu(D)\deg(L)^m}{|D|\prod_{P|D}(1+|P|)} \\ & - \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \sum_{a=1}^{m+1} j_m(a) \sum_{D\in\mathbb{A}^+_{\le[\frac{h}{2}]}} \frac{\mu(D)\deg(L)^a}{|D|\prod_{P|D}(1+|P|)} + O(g^m q^{\frac{3g}{2}}). \end{aligned} \tag{4.4}$$

Then, from (4.4), by using Lemma 4.5, we get that

$$\begin{aligned}\mathcal{M}_{h,m}(\mu) = {} & \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)} J_m([\tfrac{h}{2}]) \sum_{D\in\mathbb{A}^+} \frac{\mu(D)}{|D|\prod_{P|D}(1+|P|)} \\ & + \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \sum_{D\in\mathbb{A}^+} \frac{\mu(D)\deg(L)^m}{|D|\prod_{P|D}(1+|P|)} \\ & - \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \sum_{a=1}^{m+1} j_m(a) \sum_{D\in\mathbb{A}^+} \frac{\mu(D)\deg(L)^a}{|D|\prod_{P|D}(1+|P|)} + O(g^m q^{\frac{3g}{2}}). \end{aligned} \tag{4.5}$$

We also recall that for any integer $n \ge 0$, we have

$$\frac{G^{(n)}(s)}{(-\ln q)^n} = \sum_{D\in\mathbb{A}^+} \frac{\mu(D)\deg(D)^n}{|D|^s\prod_{P|D}(1+|P|)}. \tag{4.6}$$

Finally, by (4.5) and (4.6), we get

$$\begin{aligned}\mathcal{M}_{h,m}(\mu) = {} & \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \left(G(1)J_m([\tfrac{h}{2}]) + \frac{G^{(m)}(1)}{(-\ln q)^m} \right) \\ & - \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \sum_{a=1}^{m+1} j_m(a) \frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^m q^{\frac{3g}{2}}). \end{aligned}$$

□

Lemma 4.7. *For $h \in \{g-1, g\}$ and $m \in \{0, 1, \ldots, \mu\}$, we have*

$$\begin{aligned}\sum_{\ell=0}^{[\frac{h}{2}]} (2\ell)^m q^{-\ell} \sum_{L\in\mathbb{A}^+_\ell} \sum_{\substack{M\in\mathcal{B}_{g+1}\\(M,L)=1}} \sum_{u\in\mathcal{F}_M} 1 = {} & \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \left(G(1)J_m([\tfrac{h}{2}]) + \frac{G^{(m)}(1)}{(-\ln q)^m} \right) \\ & - \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{a=1}^{m+1} j_m(a) \frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^m q^{\frac{3g}{2}}). \end{aligned}$$

Proof. Put

$$\mathcal{N}_{h,m}(\mu) = \sum_{\ell=0}^{[\frac{h}{2}]} (2\ell)^m q^{-\ell} \sum_{L \in \mathbb{A}_\ell^+} \sum_{\substack{M \in \mathcal{B}_{g+1} \\ (M,L)=1}} \sum_{u \in \mathcal{F}_M} 1.$$

Since $\#\mathcal{F}_M = \Phi(\tilde{M})$, we have

$$\mathcal{N}_{h,m}(\mu) = \sum_{\ell=0}^{[\frac{h}{2}]} (2\ell)^m q^{-\ell} \sum_{L \in \mathbb{A}_\ell^+} \sum_{\substack{\tilde{M} \in \mathbb{A}_{g+1}^+ \\ (\tilde{M},L)=1}} \Phi(\tilde{M}).$$

Then, by using (4.1), we can get

$$\begin{aligned} \mathcal{N}_{h,m}(\mu) &= \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{\ell=0}^{[\frac{h}{2}]} \ell^m \sum_{D \in \mathbb{A}_{\leq \ell}^+} \frac{\mu(D)}{|D| \prod_{P|D}(1+|P|)} + O(g^m q^{\frac{3g}{2}}) \\ &= \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{D \in \mathbb{A}_{\leq [\frac{h}{2}]}^+} \frac{\mu(D)}{|D| \prod_{P|D}(1+|P|)} \sum_{\deg(D) \leq \ell \leq [\frac{h}{2}]} \ell^m + O(g^m q^{\frac{3g}{2}}). \end{aligned} \tag{4.7}$$

Inserting (4.3) into (4.7), we have

$$\begin{aligned} \mathcal{N}_{h,m}(\mu) = & \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} J_m([\tfrac{h}{2}]) \sum_{D \in \mathbb{A}_{\leq [\frac{h}{2}]}^+} \frac{\mu(D)}{|D| \prod_{P|D}(1+|P|)} \\ & + \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{D \in \mathbb{A}_{\leq [\frac{h}{2}]}^+} \frac{\mu(D) \deg(L)^m}{|D| \prod_{P|D}(1+|P|)} \\ & - \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{a=1}^{m+1} j_m(a) \sum_{D \in \mathbb{A}_{\leq [\frac{h}{2}]}^+} \frac{\mu(D) \deg(L)^a}{|D| \prod_{P|D}(1+|P|)} + +O(g^m q^{\frac{3g}{2}}). \end{aligned} \tag{4.8}$$

Then, from (4.8), by using Lemma 4.5, we get that

$$\begin{aligned} \mathcal{N}_{h,m}(\mu) = & \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} J_m([\tfrac{h}{2}]) \sum_{D \in \mathbb{A}^+} \frac{\mu(D)}{|D| \prod_{P|D}(1+|P|)} \\ & + \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{D \in \mathbb{A}^+} \frac{\mu(D) \deg(L)^m}{|D| \prod_{P|D}(1+|P|)} \\ & - \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{a=1}^{m+1} j_m(a) \sum_{D \in \mathbb{A}^+} \frac{\mu(D) \deg(L)^a}{|D| \prod_{P|D}(1+|P|)} + O(g^m q^{\frac{3g}{2}}). \end{aligned} \tag{4.9}$$

Finally, by (4.9) and (4.6), we get

$$\mathcal{N}_{h,m}(\mu) = \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \left(G(1) J_m([\tfrac{h}{2}]) + \frac{G^{(m)}(1)}{(-\ln q)^m} \right) - \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{a=1}^{m+1} j_m(a) \frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^m q^{\frac{3g}{2}}).$$

