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IMPULSIVE FUZZY SOLUTIONS FOR ABSTRACT SECOND ORDER PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

DIMPLEKUMAR N. CHALISHAJAR*, R. RAMESH

Abstract. This work considers the existence and uniqueness of fuzzy solutions for impulsive abstract partial neutral functional differential systems. To establish the existence and uniqueness, we apply the concept of impulse, semi group theory and suitable fixed point theorem.

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1. Introduction

Differential equation is an integral aspect of pure and applied mathematics that is used in a variety of fields. When analyzing a real-world phenomenon, it is often important to deal with vague concepts. In this case, fuzzy set theory may be one of the better non-statistical or non-probabilistic approaches, leading us to examine fuzzy differential equations theory. A rich literature in this field justifies the significance of the branch of fuzzy differential equation in fuzzy analysis. For details, see [1, 2, 4, 5, 6, 7, 9, 11].

In recent years, the theory of impulsive differential equations has become a hot topic of research. Further, the introduction of delay in fuzzy model allows to consider more general situations. For more details, we refer the reader to [3, 10].

We have tried in this work to describe the existence of fuzzy solutions for the following second order abstract differential system using α techniques of fuzzy numbers.

$$\frac{d^2}{dt^2}(p(t)-r(t,p_t)) = Ap(t)+F(t,p_t)), t\in[0,T]=J \tag{1}$$

$$p(0)=\zeta, \tag{2}$$

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$$\frac{d}{dt}(p(t) - r(t, p_t))|_{t=0} = z \tag{3}$$

$$\Delta p(t_i) = I_i{}'(p_{t_i}), t \neq t_i \tag{4}$$

where A is a fuzzy coefficient, E^n is the set of all convex upper semi-continuously fuzzy numbers on R^n. The continuous functions $r, p, F : J \times E^n \to E^n$ are non-linear, $\zeta \in E^n$, $I_i^{'} \in C(E^n, E^n)$ are bounded functions and $\Delta p(t_i) = p(t_i^+) - p(t_i^-)$, where p_t represents the history where $p_t(\Theta) = p(t + \Theta)$, $\Theta < 0$.

So far to the best of our understanding, the existence and uniqueness of solutions for the given differential System (1) − (4) defined in the abstract form are not yet studied using fuzzy techniques, and this serves as a primary motivation for this present work. This work is structured as follows: In Section 2 some preliminary concepts of fuzzy sets and fuzzy numbers are provided, and in Section 3, the existence and uniqueness of fuzzy solutions are established for System (1) − (4).

2. Preliminaries

Here, we review few basic concepts,remarks and properties of fuzzy numbers which will be used through out this work are presented. They have been introduced to deal with imprecise numerical quantities in a practical way. A fuzzy number is a generalization of a regular, real number in that it refers to a related set of possible values, each with its own weight between 0 and 1. For more on fuzzy numbers and its properties, refer [8, 12].

Let L^n be the set of all non-empty compact, convex subsets of $\mathbb{R}^n$.For $M, N \in Q^n$ and for any $\beta \in \mathbb{R}$ the addition and multiplication operation are represented as

$$M + N = \{m + n/m \in M, n \in N\}, G = \{\beta g/g \in G\}$$

In the universe set X, a fuzzy set is defined as the mapping from $m \to [0, 1]$.Here m is assigned as the degree of membership and it's value lies between 0 and 1. For the fuzzy set m defined in n-dimensional space and for $\alpha \in (0, 1]$,we denote as,

$$[m]^\alpha = \{x \in R^n/m(x) \geq \alpha\}$$

If m be a fuzzy subset of X; the support of m, denoted as $supp(m)$, is the crisp subset of X whose elements all have nonzero membership values in m i.e., $supp(m) = \{x \in X|m(x) > 0\}$.For any $\alpha \in [0, 1]$, m is called compact if $[m]^\alpha \in L^n$.

The collection of all fuzzy sets of R^n is called as E^n which satisfies the conditions such as m is normal, fuzzy convex, upper semi-continuous and $[m]^0$ is compact. For any $m, n \in E^n$ the complete metric $\bar{d}$ is defined as

$$b_\infty(m, n) = \sup_{0<\alpha\leq 1} \bar{d}([m]^\alpha, [n]^\alpha)$$

Let $m, n \in C(J : E^n)$. Then supremum metric is defined as

$$B^{'}(m, n) = \sup_{0<\alpha\leq T} b_\infty([m]^\alpha, [n]^\alpha)$$

3. Main results

In this section, we define the existence and uniqueness of fuzzy solutions using Banach fixed point theorem for $(1) - (4)$.
We shall consider a space $\Gamma = p : J \to E^n$ to define the solution for $(1) - (4)$
Let us define $\Gamma^{'} = \Gamma \cap C([0, T] : E^n)$

