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ON FUZZY MAXIMAL, MINIMAL AND MEAN OPEN SETS

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ABSTRACT. We have observed that there exist certain fuzzy topological spaces with no fuzzy minimal open sets. This observation motivates us to investigate fuzzy topological spaces with neither fuzzy minimal open sets nor fuzzy maximal open sets. We have observed if such fuzzy topological spaces exist and if it is connected are not fuzzy cut-point spaces. We also study and characterize certain properties of fuzzy mean open sets in fuzzy T_1-connected fuzzy topological spaces.

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1. Introduction

Origination of fuzzy sets by Zadeh[9] emerged many branches of mathematics for many decades. Chang[1] introduced fuzzy topology in 1968. Even though we discussed about fuzzy minimal and fuzzy maximal open sets by Ittanagi and Wali [2], there are some certain fuzzy sets which are neither fuzzy minimal nor fuzzy maximal called as fuzzy mean open sets[6]. From this consideration, a fuzzy open set α (resp. fuzzy closed set γ) of a fuzzy topological space X is called a fuzzy mean open set[5](resp. fuzzy mean closed set) if there exists two distinct proper fuzzy open sets $\lambda, \mu(\neq \alpha)$ (resp. two distinct proper fuzzy closed sets $\beta, \delta(\neq \gamma)$) such that $\lambda < \alpha < \mu$(resp. $\beta < \gamma < \delta$).The complement of fuzzy mean open sets is a fuzzy mean closed sets and vice-versa[6]. In current article, we investigate that there are some fuzzy topological spaces with no fuzzy minimal open sets and since we develope a fuzzy topological spaces having no fuzzy minimal and fuzzy maximal open sets; if such fuzzy topological spaces are fuzzy connected, then they are not fuzzy cut-point space. Finally, we distinguish fuzzy mean open sets in T_1- fuzzy connected topological space.

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2. Fuzzy Mean Open sets

Let us recall some basic definitions:

Definition 2.1. *(Swaminathan and Sivaraja[7]) A nonzero fuzzy clopen set δ of X is said to be a*
(i) fuzzy minimal clopen if ϑ is a fuzzy clopen such that $\vartheta < \delta$, then either $\vartheta = \delta$ otherwise $\vartheta = 0_X$.
(ii)fuzzy maximal clopen if ϑ is a fuzzy clopen such that $\delta < \vartheta$, then either $\delta = \vartheta$ otherwise $\vartheta = 1_X$.

Definition 2.2. *Let X be a nonzero fuzzy connected topological space. A fuzzy point $x_\alpha \in X$ is said to be a fuzzy cut-point[8] of X if $1_X - \{x_\alpha\}$ is fuzzy disconnected subset of X. A nonzero FCT X is called a fuzzy cut-point if each $\{x_\alpha\} \in X$ is a fuzzy-cut point of X.*

Example 2.3. (Pao-Ming and Ying-Ming[4]) Let $X = \{y, z\}$ with $y \neq z$. Let δ be fuzzy topology on X which has $\mathscr{B} = \{p_y^\lambda \,|\lambda \in (\frac{2}{3}, 1]\} \bigvee \{p_z^\lambda \,|\lambda \in (0, 1]\} \bigvee \{\phi\}$ as a base. It is observed that X is fuzzy connected topological space but not a fuzzy cut-point space. Obviously, (X, δ) is a fuzzy T_2 space.

Theorem 2.4. [8] *Let X be a fuzzy connected topological space and let x_α be a fuzzy-cut point of X such that $1_X - \{x_\alpha\} = \delta|\vartheta$. Then x_α is a fuzzy open or fuzzy closed. If $\{x_\alpha\}$ is fuzzy open (or fuzzy closed), then δ, ϑ are fuzzy closed (or fuzzy open).*

Lemma 2.5. *Let X be a T_1-fuzzy connected topological space. Then every proper fuzzy open set γ of X is both infinite and not fuzzy minimal open in X.*

Proof. Let γ be any proper fuzzy open set of T_1-fuzzy connected topological space X. If γ is finite, then $\{x_\alpha\}$ is fuzzy closed $\forall x_\alpha \in \gamma$ as X is T_1-fuzzy connected topological space implies that γ is fuzzy clopen in X and X is fuzzy disconnected which a contradiction.