□

Lemma 4.8. *For $h \in \{g-1, g\}$, we have*

$$q^{-\frac{h+1}{2}} \sum_{\ell=0}^{[\frac{h}{2}]} \sum_{L \in \mathbb{A}_\ell^+} \sum_{\substack{M \in \mathcal{B}_{g+1} \\ (M,L)=1}} \sum_{u \in \mathcal{F}_M} 1 = G(1) q^{2g+[\frac{h}{2}]-\frac{h}{2}+\frac{3}{2}} + O(g q^{\frac{3g}{2}}).$$

Proof. Put

$$\mathcal{L}_{h,m}(\mu) = q^{-\frac{h+1}{2}} \sum_{\ell=0}^{[\frac{h}{2}]} \sum_{L \in \mathbb{A}_\ell^+} \sum_{\substack{M \in \mathcal{B}_{g+1} \\ (M,L)=1}} \sum_{u \in \mathcal{F}_M} 1.$$

Since $\#\mathcal{F}_M = \Phi(\tilde{M})$, we have

$$\mathcal{L}_{h,m}(\mu) = q^{-\frac{h+1}{2}} \sum_{\ell=0}^{[\frac{h}{2}]} \sum_{L \in \mathbb{A}_\ell^+} \sum_{\substack{\tilde{M} \in \mathbb{A}_{g+1}^+ \\ (\tilde{M},L)=1}} \Phi(\tilde{M}).$$

Then, by using (4.1), we can get

$$\begin{aligned} \mathcal{L}_{h,m}(\mu) &= \frac{q^{2g-\frac{h}{2}+\frac{3}{2}}}{\zeta_{\mathbb{A}}(2)} \sum_{\ell=0}^{[\frac{h}{2}]} q^\ell \sum_{D \in \mathbb{A}_{\le \ell}^+} \frac{\mu(D)}{|D| \prod_{P|D}(1+|P|)} + O(q^{\frac{3g}{2}}) \\ &= \frac{q^{2g-\frac{h}{2}+\frac{3}{2}}}{\zeta_{\mathbb{A}}(2)} \sum_{D \in \mathbb{A}_{\le [\frac{h}{2}]}^+} \frac{\mu(D)}{|D| \prod_{P|D}(1+|P|)} \sum_{\deg(D) \le \ell \le [\frac{h}{2}]} q^\ell + O(q^{\frac{3g}{2}}). \end{aligned} \tag{4.10}$$

Since

$$\sum_{\deg(D) \le \ell \le [\frac{h}{2}]} q^\ell = \zeta_{\mathbb{A}}(2) \left(q^{[\frac{h}{2}]} - q^{\deg(D)-1} \right),$$

we have

$$\begin{aligned} \mathcal{L}_{h,m}(\mu) &= q^{2g+[\frac{h}{2}]-\frac{h}{2}+\frac{3}{2}} \sum_{D \in \mathbb{A}_{\le [\frac{h}{2}]}^+} \frac{\mu(D)}{|D| \prod_{P|D}(1+|P|)} \\ &\quad - q^{2g-\frac{h}{2}+\frac{1}{2}} \sum_{D \in \mathbb{A}_{\le [\frac{h}{2}]}^+} \frac{\mu(D)}{\prod_{P|D}(1+|P|)} + O(q^{\frac{3g}{2}}). \end{aligned} \tag{4.11}$$

By Lemma 4.5 and (4.6), we have

$$\sum_{D\in\mathbb{A}^+_{\leq[\frac{h}{2}]}} \frac{\mu(D)}{|D|\prod_{P|D}(1+|P|)} = G(1) + O(q^{-\frac{g}{2}}). \tag{4.12}$$

We also have

$$\Bigg|\sum_{D\in\mathbb{A}^+_{\leq[\frac{h}{2}]}} \frac{\mu(D)}{\prod_{P|D}(1+|P|)}\Bigg| \ll \sum_{\ell=0}^{[\frac{h}{2}]}\sum_{D\in\mathcal{H}_\ell}\frac{1}{|D|} \ll g. \tag{4.13}$$

By inserting (4.12) and (4.13) into (4.11), we get

$$\mathcal{L}_{h,m}(\mu) = G(1)q^{2g+[\frac{h}{2}]-\frac{h}{2}+\frac{3}{2}} + O(gq^{\frac{3g}{2}}).$$

□

5. Proof of Theorem 3.1

In this section we give a proof of Theorem 3.1.

5.1. μ-th derivative of $L(s,\chi_u)$ for $u\in\mathcal{I}_{g+1}$

Let $u\in\mathcal{I}_{g+1}$. Then $L(s,\chi_u)$ can be represented as

$$L(s,\chi_u) = \sum_{f\in\mathbb{A}^+_{\leq g}}\chi_u(f)|f|^{-s} + q^{(1-2s)g}\sum_{f\in\mathbb{A}^+_{\leq g-1}}\chi_u(f)|f|^{s-1}.$$

Lemma 5.1. *Let $u\in\mathcal{I}_{g+1}$. For any integer $\mu\geq 0$, we have*

$$\begin{aligned}\frac{L^{(\mu)}(s,\chi_u)}{(\ln q)^\mu} =& \sum_{n=0}^{g}(-n)^\mu A_n(u)q^{-ns}\\ &+ q^{(1-2s)g}\sum_{m=0}^{\mu}\binom{\mu}{m}(-2g)^{\mu-m}\sum_{n=0}^{g-1}n^m A_n(u)q^{(s-1)n},\end{aligned}$$

where $A_n(u)=\sum_{f\in\mathbb{A}^+_n}\chi_u(f)$. In particular, we also have

$$\begin{aligned}\frac{L^{(\mu)}(\frac{1}{2},\chi_u)}{(\ln q)^\mu} =& \sum_{n=0}^{g}(-n)^\mu A_n(u)q^{-\frac{n}{2}}\\ &+ \sum_{m=0}^{\mu}\binom{\mu}{m}(-2g)^{\mu-m}\sum_{n=0}^{g-1}n^m A_n(u)q^{-\frac{n}{2}}.\end{aligned} \tag{5.1}$$

Proof. See the proof of Lemma 5.1 in [3]. □

Write

$$\mathcal{S}^{\mathrm{o}}_{h,m}(\mu) = \sum_{n=0}^{h} n^m q^{-\frac{n}{2}}\sum_{f\in\mathbb{A}^+_n}\sum_{u\in\mathcal{I}_{g+1}}\chi_u(f)$$

for $h \in \{g-1, g\}$ and $m \in \{0, 1, \ldots, \mu\}$. Then, by (5.1), we can write

$$\sum_{u \in \mathcal{I}_{g+1}} \frac{L^{(\mu)}(\frac{1}{2}, \chi_u)}{(\ln q)^\mu} = (-1)^\mu \mathcal{S}^{\mathrm{o}}_{g,\mu}(\mu) + \sum_{m=0}^{\mu} \binom{\mu}{m} (-2g)^{\mu-m} \mathcal{S}^{\mathrm{o}}_{g-1,m}(\mu).$$

5.2. Averaging $\mathcal{S}^{\mathrm{o}}_{h,m}(\mu)$

In this subsection we obtain an asymptotic formula of $\mathcal{S}^{\mathrm{o}}_{h,m}(\mu)$.