Definition 3.1. A function $p : J \to E^n$ is an integral solution of System $(1)-(4)$, then

$$\begin{aligned} p(t) &= C(t)(\zeta(0) - r(0, \zeta)) + N(t)z + r(t, p_t) \\ &+ \int_0^t AN(t-s)r(s, p_s)ds + \int_0^t N(t-s)F(s, p_s))ds \\ &+ \sum_{0<t_i<t} C(t-t_i)I_i^{'}(p_{t_i}), t \in J. \end{aligned} \tag{5}$$

For the fuzzy numbers $N(t)$ and $C(t)$, we assume the followings:

(H1) Let $[N(t)]^\alpha = [N_l^\alpha(t), N_r^\alpha(t)]$, $N(0) = I$, $|N_j^\alpha(t)| \leq m_1$, $m_1 > 0$, $|AN(t)| \leq m_0, m_0 > 0\ \forall\ t \in J = [0, T]$.

(H2) Let $[C(t)]^\alpha = [C_l^\alpha(t), C_r^\alpha(t)]$, $C(0) = I$ and $C_j^\alpha(t)(j = l, r)$, $|C_j^\alpha(t)| \leq m_2$, $m_2 > 0\ \forall\ t \in J = [0, T]$.

(H3) $\exists$ positive constants $d_g, d_f > 0$ for the functions r and F which are strongly measurable satisfying the Lipschitz conditions

$$\bar{d}([r(t, p)]^\alpha, [r(t, q)]^\alpha) \leq d_g \bar{d}([p(t)]^\alpha, [q(t)]^\alpha)$$
$$\bar{d}([F(t, p)]^\alpha, [F(t, q)]^\alpha) \leq d_f \bar{d}([p(t)]^\alpha, [q(t)]^\alpha)$$

(H4) $\exists$ positive constant d_i such that

$$\bar{d}\big([I_i^{'}(p(t_i^-))]^\alpha, [I_i^{'}(q(t_i^-))]^\alpha\big) \leq d_i \bar{d}([p(t)]^\alpha, [q(t)]^\alpha)$$

(H5) $\Bigg(m_1(m_0 d_g + d_f)T + d_g + d_i\Bigg) < 1$, then System $(1) - (4)$ has a fuzzy solution which is unique.

Theorem 3.2. *If the Hypotheses $(H1) - (H5)$ holds, then the System $(1) - (4)$ has a fuzzy solution which is unique.*

Proof.

We have,

$$\begin{aligned} E_0p(t) &= C(t)(\zeta(0) - r(0,\zeta)) + N(t)z + r(t,p_t) \\ &\quad + \int_0^t AN(t-s)r(s,p_s)ds + \int_0^t N(t-s)F(s,p_s)ds \\ &\quad + \sum_{0<t_i<t} C(t-t_i)I_i^{'}(p_{t_i}), t \in J. \end{aligned}$$

Similarly,

$$\begin{aligned} E_0q(t) &= C(t)(\zeta(0) - r(0,\zeta)) + N(t)z + r(t,q_t) \\ &\quad + \int_0^t AN(t-s)r(s,q_s)ds + \int_0^t N(t-s)F(s,q_s)ds \\ &\quad + \sum_{0<t_i<t} C(t-t_i)I_i^{'}(q_{t_i}), t \in J. \end{aligned}$$

Now,

$$\begin{aligned} &\bar{d}([E_0p(t)]^\alpha, [E_0q(t)]^\alpha) \\ &\le \bar{d}\Bigg(\Bigg[C(t)(\zeta(0) - r(0,\zeta)) + N(t)z + r(t,p_t) \\ &\quad + \int_0^t AN(t-s)r(s,p_s)ds + \int_0^t N(t-s)F(s,p_s)ds + \sum_{0<t_i<t} C(t-t_i)I_i^{'}(p_{t_i})\Bigg]^\alpha, \\ &\quad \Bigg[C(t)(\zeta(0) - r(0,\zeta)) + N(t)z + r(t,q_t) \\ &\quad + \int_0^t AN(t-s)r(s,q_s)ds + \int_0^t N(t-s)F(s,q_s)ds + \sum_{0<t_i<t} C(t-t_i)I_i^{'}(q_{t_i})\Bigg]^\alpha\Bigg) \\ &\le \bar{d}\Bigg([C(t)\zeta(0)]^\alpha + [C(t)r(0,\zeta)]^\alpha + [N(t)z]^\alpha + [r(t,p_t)]^\alpha \\ &\quad + \left[\int_0^t AN(t-s)r(s,p_s)ds\right]^\alpha + \left[\int_0^t N(t-s)F(s,p_s)ds\right]^\alpha \\ &\quad + \left[\sum_{0<t_i<t} C(t-t_i)I_i^{'}(p_{t_i})\right]^\alpha, \\ &\quad [C(t)\zeta(0)]^\alpha + [C(t)r(0,\zeta)]^\alpha + [N(t)z]^\alpha + [r(t,q_t)]^\alpha \\ &\quad + \left[\int_0^t AN(t-s)r(s,q_s)ds\right]^\alpha + \left[\int_0^t N(t-s)F(s,q_s)ds\right]^\alpha \\ &\quad + \left[\sum_{0<t_i<t} C(t-t_i)I_i^{'}(q_{t_i})\right]^\alpha\Bigg) \end{aligned}$$