If γ is infinite, then $\gamma - \{x_\alpha\} \neq 0_X$ is a fuzzy open set such that $\gamma - \{x_\alpha\} \leq \gamma$ $\forall x_\alpha \in \gamma$. Hence γ is not fuzzy minimal open in X. □

Theorem 2.6. *A proper fuzzy open set γ of a T_1-fuzzy connected topological space X is a fuzzy mean open of X iff $\gamma \neq 1_X - \{x_\alpha\}$ $\forall$ $x_\alpha \in X$.*

Proof. Let γ be a proper fuzzy open set in fuzzy connected topological space X such that $\gamma \neq 1_X - \{x_\alpha\}$, $\forall$ $x_\alpha \in X$. By deploying lemma 2.5 X is infinite and $1_X - \{x_\alpha\} \neq 0_X$ $\forall$ $x_\alpha \in X$. Hence $\gamma \neq 0_X$ and γ is not fuzzy minimal open by lemma 2.5. Again as X is infinite and $\gamma \neq 0_X, 1_X - \{x_\alpha\}, \forall x_\alpha \in X$ $\exists$ an element $y_\beta \in X$ except x_α such that $\gamma = 1_X - \{x_\alpha, y_\beta\}$ or $\gamma \lneq 1_X - \{x_\alpha, y_\beta\}$. Clearly, $1_X - \{x_\alpha, y_\beta\}$ is a proper fuzzy open set in X as X is a T_1-fuzzy topological space and $1_X - \{x_\alpha, y_\beta\}$ is not fuzzy maximal open in X. So γ is not fuzzy maximal open in X.

Contrarily, let γ be a fuzzy mean open in X. Clearly γ is neither fuzzy maximal open nor fuzzy minimal open in fuzzy connected topological space X.

As X is a T_1-fuzzy connected topological space and for any $x_\alpha \in X$, $1_X - \{x_\alpha\}$ is a fuzzy maximal open in X for any fuzzy minimal closed $\{x_\alpha\}$ in X. This implies that $\gamma \neq 1_X - \{x_\alpha\}$ $\forall x_\alpha \in X$. □

Example 2.7. (Lowen([3])) Let $E = I$ and δ be the FT generated by the subbase

$$\{\chi_0\} \bigvee \{\alpha : \alpha \textit{ is a constant}\} \bigvee \{\nu \in I^x : \nu(x) \leq x \textit{ or } 0 \ , \forall x \in X\}$$

where χ_0 is Dirac function 0. It is clear that $(X, \mathfrak{F})$ is fuzzy connected topological space and all proper fuzzy open sets are fuzzy mean open.

Theorem 2.8. *Let X be a T_1-fuzzy connected topological space and let x_α be a fuzzy cut point of X such that $1_X - \{x_\alpha\} = \delta \mid \vartheta$. Then δ, ϑ are fuzzy mean open in fuzzy topological space X.*

Proof. By deploying theorem 2.4, δ,ϑ are fuzzy open in X as $\{x_\alpha\}$ is fuzzy closed in X .Obviously $\delta \lneqq 1_X - \{x_\alpha\}$ and $\vartheta \lneqq 1_X - \{x_\alpha\}$ as$1_X - \{x_\alpha\} = \delta| \ \vartheta$. Since $1_X - \{x_\alpha\}$ is fuzzy maximal open in fuzzy connected topological space X $\forall$ $x_\alpha \in X$, δ and ϑ are not fuzzy maximal open in X. By deploying lemma 2.5 δ and ϑ are not fuzzy minimal open in X. Hence δ, ϑ are fuzzy mean open in fuzzy topological space X. □

Theorem 2.9. *Let $(X, \mathfrak{F})$ be a T_1-fuzzy connected topological space and $\mathscr{S}_{mo}$ denote the family of all fuzzy mean open set in X. Then $\mathscr{M} = \{0_X\} \vee \mathscr{S}_{mo}$ forms a basis of the FT $\mathfrak{F}$ on X.*

Proof. Let γ be any proper fuzzy open set in X. If γ is fuzzy mean open, then the proof is obvious. Otherwise,$\exists$ $x_\alpha \in X$ such that $\gamma = 1_X - \{x_\alpha\}$. Now $1_X - \{x_\alpha\}$ is proper fuzzy open in X and hence by lemma 2.5, $1_X - \{x_\alpha\}$ is infinite. Choose any $y_\beta \in 1_X - \{x_\alpha\}$ implies that $y_\beta \neq x_\alpha$. $1_X - \{x_\alpha, y_\beta\}$ is again a nonzero fuzzy open in X and hence $1_X - \{x_\alpha, y_\beta\}$ is infinite by lemma 2.5. Again choose $z_\omega \in 1_X - \{x_\alpha, y_\beta\}$, then $z_\omega \neq y_\beta \neq x_\alpha$. Hence, $\exists$ $1_X - \{z_\omega, x_\alpha\}$ a fuzzy mean open in X such that $y_\beta \in 1_X - \{z_\omega, x_\alpha\} < \gamma$. □