Proposition 5.2. *For $h \in \{g-1, g\}$ and $m \in \{0, 1, \ldots, \mu\}$, we have*

$$\begin{aligned} \mathcal{S}^{\mathrm{o}}_{h,m}(\mu) = {} & \frac{2^{m+1} q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \left(G(1) J_m([\tfrac{h}{2}]) + \frac{G^{(m)}(1)}{(-\ln q)^m} \right) \\ & - \frac{2^{m+1} q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \sum_{a=1}^{m+1} j_m(a) \frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^m 2^{\frac{g}{2}} q^{\frac{3}{2} g}). \end{aligned} \tag{5.2}$$

Proof. We split the sum over f with f being a perfect square of a polynomial or not. Then we can write

$$\mathcal{S}^{\mathrm{o}}_{h,m}(\mu) = \mathcal{S}^{\mathrm{o}}_{h,m}(\mu)_{\square} + \mathcal{S}^{\mathrm{o}}_{h,m}(\mu)_{\neq\square},$$

where

$$\mathcal{S}^{\mathrm{o}}_{h,m}(\mu)_{\square} = \sum_{n=0}^{h} n^m q^{-\frac{n}{2}} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \sum_{u \in \mathcal{I}_{g+1}} \chi_u(f) \tag{5.3}$$

and

$$\mathcal{S}^{\mathrm{o}}_{h,m}(\mu)_{\neq\square} = \sum_{n=0}^{h} n^m q^{-\frac{n}{2}} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \sum_{u \in \mathcal{I}_{g+1}} \chi_u(f). \tag{5.4}$$

For the contribution of non-squares, from (5.4) by using Lemma 4.1, we have

$$\begin{aligned} |\mathcal{S}^{\mathrm{o}}_{h,m}(\mu)_{\neq\square}| & \ll \sum_{n=0}^{h} n^m q^{-\frac{n}{2}} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \Bigg| \sum_{u \in \mathcal{I}_{g+1}} \chi_u(f) \Bigg| \\ & \ll g q^g \sum_{n=0}^{h} n^m q^{-\frac{n}{2}} \sum_{f \in \mathbb{A}_n^+} 2^{\frac{n}{2}} \ll g^m 2^{\frac{g}{2}} q^{\frac{3}{2} g}. \end{aligned} \tag{5.5}$$

Now, we consider the contribution of squares. We can write

$$\mathcal{S}^{\mathrm{o}}_{h,m}(\mu)_{\square} = \sum_{n=0}^{h} n^m q^{-\frac{n}{2}} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \sum_{s=0}^{g} \sum_{u \in \mathcal{I}_{(s, g+1-s)}} \chi_u(f).$$

Note that $\mathcal{I}_{(0,g+1)} = \mathcal{G}_{g+1}$. For $1 \le s \le g$ and $M \in \mathcal{B}_s$, let $\mathcal{I}_M = \{v + F : v \in \mathcal{F}_M \text{ and } F \in \mathcal{G}_{g+1-s}\}$. Then $\mathcal{I}_{(s,g+1-s)}$ is the disjoint union of the $\mathcal{I}_M$'s, where M runs over $\mathcal{B}_r$. Hence, we see that

$$\begin{aligned}
\mathcal{S}^{\mathrm{o}}_{h,m}(\mu)_{\square} &= \sum_{\ell=0}^{[\frac{h}{2}]}(2\ell)^m q^{-\ell} \sum_{L\in\mathbb{A}_\ell^+}\sum_{F\in\mathcal{G}_{g+1}} \left\{\frac{F}{L^2}\right\} \\
&\quad + \sum_{\ell=0}^{[\frac{h}{2}]}(2\ell)^m q^{-\ell} \sum_{L\in\mathbb{A}_\ell^+}\sum_{s=1}^{g}\sum_{M\in\mathcal{B}_s}\sum_{u\in\mathcal{I}_M} \left\{\frac{u}{L^2}\right\} \\
&= \sum_{\ell=0}^{[\frac{h}{2}]}(2\ell)^m q^{-\ell} \sum_{L\in\mathbb{A}_\ell^+}\sum_{F\in\mathcal{G}_{g+1}} 1 + \sum_{\ell=0}^{[\frac{h}{2}]}(2\ell)^m q^{-\ell} \sum_{L\in\mathbb{A}_\ell^+}\sum_{s=1}^{g}\sum_{\substack{M\in\mathcal{B}_s\\(M,L)=1}}\sum_{u\in\mathcal{I}_M} 1. \qquad (5.6)
\end{aligned}$$

Since $\#\mathcal{G}_{g+1} = \frac{2}{\zeta_{\mathbb{A}}(2)}q^{g+1}$, we have

$$\sum_{\ell=0}^{[\frac{h}{2}]}(2\ell)^m q^{-\ell} \sum_{L\in\mathbb{A}_\ell^+}\sum_{F\in\mathcal{G}_{g+1}} 1 = \frac{2q^{g+1}}{\zeta_{\mathbb{A}}(2)}\sum_{\ell=0}^{[\frac{h}{2}]}(2\ell)^m \ll g^m q^g. \qquad (5.7)$$

We also have, by Lemma 4.6, that

$$\begin{aligned}
&\sum_{\ell=0}^{[\frac{h}{2}]}(2\ell)^m q^{-\ell} \sum_{L\in\mathbb{A}_\ell^+}\sum_{s=1}^{g}\sum_{\substack{M\in\mathcal{B}_s\\(M,L)=1}}\sum_{u\in\mathcal{I}_M} 1 \\
&= \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\left(G(1)J_m([\tfrac{h}{2}]) + \frac{G^{(m)}(1)}{(-\ln q)^m}\right) \\
&\quad - \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\sum_{a=1}^{m+1} j_m(a)\frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^m q^{\frac{3g}{2}}). \qquad (5.8)
\end{aligned}$$