$$
\begin{aligned}
&\le \bar{d}([r(t,p_t)]^\alpha, [r(t,q_t)]^\alpha) \\
&\quad + \bar{d}\bigg(\bigg[\int_0^t AN(t-s)r(s,p_s)ds\bigg]^\alpha, \bigg[\int_0^t AN(t-s)r(s,q_s)ds\bigg]^\alpha\bigg) \\
&\quad + \bar{d}\bigg(\bigg[\int_0^t N(t-s)F(s,p_s)ds\bigg]^\alpha, \bigg[\int_0^t N(t-s)F(s,q_s)ds\bigg]^\alpha\bigg) \\
&\quad + \bar{d}\bigg(\bigg[\sum_{0<t_i<t} C(t-t_i)I_i^{'}(p_{t_i})\bigg]^\alpha, \bigg[\sum_{0<t_i<t} C(t-t_i)I_i^{'}(q_{t_i})\bigg]^\alpha\bigg) \\
&\le d_g\bar{d}([p(t+\Theta)]^\alpha, [q(t+\Theta)]^\alpha) \\
&\quad + m_0m_1\int_0^t d_g\bar{d}([p(s+\Theta)]^\alpha, [q(s+\Theta)]^\alpha)ds \\
&\quad + m_1\int_0^t d_f\bar{d}([p(s+\Theta)]^\alpha, [q(s+\Theta)]^\alpha)ds + d_i\bar{d}([p(t+\Theta)]^\alpha, [q(t+\Theta)]^\alpha)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&b_\infty(E_0p(t), E_0q(t)) \\
&= \sup_{0<\alpha\le 1} \bar{d}([E_0p(t)]^\alpha, [E_0q(t)]^\alpha) \\
&\quad \le d_g \sup_{0<\alpha\le 1} \bar{d}([p(t+\Theta)]^\alpha, [q(t+\Theta)]^\alpha) \\
&\quad + m_0m_1\int_0^t d_g \sup_{0<\alpha\le 1} \bar{d}([p(s+\Theta)]^\alpha, [q(s+\Theta)]^\alpha)ds \\
&\quad + m_1\int_0^t d_f \sup_{0<\alpha\le 1} \bar{d}([p(s+\Theta)]^\alpha, [q(s+\Theta)]^\alpha)ds \\
&\quad + \sup_{0<\alpha\le 1} d_i\bar{d}([p(t+\Theta)]^\alpha, [q(t+\Theta)]^\alpha)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
B^{'}(E_0p, E_0q) &= b_\infty(E_0p(t), E_0q(t)) \\
&\le d_gb_\infty([p(t+\Theta)], [q(t+\Theta)]) \\
&\quad + m_0m_1d_g\int_0^t b_\infty([p(s+\Theta)], [q(s+\Theta)])ds \\
&\quad + m_1d_f\int_0^t b_\infty([p(s+\Theta)], [q(s+\Theta)])ds \\
&\quad + d_ib_\infty([p(t+\Theta)], [q(t+\Theta)]) \\
&\le \Big(m_1(m_0d_g + d_f)T + d_g + d_i\Big)B^{'}(p,q)
\end{aligned}
$$

From Hypothesis $(H5)$, E_0 is a contraction. Therefore, by fixed point theorem due to Banach, System $(1)-(4)$ has a fuzzy solution which is unique.

4. Conclusion

In this paper, Banach fixed point theorem is employed to get the existence results for impulsive fuzzy solutions for abstract partial neutral functional differential equations. The existence theorem of solutions in the metric space of normal fuzzy convex sets with distance given by the maximum of the Hausdorff distance between level sets is also obtained using alpha cut ideas. Sufficient conditions are also determined to get the desired result. One can extend the same findings to study the controllability nature of the above systems by applying suitable fixed point theorem. Existence, uniqueness and controllability of fuzzy solutions of an of fractional impulsive abstract partial neutral functional differential equations will be the future work for the researchers in the field.