Theorem 2.10. *Let $(X, \mathfrak{F})$ be a fuzzy topological space such that $\mathfrak{F}$ is a finite family in X, then there exist a fuzzy minimal open set in X.*

Proof. Let F_j, $j \in \{1, 2, \ ...... \ , m\}$ be a proper fuzzy open set in fuzzy topological space $(X, \mathfrak{F})$,$\mathfrak{F} = \{0_X, F_1, F_2, ..., F_m, 1_X\}$. Let $F_k \in \{F_1, F_2, ..., F_m\}$ be arbitrary. If F_k is fuzzy minimal open for some $k \in \{1, 2, ..., m\}$, then proof is obvious. Otherwise, $\exists$ F_l a proper fuzzy open set such that $F_l < F_k$ where $l \in \{1, 2, \ ... \ , m\} - \{k\}$. If F_l is fuzzy minimal open, then proof is obvious. Otherwise, $\exists$ F_i a proper fuzzy open set such that $F_i < F_l$ where $i \in \{1, 2, ..., m\} - \{l, k\}$. Proceeding similarly, after a finite number of steps, we obtain there exists a proper fuzzy open sets $F_{k_{m-1}}$(say) such that $F_{k_{m-1}} < F_{k_{m-2}} < ..... < F_i < F_l < F_k$ where $k_{m-1} \in \{1, 2, ..., m\} - \{k, l, i...., k_{m-2}\}$. As $\{k, l, i, ..., k_{m-2}\}$ is a permutation of $\{1, 2, ..., m-1\}$ of $(m-1)$ elements, $F_{k_{m-1}}$ is a fuzzy minimal open in fuzzy topological space X. □

Corollary 2.11. *Let $(X, \mathfrak{F})$ be a fuzzy topological space such that $\mathfrak{F}$ is a finite family. Then X contains a fuzzy maximal open.*

Proof. Analogous to Theorem 2.10. □

Corollary 2.12. *Let $(X, \mathfrak{F})$ be a fuzzy disconnected topological space space X with finite number of finite fuzzy clopen sets. Then X contains both fuzzy minimal clopen and fuzzy maximal clopen sets.*

Proof. Analogous to Theorem 2.10. □

Remark 2.1. If a fuzzy topological space $(X, \mathfrak{F})$ has neither a fuzzy minimal open nor a fuzzy maximal open, then X must contain infinite number of fuzzy open sets. Further, if $(X, \mathfrak{F})$ is a fuzzy disconnected topological space having neither a fuzzy maximal clopen nor a fuzzy minimal clopen set, then X must contains an infinite number of fuzzy clopen sets.

Theorem 2.13. *Let X be a T_1-fuzzy connected topological space and $\mathscr{D}$ denote the collection of all fuzzy maximal open in X. Then $|X| = |\mathscr{D}|$.*

Proof. We know that $\mathscr{D} = \{\gamma \in \mathfrak{F} \mid \gamma = 1_X - \{x_\alpha\}, x_\alpha \in X\}$. Let us define a function $f : \mathscr{D} \to X$ such that $f(1_X - \{x_\alpha\}) = x_\alpha \ \forall \ 1_X - \{x_\alpha\} \in \mathscr{D}$. Clearly f is bijective. Consequently, $|X| = |\mathscr{D}|$. □

3. Fuzzy topological space wherein all fuzzy open sets are fuzzy mean open

Through Example 2.7, we have shown that every proper fuzzy open sets are fuzzy mean open which induces to study fuzzy topological space wherein all proper fuzzy open sets are fuzzy mean open. Since δ is fuzzy mean closed in X iff $1_X - \delta$ is fuzzy mean open . Hence, if each and every proper fuzzy open sets of a fuzzy topological space X are fuzzy mean open, then every proper fuzzy closed sets in X are fuzzy mean closed and vice versa.

Theorem 3.1. *If all proper fuzzy open sets of fuzzy connected topological space X is fuzzy mean open, then X is not a fuzzy cut-point space.*

Proof. As every proper fuzzy open sets of X are fuzzy mean open, then every proper fuzzy closed sets of X are fuzzy mean closed. Let $x_\alpha \in X$ be a fuzzy cut-point of X. By Theorem 2.4, $\{x_\alpha\}$ is either fuzzy minimal open or fuzzy minimal closed, a contradiction to the hypothesis that all proper fuzzy open sets and fuzzy closed sets are fuzzy mean open. □

From example 2.3, converse of Theorem 3.1 fails.