By inserting (5.7) and (5.8) into (5.6), we get

$$\begin{aligned}
\mathcal{S}^{\mathrm{o}}_{h,m}(\mu)_{\square} &= \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\left(G(1)J_m([\tfrac{h}{2}]) + \frac{G^{(m)}(1)}{(-\ln q)^m}\right) \\
&\quad - \frac{2^{m+1}q^{2g+1}}{\zeta_{\mathbb{A}}(2)}\sum_{a=1}^{m+1} j_m(a)\frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^m q^{\frac{3g}{2}}). \qquad (5.9)
\end{aligned}$$

Finally, combining (5.5) and (5.9), we obtain the result. □

5.3. Completing the proof

Recall that

$$\sum_{u\in\mathcal{I}_{g+1}} \frac{L^{(\mu)}(\frac{1}{2},\chi_u)}{(\ln q)^\mu} = (-1)^\mu \mathcal{S}^{\mathrm{o}}_{g,\mu}(\mu) + \sum_{m=0}^{\mu}\binom{\mu}{m}(-2g)^{\mu-m}\mathcal{S}^{\mathrm{o}}_{g-1,m}(\mu). \qquad (5.10)$$

By (5.9) with $h = g$ and $m = \mu$, we have that

$$(-1)^\mu \mathcal{S}^{\circ}_{g,\mu}(\mu) = \frac{(-1)^\mu 2^{\mu+1} q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \left(G(1) J_\mu([\tfrac{g}{2}]) + \frac{G^{(\mu)}(1)}{(-\ln q)^\mu} \right)$$

$$+ \frac{(-2)^{\mu+1} q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \sum_{a=1}^{\mu+1} j_\mu(a) \frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^\mu 2^{\frac{g}{2}} q^{\frac{3}{2}g}). \tag{5.11}$$

We also, by (5.9), have that

$$\sum_{m=0}^{\mu} \binom{\mu}{m} (-2g)^{\mu-m} \mathcal{S}^{\circ}_{g-1,m}(\mu)$$

$$= \frac{2^{\mu+1} q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \sum_{m=0}^{\mu} \binom{\mu}{m} (-g)^{\mu-m} \left(G(1) J_m([\tfrac{g-1}{2}]) + \frac{G^{(m)}(1)}{(-\ln q)^m} \right)$$

$$- \frac{2^{\mu+1} q^{2g+1}}{\zeta_{\mathbb{A}}(2)} \sum_{m=0}^{\mu} \binom{\mu}{m} (-g)^{\mu-m} \sum_{a=1}^{m+1} j_m(a) \frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^\mu 2^{\frac{g}{2}} q^{\frac{3}{2}g}). \tag{5.12}$$

By inserting (5.11) and (5.12) into (5.10), we complete the proof.

6. Proof of Theorem 3.2

In this section we give a proof of Theorem 3.2.

6.1. μ-th derivative of $L(s, \chi_u)$ for $u \in \mathcal{F}_{g+1}$

For $u \in \mathcal{F}_{g+1}$, $L(s, \chi_u)$ can be represented as

$$L(s, \chi_u) = \sum_{f \in \mathbb{A}^+_{\leq g}} \chi_u(f) |f|^{-s} - q^{-(g+1)s} \sum_{f \in \mathbb{A}^+_{\leq g}} \chi_u(f)$$
$$+ q^{(1-2s)g} \delta(s) \sum_{f \in \mathbb{A}^+_{\leq g-1}} \chi_u(f) |f|^{s-1} - q^{-gs} \delta(s) \sum_{f \in \mathbb{A}^+_{\leq g-1}} \chi_u(f),$$

where $\delta(s) = \frac{1-q^{-s}}{1-q^{s-1}}$.

Lemma 6.1. *Let $u \in \mathcal{F}_{g+1}$. For any integer $\mu \geq 0$, we have*

$$\frac{L^{(\mu)}(s, \chi_u)}{(-\ln q)^\mu} = \sum_{n=0}^{g} n^\mu A_n(u) q^{-ns} - (g+1)^\mu q^{-(g+1)s} \sum_{n=0}^{g} A_n(u)$$
$$+ q^{(1-2s)g} \sum_{a+b+c=\mu} \frac{\mu!}{a!b!c!} \frac{(2g)^a \delta^{(b)}(s)}{(-\ln q)^b} \sum_{n=0}^{g-1} (-n)^c A_n(u) q^{n(s-1)}$$
$$- q^{-gs} \sum_{m=0}^{\mu} \binom{\mu}{m} \frac{g^{\mu-m} \delta^{(m)}(s)}{(-\ln q)^m} \sum_{n=0}^{g-1} A_n(u),$$

where $A_n(u) = \sum_{f\in\mathbb{A}_n^+} \chi_u(f)$. *In particular, we also have*

$$\begin{aligned}\frac{L^{(\mu)}(\frac{1}{2},\chi_u)}{(-\ln q)^\mu} &= \sum_{n=0}^{g} n^\mu A_n(u) q^{-\frac{n}{2}} - (g+1)^\mu q^{-\frac{g+1}{2}} \sum_{n=0}^{g} A_n(u) \\ &\quad + \sum_{a+b+c=\mu} \frac{\mu!}{a!b!c!} \frac{(2g)^a \delta^{(b)}(\frac{1}{2})}{(-\ln q)^b} \sum_{n=0}^{g-1} (-n)^c A_n(u) q^{-\frac{n}{2}} \\ &\quad - q^{-\frac{g}{2}} \sum_{m=0}^{\mu} \binom{\mu}{m} \frac{g^{\mu-m}\delta^{(m)}(\frac{1}{2})}{(-\ln q)^m} \sum_{n=0}^{g-1} A_n(u). \end{aligned} \tag{6.1}$$