References

1. T. Allahviranloo and B. Ghanbari, *On the fuzzy fractional differential equation with interval Atangana–Baleanu fractional derivative approach*, Chaos, Solitons & Fractals **130** (2020), 93-97.
2. D.N. Chalishajar and R. Ramesh, *Fuzzy Solutions to Second Order Three Point Boundary Value Problem*, Applications & Applied Mathematics **15** (2020), 916-927.
3. E. Hernández, *Existence results for a partial second order functional differential equation with impulses*, Dynamics of continuous discrete and impulsive systems series A **14** (2007), 229-250.
4. O. Kaleva, *The Cauchy problem for fuzzy differential equations*, Fuzzy sets and systems **35** (1990), 389-396.
5. A. Khastan, & R. Rodríguez-López, *An existence and uniqueness result for fuzzy Goursat partial differential equation*, Fuzzy Sets and Systems **375** (2019), 141-160.
6. V. Lakshmikantham, & R.N. Mohapatra, *Theory of fuzzy differential equations and inclusions*, CRC press, 2004.
7. H.V. Long, J.J. Nieto, & N. Son, *New approach for studying nonlocal problems related to differential systems and partial differential equations in generalized fuzzy metric spaces*, Fuzzy Sets and Systems **331** (2018), 26-46.
8. M. Mizumoto and J. Tanaka, *Some properties of fuzzy numbers*, M.M. Gupta et al., Eds., Advances in Fuzzy Set Theory and Applications (North-Holland, New York, 1979) 153-164.
9. M. Puri, D. Ralescu, *Fuzzy random variables*, J. Math. Anal. Appl. **114** (1986), 409-422.
10. R. Ramesh, & S. Vengataasalam, *Existence and uniqueness theorem for a solution of fuzzy impulsive differential equations*, Italian Journal of Pure and Applied Mathematics **33** (2014), 345-366.
11. S. Tapaswini, S. Chakraverty, & T. Allahviranloo, *A new approach to nth order fuzzy differential equations*, Computational Mathematics and Modeling **28** (2017), 278-300.
12. G. Wang, Y. Li, C. Wen, *On fuzzy n-cell numbers and n-dimensional fuzzy vectors*, Fuzzy sets and systems **158** (2007), 71-84.

COL. Dimplekumar N. Chalishajar is a Professor of Applied Mathematics at Virginia Military Institute (VMI), USA. He did his Ph.D. from the Indian Institute of Science (IISc) and the University of Baroda, India. His fields of interest are Control Theory, Dynamical

Systems/Inclusions, Fractional-order Systems, Time and State Delay systems PDEs, Mathematical Biology, Functional Analysis, etc. He has published 75 research articles in several peer-reviewed international journals with one monogram. He is on the editorial board of more than 11 international journals and he has been serving as a reviewer in more than 25 reputed international journals. He has reviewed 170 research articles so far to help the mathematical community. He has been invited to deliver an expert lecture by several universities nationally and recognized internationally. He has delivered 70 research talks so far. He has an overall teaching experience of 26 years and he has been involved in the research for the last 27 years. He is an author/co-author of five books including APEX. He worked as a professor and head of the department for 12 years in India and he has been serving at VMI since 2008. Apart from this, he is actively involved in several administrative committees at VMI like Chair of the International Program Committee, Chair of VMIRL Liaison Committee, and Institute Review Board Committee. He is a life member of Society of Industrial and Applied Mathematics (SIAM), Mathematical Association of America (MAA), American Mathematical Society (AMS). He has visited several counties for academic purposes like France, Spain, Portugal, Romania, Ukraine, Russia, Bulgaria, Turkey, Nepal, South Korea, Japan, UK, Canada.

Department of Applied Mathematics, Virginia Military Institute(VMI), Lexington, VA 24450, USA.
e-mail: chalishajardn@vmi.edu, dipu17370@gmail.com

R. Ramesh is currently an Associate Professor at Sri Krishna College of Engineering and Technology, Coimbatore, India since 2019. He was an Associate Professor at K.S. Rangasamy College of Technology, Tiruchengode, India, from 2008 to 2019. He did his Ph.D. from Anna University, Chennai, India, and M.Phil. in Mathematics from Alagappa University, India. His research interests are Fuzzy theory and applications, functional analysis, dynamical systems, etc. He has published 30 research papers in peer reviewed international journals and reviewed 35 articles.

Department of Science and Humanities, Sri Krishna College of Engineering and Technology, Coimbatore-641008, Tamilnadu, India.
e-mail: rameshrajappa1982@gmail.com