Remark 3.1. If a fuzzy topological space X is T_1, then every singleton set in X is fuzzy minimal closed. Hence every proper fuzzy open sets of X are not fuzzy mean open. Contradictorily, if in a fuzzy topological space X all proper fuzzy open sets are fuzzy mean open, then X is not fuzzy T_1 and hence X is not fuzzy T_i for $i \in \{2, 3, 4, 5\}$.

Remark 3.2. Consider the subspace (E, δ_E) of the fuzzy topological space (I, δ) of Example 2.7, where $E = \{\beta : \beta(x) > \frac{1}{2}, \forall x \in X\} \vee \{\beta : \beta(x) \leq \frac{1}{2}, \forall x \in X\}$. Now $\{\frac{1}{2}\}$ is a fuzzy maximal open in (E, δ_E). It violates heriditary property that every "fuzzy open sets of a fuzzy topological space (I, δ) are not fuzzy mean open."

Theorem 3.2. *Let $(X, \mathfrak{F})$, (Y, ω) be any two fuzzy topological space such that $f : X \to Y$ be a homeomorphism. All proper fuzzy open sets of $(X, \mathfrak{F})$ and (Y, ω) are fuzzy mean open set.*

Proof. Since f is fuzzy continuous, $f^{-1}(\gamma)$ is fuzzy open in X for any fuzzy open set γ in Y.Clearly $\gamma \neq 1_Y, 0_Y, f^{-1}(\gamma) \neq 1_X, 0_X$ as f is bijective. As every proper fuzzy open sets of X are fuzzy mean open, $\exists\ \lambda \neq 0_X, f^{-1}(\gamma)$ and $\mu \neq 1_X, f^{-1}(\gamma)$ is fuzzy open in X such that $\lambda < f^{-1}(\gamma) < \mu$. Hence $f(\lambda) < \gamma < f(\mu)$ as f is bijective then f is both fuzzy open and bijective, $f(\lambda) \neq 0_Y, \gamma$; $f(\mu) \neq 1_Y, \gamma$. Since $f(\lambda), f(\mu)$ are fuzzy open in Y. Hence γ is fuzzy mean open set in Y. □

Remark 3.3. From Theorem 3.2, let $(X, \mathfrak{F})$ and (Y, ω) be any two fuzzy topological space such that all proper fuzzy open sets of $(X, \mathfrak{F})$ are fuzzy mean open and $f : X \to Y$ is not a homeomorphism and all proper fuzzy open sets then (Y, ω) contains a fuzzy minimal open or a fuzzy maximal open set.

Theorem 3.3. *Let all proper fuzzy open sets of a fuzzy topological space $(X, \mathfrak{F})$ and (Y, ω) be fuzzy mean open. Then for any fuzzy product topology $\mathfrak{I}$ and all proper fuzzy open sets of $(X \times Y, \mathfrak{I})$ are also fuzzy mean open set.*

Proof. Obviously λ, μ are nonzero fuzzy open in X, Y respectively for any nonzero fuzzy open sets $\lambda \times \mu$ in $X \times Y$. Clearly, if $\lambda = 1_X$, then $\mu \neq 1_Y$ and $\lambda \neq 1_X$ when $\mu = 1_Y$. If $\lambda \neq 1_X$ then $\exists\ \sigma, \omega$ proper fuzzy open sets in X such that $\sigma \lneq \lambda \lneq \omega$ implies that $\sigma \times \mu \lneq \lambda \times \mu \lneq \omega \times \mu$. Hence $\lambda \times \mu$ is fuzzy mean open in $X \times Y$. Similarly, we can prove the same for all possible cases $\mu \neq 1_Y$ and $\lambda \neq 1_X, \mu \neq 1_Y$. □

Theorem 3.4. *Let $(X, \mathfrak{F})$ be a fuzzy topological space wherein every proper fuzzy open sets are fuzzy mean open. Then the family of all fuzzy mean open sets of X forms a fuzzy open cover of X.*

Proof. Let $\gamma = \bigvee\{\lambda \mid \lambda \in \mathscr{S}_{mo}\}$ for any $\mathscr{S}_{mo}$, the family of all fuzzy mean open sets in X . If $\gamma \neq X$, then γ is fuzzy maximal open set in X, a contradiction. □

4. Conclusion

The notion of fuzzy minimal and maximal open sets having magnificant role in order to eatablish separation axioms, compactness and connectedness. There are some family between fuzzy maximal and minimal sets which is called as fuzzy mean open sets. When we are dealing with fuzzy space X, suppose a fuzzy point or subset is removed that will be disconnected. That is personified by means of fuzzy mean open sets. As such chioce of rejection and covering of a space, one can conclude and study the properties of connectedness and compactness respectively. Therefore the fuzzy minimal, maximal and mean open sets play dominant role and in further study these notions can be investigated via various kinds of open sets.

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