Proof. See the proof of Lemma 6.1 in [3]. □

Write

$$\mathcal{S}^{\mathrm{e}}_{h,m}(\mu) = \sum_{n=0}^{h} n^m q^{-\frac{n}{2}} \sum_{f\in\mathbb{A}_n^+} \sum_{u\in\mathcal{F}_{g+1}} \chi_u(f)$$

and

$$\mathcal{T}_h(\mu) = q^{-\frac{h+1}{2}} \sum_{n=0}^{h} \sum_{f\in\mathbb{A}_n^+} \sum_{u\in\mathcal{F}_{g+1}} \chi_u(f)$$

for $h \in \{g-1, g\}$ and $m \in \{0, 1, \dots, \mu\}$. Then, by (6.1), we can write

$$\begin{aligned}\sum_{u\in\mathcal{F}_{g+1}} \frac{L^{(\mu)}(\frac{1}{2},\chi_u)}{(-\ln q)^\mu} &= \mathcal{S}^{\mathrm{e}}_{g,\mu}(\mu) - (g+1)^\mu \mathcal{T}_g(\mu) \\ &\quad + \sum_{a+b+c=\mu} \frac{(-1)^c\mu!}{a!b!c!} \frac{(2g)^a \delta^{(b)}(\frac{1}{2})}{(-\ln q)^b} \mathcal{S}^{\mathrm{e}}_{g-1,c}(\mu) \\ &\quad - \sum_{m=0}^{\mu} \binom{\mu}{m} \frac{g^{\mu-m}\delta^{(m)}(\frac{1}{2})}{(-\ln q)^m} \mathcal{T}_{g-1}(\mu).\end{aligned}$$

6.2. Averaging $\mathcal{S}^{\mathrm{e}}_{h,m}(\mu)$

In this subsection we obtain an asymptotic formula of $\mathcal{S}^{\mathrm{e}}_{h,m}(\mu)$.

Proposition 6.2. *For* $h \in \{g-1, g\}$ *and* $m \in \{0, 1, \dots, \mu\}$, *we have*

$$\begin{aligned}\mathcal{S}^{\mathrm{e}}_{h,m}(\mu) &= \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \left(G(1) J_m([\tfrac{h}{2}]) + \frac{G^{(m)}(1)}{(-\ln q)^m} \right) \\ &\quad - \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{a=1}^{m+1} j_m(a) \frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^m 2^{\frac{g}{2}} q^{\frac{3}{2}g}).\end{aligned} \tag{6.2}$$

Proof. We can write $\mathcal{S}^{\mathrm{e}}_{h,m}(\mu) = \mathcal{S}^{\mathrm{e}}_{h,m}(\mu)_{\square} + \mathcal{S}^{\mathrm{e}}_{h,m}(\mu)_{\neq\square}$, where

$$\mathcal{S}^{\mathrm{e}}_{h,m}(\mu)_{\square} = \sum_{n=0}^{h} n^m q^{-\frac{n}{2}} \sum_{\substack{f\in\mathbb{A}_n^+ \\ f=\square}} \sum_{u\in\mathcal{F}_{g+1}} \chi_u(f) \tag{6.3}$$

and

$$\mathcal{S}^{\mathrm{e}}_{h,m}(\mu)_{\neq\square} = \sum_{n=0}^{h} n^m q^{-\frac{n}{2}} \sum_{\substack{f\in\mathbb{A}_n^+ \\ f\neq\square}} \sum_{u\in\mathcal{F}_{g+1}} \chi_u(f). \tag{6.4}$$

For the contribution of non-squares, from (6.4) by using Lemma 4.2, we have

$$\begin{aligned} |\mathcal{S}^{\mathrm{e}}_{h,m}(\mu)_{\neq\square}| &\ll \sum_{n=0}^{h} n^m q^{-\frac{n}{2}} \sum_{\substack{f\in\mathbb{A}_n^+ \\ f\neq\square}} \Bigg| \sum_{u\in\mathcal{F}_{g+1}} \chi_u(f) \Bigg| \\ &\ll q^g \sum_{n=0}^{h} n^m q^{-\frac{n}{2}} \sum_{f\in\mathbb{A}_n^+} 2^{\frac{n}{2}} \ll g^m 2^{\frac{g}{2}} q^{\frac{3}{2}g}. \end{aligned} \tag{6.5}$$

Now, we consider the contribution of square parts. For each $M \in \mathcal{B}_{g+1}$, let $\mathcal{F}_M$ be the set of rational functions $u \in \mathcal{F}_{g+1}$ whose denominator is M. Then $\mathcal{F}_{g+1}$ is a disjoint union of the $\mathcal{F}_M$'s, where M runs over $\mathcal{B}_{g+1}$. Hence, we can write

$$\begin{aligned} \mathcal{S}^{\mathrm{e}}_{h,m}(\mu)_{\square} &= \sum_{n=0}^{h} n^m q^{-\frac{n}{2}} \sum_{\substack{f\in\mathbb{A}_n^+ \\ f=\square}} \sum_{M\in\mathcal{B}_{g+1}} \sum_{u\in\mathcal{F}_M} \chi_u(f) \\ &= \sum_{\ell=0}^{[\frac{h}{2}]} (2\ell)^m q^{-\ell} \sum_{L\in\mathbb{A}_\ell^+} \sum_{\substack{M\in\mathcal{B}_{g+1} \\ (M,L)=1}} \sum_{u\in\mathcal{F}_M} 1. \end{aligned}$$

Then, by Lemma 4.7, we have

$$\begin{aligned} \mathcal{S}^{\mathrm{e}}_{h,m}(\mu)_{\square} = {}& \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \left(G(1) J_m([\tfrac{h}{2}]) + \frac{G^{(m)}(1)}{(-\ln q)^m} \right) \\ & - \frac{2^m q^{2g+2}}{\zeta_{\mathbb{A}}(2)} \sum_{a=1}^{m+1} j_m(a) \frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^m q^{\frac{3g}{2}}). \end{aligned} \tag{6.6}$$

Finally, combining (6.5) and (6.6), we obtain the result. □

6.3. Averaging $\mathcal{T}_h(\mu)$

In this subsection we obtain an asymptotic formula of $\mathcal{T}_h(\mu)$.

Proposition 6.3. *For $h \in \{g-1, g\}$, we have*

$$\mathcal{T}_h(\mu) = G(1) q^{2g+[\frac{h}{2}]-\frac{h}{2}+\frac{3}{2}} + O(2^{\frac{g}{2}} q^{\frac{3}{2}g}). \tag{6.7}$$

Proof. We can write $\mathcal{T}_h(\mu) = \mathcal{T}_h(\mu)_\square + \mathcal{T}_h(\mu)_{\neq\square}$, where

$$\mathcal{T}_h(\mu)_\square = q^{-\frac{h+1}{2}} \sum_{n=0}^{h} \sum_{\substack{f\in\mathbb{A}_n^+\\ f=\square}} \sum_{u\in\mathcal{F}_{g+1}} \chi_u(f) \tag{6.8}$$

and

$$\mathcal{T}_h(\mu)_{\neq\square} = q^{-\frac{h+1}{2}} \sum_{n=0}^{h} \sum_{\substack{f\in\mathbb{A}_n^+\\ f\neq\square}} \sum_{u\in\mathcal{F}_{g+1}} \chi_u(f). \tag{6.9}$$

For the contribution of non-squares, from (6.9) by using Lemma 4.2, we have

$$\begin{aligned} |\mathcal{T}_h(\mu)_{\neq\square}| &\ll q^{-\frac{h+1}{2}} \sum_{n=0}^{h} \sum_{\substack{f\in\mathbb{A}_n^+\\ f\neq\square}} \Big| \sum_{u\in\mathcal{F}_{g+1}} \chi_u(f)\Big| \\ &\ll q^{g-\frac{h+1}{2}} \sum_{n=0}^{h} \sum_{f\in\mathbb{A}_n^+} 2^{\frac{n}{2}} \ll 2^{\frac{g}{2}} q^{\frac{3}{2}g}. \end{aligned} \tag{6.10}$$

Now, we consider the contribution of square parts. Since $\mathcal{F}_{g+1}$ is a disjoint union of the $\mathcal{F}_M$'s, where M runs over $\mathcal{B}_{g+1}$, we can write

$$\mathcal{T}_h(\mu)_\square = q^{-\frac{h+1}{2}} \sum_{n=0}^{h} \sum_{\substack{f\in\mathbb{A}_n^+\\ f=\square}} \sum_{M\in\mathcal{B}_{g+1}} \sum_{u\in\mathcal{F}_M} \chi_u(f) = q^{-\frac{h+1}{2}} \sum_{\ell=0}^{[\frac{h}{2}]} \sum_{L\in\mathbb{A}_\ell^+} \sum_{\substack{M\in\mathcal{B}_{g+1}\\ (M,L)=1}} \sum_{u\in\mathcal{F}_M} 1.$$

Then, by Lemma 4.8, we have

$$\mathcal{T}_h(r)_\square = G(1) q^{2g+[\frac{h}{2}]-\frac{h}{2}+\frac{3}{2}} + O(gq^{\frac{3g}{2}}). \tag{6.11}$$

Finally, combining (6.10) and (6.11), we obtain the result. $\square$

6.4. Completing the proof

Recall that

$$\begin{aligned} \sum_{u\in\mathcal{F}_{g+1}} \frac{L^{(\mu)}(\frac{1}{2},\chi_u)}{(-\ln q)^\mu} &= \mathcal{S}^{\mathrm{e}}_{g,\mu}(\mu) - (g+1)^\mu \mathcal{T}_g(\mu) \\ &\quad + \sum_{a+b+c=\mu} \frac{(-1)^c \mu!}{a!b!c!} \frac{(2g)^a \delta^{(b)}(\frac{1}{2})}{(-\ln q)^b} \mathcal{S}^{\mathrm{e}}_{g-1,c}(\mu) \\ &\quad - \sum_{m=0}^{\mu} \binom{\mu}{m} \frac{g^{\mu-m}\delta^{(m)}(\frac{1}{2})}{(-\ln q)^m} \mathcal{T}_{g-1}(\mu). \end{aligned} \tag{6.12}$$

By using (6.2), we have that

$$\mathcal{S}^{\mathrm{e}}_{g,\mu}(\mu) = \frac{2^\mu q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\left(G(1)J_\mu([\tfrac{g}{2}]) + \frac{G^{(\mu)}(1)}{(-\ln q)^\mu}\right)$$
$$- \frac{2^\mu q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\sum_{a=1}^{\mu+1} j_\mu(a)\frac{G^{(a)}(1)}{(-\ln q)^a} + O(g^\mu 2^{\frac{g}{2}} q^{\frac{3g}{2}}) \tag{6.13}$$

and

$$\sum_{a+b+c=\mu}\frac{(-1)^c\mu!}{a!b!c!}\frac{(2g)^a\delta^{(b)}(\frac{1}{2})}{(-\ln q)^b}\mathcal{S}^{\mathrm{e}}_{g-1,c}(\mu)$$
$$= \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\sum_{a+b+c=\mu}\frac{\mu!}{a!b!c!}\frac{(2g)^a(-2)^c\delta^{(b)}(\frac{1}{2})}{(-\ln q)^b}\left(G(1)J_c([\tfrac{g-1}{2}]) + \frac{G^{(c)}(1)}{(-\ln q)^c}\right)$$
$$- \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}\sum_{a+b+c=\mu}\frac{\mu!}{a!b!c!}\frac{(2g)^a(-2)^c\delta^{(b)}(\frac{1}{2})}{(-\ln q)^b}\sum_{a=1}^{c+1} j_c(a)\frac{G^{(a)}(1)}{(-\ln q)^a}$$
$$+ O(g^\mu 2^{\frac{g}{2}} q^{\frac{3g}{2}}). \tag{6.14}$$

By using (6.7), we also have

$$(g+1)^\mu \mathcal{T}_g(\mu) = G(1)(g+1)^\mu q^{2g+[\frac{g}{2}]-\frac{g}{2}+\frac{3}{2}} + O(g^\mu 2^{\frac{g}{2}} q^{\frac{3g}{2}}) \tag{6.15}$$

and

$$\sum_{m=0}^{\mu}\binom{\mu}{m}\frac{g^{\mu-m}\delta^{(m)}(\frac{1}{2})}{(-\ln q)^m}\mathcal{T}_{g-1}(\mu)$$
$$= G(1)q^{2g+[\frac{g-1}{2}]-\frac{g-1}{2}+\frac{3}{2}}\sum_{m=0}^{\mu}\binom{\mu}{m}\frac{g^{\mu-m}\delta^{(m)}(\frac{1}{2})}{(-\ln q)^m} + O(g^\mu 2^{\frac{g}{2}} q^{\frac{3g}{2}}). \tag{6.16}$$

By inserting (6.13), (6.14), (6.15) and (6.16) into (6.12), we complete the proof.

7. Appendix: Non-maximal order case

In this appendix, we consider the case of non-maximal orders, as was done in [1] in odd characteristic case. We use the same notations as in [7], with minor changes.

Let $\mathcal{D}$ be the set of rational functions $\frac{D}{M^2} \in k$ such that $M \in \mathbb{A}^+$, $\gcd(D, M) = 1$, $\frac{D}{M^2} \notin \wp(k)$, $\mathrm{sgn}(D) = \xi$ if $\deg(D) = 2\deg(M)$ and $2 \nmid \deg(D)$ if $\deg(D) > 2\deg(M)$, where $\mathrm{sgn}(D)$ denotes the leading coefficient of D. Define $\sim$ on $\mathcal{D}$ by

$$\frac{D}{M^2} \sim \frac{D'}{M'^2} \quad \text{if } M = M' \text{ and } \frac{D}{M^2} + \frac{D'}{M'^2} \in \wp(k).$$

Then $\sim$ is an equivalence relation and let $[(D, M)]$ be the equivalence class containing $\frac{D}{M^2}$. Let

$$\mathcal{D}_m = \left\{[(D, M)] : M \in \mathbb{A}^+_m,\ \deg(D) < 2m\right\},$$

$$\mathcal{D}'_m = \left\{[(D,M)] : M \in \mathbb{A}_m^+,\ \deg(D) = 2m\right\},$$
$$\tilde{\mathcal{D}}_{\ell,d} = \left\{[(D,M)] : M \in \mathbb{A}_\ell^+,\ \deg(D) = 2\ell + 2d + 1\right\}$$

and

$$\tilde{\mathcal{D}}_m = \bigcup_{\ell+d=m-1} \tilde{\mathcal{D}}_{\ell,d}.$$

It is shown in [7] that

$$\#\mathcal{D}_m = \#\mathcal{D}'_m = q^{2m} - q^m, \quad \#\tilde{\mathcal{D}}_{\ell,d} = 2(q-1)q^{2\ell+d} \quad \text{and}$$
$$\#\tilde{\mathcal{D}}_m = 2(q^{2m-1} - q^{m-1}).$$

Note that $q^{2m} - q^m$ is the number of monic polynomials of degree $2m$ which is not a square. The reason for 2 in $\#\tilde{\mathcal{D}}_m$ is as follows; Write $D = AM^2 + B$ with $\deg(B) < 2\deg(M)$. 2 is the factor that whether the constant term of A is contained in $\wp(k)$ or not, just like, when $\deg(D)$ is odd, whether $\mathrm{sgn}(D)$ is a square or not in odd characteristic case. We also note that the element $[(D,M)] \in \mathcal{D}_m$ (resp. $\mathcal{D}'_m$, resp. $\tilde{\mathcal{D}}_m$) corresponds to monic nonsquare polynomial M of degree $2m$ (resp. γM, M monic polynomial of degree $2m$ for γ a generator of $\mathbb{F}_q^*$, resp. polynomial M of degree $2m-1$ with $\mathrm{sgn}(M) = 1$ or γ) in odd characteristic case.

Let

$$\sigma_n(D,M) = \sum_{N \in \mathbb{A}_n^+} \left\{\frac{D/M^2}{N}\right\}.$$

Then

$$L(s, \chi_{(D,M)}) = \sum_n \sigma_n(D,M) q^{-ns}.$$

Let

$$S_{m,n} = \sum_{[(D,M)] \in \mathcal{D}_m} \sigma_n(D,M), \quad S'_{m,n} = \sum_{[(D,M)] \in \mathcal{D}'_m} \sigma_n(D,M),$$

$$\tilde{S}_{\ell,d,n} = \sum_{[(D,M)] \in \tilde{\mathcal{D}}_{\ell,d}} \sigma_n(D,M) \quad \text{and} \quad \tilde{S}_{m,n} = \sum_{[(D,M)] \in \tilde{\mathcal{D}}_m} \sigma_n(D,M).$$

It is shown in [7, Proposition 4.3, Fact 4.8 and Proposition 4.13] that

$$S_{m,n} = S'_{m,n} = \tilde{S}_{m,n} = 0, \text{ if } n \geq 2m,$$

and, for $n < 2m$,

$$S_{m,n} = q^m \Phi(\tfrac{n}{2}, m) - \Phi(n,m) = \Phi(\tfrac{n}{2}, 2m) - \Phi(n,m), \tag{7.1}$$

$$S'_{m,n} = q^m \Phi(\tfrac{n}{2}, m) - (-1)^n \Phi(n,m) = \Phi(\tfrac{n}{2}, 2m) - (-1)^n \Phi(n,m), \tag{7.2}$$

$$\tilde{S}_{\ell,d,n} = 2(q-1)q^{\ell+d} \Phi(\tfrac{n}{2}, \ell) \quad \text{and} \quad \tilde{S}_{m,n} = 2(q-1)q^{m-1} \sum_{\ell=0}^{m-1} \Phi(\tfrac{n}{2}, \ell), \tag{7.3}$$

where

$$\Phi(a,m)=\begin{cases} q^m & \text{if } a=0,\\ \frac{q^{m+a}}{\zeta_{\mathbb{A}}(2)} & \text{if } a\neq 0, \text{ an integer},\\ 0 & \text{if } a \text{ is not an integer,}\end{cases} \tag{7.4}$$

as defined in [7, Proposition 4.4] and [1, Proposition 2.2].

Remark 7.1. We note that $S_{m,n}$ is equal to

$$\sum_{M\in\mathbb{A}_{2m}^+}\sum_{N\in\mathbb{A}_n^+}\chi_M(N),$$

and $S'_{m,n}$ is equal to

$$\sum_{M\in\gamma\mathbb{A}_{2m}^+}\sum_{N\in\mathbb{A}_n^+}\chi_M(N)$$

in odd characteristic case [1, §4, §5]. Also one can see easily ([1, §3]) that $\tilde{S}_{m,n}$ is equal to

$$2\sum_{M\in\mathbb{A}_{2m-1}^+}\sum_{N\in\mathbb{A}_n^+}\chi_M(N)+O(\delta_n q^m),$$

where δ_n is 1 or 0 according to n is even or odd.

Let B_j be the jth Bernoulli number and $\Phi(z,s,\alpha)$ be the Lerch transcendent function given by

$$\Phi(z,s,\alpha)=\sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s},$$

and $Li_s(z)$ be the polylogarithm function given by

$$Li_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s}.$$

Lemma 7.2 (Faulharber's formula ([1], (2.5), (2.6), (2.7)))**.** *We have*

$$\sum_{n=1}^{m}n^\mu=\frac{1}{\mu+1}\sum_{j=0}^{\mu}(-1)^j\binom{\mu+1}{j}B_j m^{\mu+1-j},$$

$$\sum_{n=1}^{2m-1}n^\mu q^{\frac{n}{2}}=-q^m\Phi(\sqrt{q},-\mu,2m)+Li_{-\mu}(\sqrt{q})$$

and

$$\sum_{n=1}^{2m-1}(-1)^n n^\mu q^{\frac{n}{2}}=-q^m\Phi(-\sqrt{q},-\mu,2m)+Li_{-\mu}(-\sqrt{q}).$$

Note that there is a minor error in the formula (2.6) of [1].

Theorem 7.3. *Let μ be a positive integer. Then we have*

(i)
$$\sum_{[(D,M)]\in\tilde{\mathcal{D}}_{\ell,d}} L^{(\mu)}(\tfrac{1}{2},\chi_{(D,M)})$$
$$= 2\frac{(-2\log q)^{\mu}}{\zeta_{\mathbb{A}}(2)^2(\mu+1)}q^{2\ell+d+1}\sum_{n=0}^{\mu}(-1)^n\binom{\mu+1}{n}B_n(\ell+d)^{\mu+1-n}$$
and
$$\sum_{[(D,M)]\in\tilde{\mathcal{D}}_{m}} L^{(\mu)}(\tfrac{1}{2},\chi_{(D,M)})$$
$$= 2\frac{(-2\log q)^{\mu}}{\zeta_{\mathbb{A}}(2)(\mu+1)}q^{m-1}(q^m-1)\sum_{n=0}^{\mu}(-1)^n\binom{\mu+1}{n}B_n(m-1)^{\mu+1-n}.$$

(ii)
$$\sum_{[(D,M)]\in\mathcal{D}_{m}} L^{(\mu)}(\tfrac{1}{2},\chi_{(D,M)})$$
$$= \frac{(-2\log q)^{\mu}}{\zeta_{\mathbb{A}}(2)(\mu+1)}q^{2m}\sum_{n=0}^{\mu}(-1)^n\binom{\mu+1}{n}B_n(m-1)^{\mu+1-n}$$
$$+\frac{(-\log q)^{\mu}}{\zeta_{\mathbb{A}}(2)}q^m\left(q^m\Phi(\sqrt{q},-\mu,2m)-Li_{-\mu}(\sqrt{q})\right).$$

(iii)
$$\sum_{[(D,M)]\in{\mathcal{D}'}_{m}} L^{(\mu)}(\tfrac{1}{2},\chi_{(D,M)})$$
$$= \frac{(-2\log q)^{\mu}}{\zeta_{\mathbb{A}}(2)(\mu+1)}q^{2m}\sum_{n=0}^{\mu}(-1)^n\binom{\mu+1}{n}B_n(m-1)^{\mu+1-n}$$
$$+\frac{(-\log q)^{\mu}}{\zeta_{\mathbb{A}}(2)}q^m\left(q^m\Phi(-\sqrt{q},-\mu,2m)-Li_{-\mu}(-\sqrt{q})\right).$$

Proof. Once we have the formulas for $S_{m,n}$, $S'_{m,n}$ and $\tilde{S}_{\ell,d,n}$, the computations in [1, §3-5], can be applied to this case. We will prove the first formula for the convenience of the reader. We have

$$\sum_{[(D,M)]\in\tilde{\mathcal{D}}_{\ell,d}} L^{(\mu)}(\tfrac{1}{2},\chi_{(D,M)})$$
$$= (-1)^{\mu}(\log q)^{\mu}\sum_{n=0}^{2\ell+2d+1}\tilde{S}_{\ell,d,n}n^{\mu}q^{-n/2}$$
$$= 2(q-1)(-1)^{\mu}(\log q)^{\mu}q^{\ell+d}\sum_{n=0}^{2\ell+2d+1}\Phi(\tfrac{n}{2},\ell)n^{\mu}q^{-n}\quad\text{(by (7.3))}$$
$$= 2(q-1)\frac{(-1)^{\mu}(\log q)^{\mu}}{\zeta_{\mathbb{A}}(2)}q^{2\ell+d}\sum_{n=1}^{\ell+d}(2n)^{\mu}\quad\text{(by (7.4))}.$$

We get the result by Foulhaber's formula. $\square$

References

[1] J. Andrade, *Mean values of derivatives of L-functions in function fields: II*, J. Number Theory **183** (2018), 24–39. https://doi.org/10.1016/j.jnt.2017.08.038

[2] J. Andrade, *Mean values of derivatives of L-functions in function fields: III*, Proc. Roy. Soc. Edinburgh Sect. A **149** (2019), no. 4, 905–913. https://doi.org/10.1017/prm.2018.53

[3] J. Andrade and H. Jung, *Mean values of derivatives of L-functions in function fields: IV*, J. Korean Math. Soc. **58** (2021), no. 6, 1529–1547. https://doi.org/10.4134/JKMS.j210243

[4] J. C. Andrade and J. P. Keating, *The mean value of $L(\frac{1}{2},\chi)$ in the hyperelliptic ensemble*, J. Number Theory **132** (2012), no. 12, 2793–2816. https://doi.org/10.1016/j.jnt.2012.05.017

[5] J. Andrade and S. Rajagopal, *Mean values of derivatives of L-functions in function fields: I*, J. Math. Anal. Appl. **443** (2016), no. 1, 526–541. https://doi.org/10.1016/j.jmaa.2016.05.019

[6] S. Bae and H. Jung, *Average values of L-functions in even characteristic*, J. Number Theory **186** (2018), 269–303. https://doi.org/10.1016/j.jnt.2017.10.006

[7] Y.-M. J. Chen, *Average values of L-functions in characteristic two*, J. Number Theory **128** (2008), no. 7, 2138–2158. https://doi.org/10.1016/j.jnt.2007.12.011

[8] J. B. Conrey, *The fourth moment of derivatives of the Riemann zeta-function*, Quart. J. Math. Oxford Ser. (2) **39** (1988), no. 153, 21–36. https://doi.org/10.1093/qmath/39.1.21

[9] J. B. Conrey, M. O. Rubinstein, and N. C. Snaith, *Moments of the derivative of characteristic polynomials with an application to the Riemann zeta function*, Comm. Math. Phys. **267** (2006), no. 3, 611–629. https://doi.org/10.1007/s00220-006-0090-5

[10] S. M. Gonek, *Mean values of the Riemann zeta function and its derivatives*, Invent. Math. **75** (1984), no. 1, 123–141. https://doi.org/10.1007/BF01403094

[11] A. E. Ingham, *Mean-value theorems in the theory of the Riemann zeta-function*, Proc. London Math. Soc. (2) **27** (1927), no. 4, 273–300. https://doi.org/10.1112/plms/s2-27.1.273

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