J. Appl. & Pure Math. Vol. **4***(2022), No. 1 - 2, pp. 1 - 7*
https://doi.org/10.23091/japm.2022.001

MULTIPLICATIVE (GENERALIZED) (α, β)-DERIVATIONS ON LEFT IDEALS IN PRIME RINGS

FAIZA SHUJAT

Abstract. A mapping $T : R \to R$ (not necessarily additive) is called multiplicative left α-centralizer if $T(xy) = T(x)\alpha(y)$ for all $x, y \in R$. A mapping $F : R \to R$ (not necessarily additive) is called multiplicative (generalized)(α, β)-derivation if there exists a map (neither necessarily additive nor derivation) $f : R \to R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)f(y)$ for all $x, y \in R$, where α and β are automorphisms on R. The main purpose of this paper is to study some algebraic identities with multiplicative (generalized) (α, β)-derivations and multiplicative left α-centralizer on the left ideal of a prime ring R.

AMS Mathematics Subject Classification : 16W25, 16N60, 16U80.
Key words and phrases : Prime rings, multiplicative (generalized) (α, β)-derivations, multiplicative left α-centralizer, left ideals.

1. Introduction

Throughout R will represent an associative ring. A ring R is n-torsion free, if $nx = 0$ implies $x = 0$ for all $x \in R$, where $n > 1$ is an integer. A ring R is called prime if $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is called semiprime if $aRa = (0)$ implies that $a = 0$. An additive mapping $\delta : R \to R$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ holds for all pairs $x, y \in R$. We make frequent use of following commutator identities:

$$[xy, z] = [x, z]y + x[y, z] \ [x, yz] = [x, y]z + y[x, z].$$

An additive mapping $F : R \to R$ is said to be a generalized derivation if there exists an associated derivation $\delta : R \to R$ such that $F(xy) = F(x)y + x\delta(y)$ holds for all pairs $x, y \in R$. Of course a generalized derivation is a generalization of derivation. Inspired by Martindale [6], in 1991, Daif [2] has given a generalization of derivation as multiplicative derivation which is defined as: a mapping $d : R \to R$ (not necessarily additive) is said to be multiplicative derivation if

Received August 30, 2021. Revised November 14, 2021. Accepted December 7, 2021.

$d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Similar type of notation is defined in [5]. Later, in [3] Daif and Tammam have extended this notation to multiplicative generalized derivation as follows: a mapping $G : R \to R$ is a multiplicative generalized derivation if there exists a derivation g such that $G(xy) = G(x)y+xg(y)$ for all $x, y \in R$. If we consider g is any map neither necessarily derivation nor additive, then G is called multiplicative (generalized)-derivation. The concept of multiplicative (generalized)-derivation covers the concept of multiplicative generalized derivation and multiplicative centralizer (if $g = 0$). A mapping f from R to R is centralizing if $[f(x), x] \in Z(R)$ and particularly commuting if $[f(x), x] = 0$ for all $x \in R$. A mapping $T : R \to R$ (not necessarily additive) is called multiplicative left α-centralizer if $T(xy) = T(x)\alpha(y)$ for all $x, y \in R$. The multiplicative left α-centralizer covers the concepts of multiplicative left centralizer, left α-centralizer and centralizers. Posner [8] has initiated to investigate the commutativity of rings with derivations. In details, he proved that a prime ring R is commutative if there is a non zero derivation δ which is centralizing on R.

Ashraf and Rehman [1] showed that a prime ring R with a nonzero ideal I must be commutative if it admits a derivation d satisfying either of the properties $d(xy)+xy \in Z(R)$ or $d(xy)-xy \in Z(R)$ for all $x, y \in R$. A number of generalization of such kind of commutativity ideas found in [4, 10]. Recently Rehman et al. in [9] introduced a concept of multiplicative (generalized) (α, β)-derivations in rings as follows: A mapping $F : R \to R$ (not necessarily additive) is called multiplicative (generalized)(α, β)-derivation if there exists a map (neither necessarily additive nor derivation) $f : R \to R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)f(y)$ for all $x, y \in R$, where α and β are automorphisms on R. For better realization of the concept we explore the following examples.

Example 1.1. Consider the ring of all real or complex valued continuous function $\mathbb{C}[0, 1]$ and define a map such that $F(h)(y) = \left\{ \begin{array}{cc} F(y)log|h(y)|, & \text{if } h(y) \neq 0 \\ 0, & \text{otherwise} \end{array} \right\}$. In above defined function we can see $F(hg) = F(h)g+hF(g)$, for all $h, g \in \mathbb{C}[0, 1]$. But F is not additive. Also notice that we can verify the definition of multiplicative (generalized) (α, β)-derivations with $\alpha = I_{id} = \beta$, where I_{id} is an identity map.

Example 1.2. Consider a 3×3 matrix ring R as $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$.

Define $F, f : R \to R$ as $F\left[\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & a^2c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and

$$f\left[\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & a^2 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ We observe that F is a multiplicative (generalized)- derivation on R associated with the mapping f. But F is not an additive mapping and obviously neither a generalized derivation on R.

Motivated by the above line of investigation, we intrigued the structure of multiplicative (generalized) (α, β)-derivations and multiplicative left α-centralizer on the left ideal of a prime ring R. We begin with following lemma:

Lemma 1.3 (Muthana and Alkhamisi [7, Lemma 2.1])**.** *Let R be a semiprime ring, I be a non zero left ideal of R. Suppose that $\alpha, \beta : R \to R$ are two mappings such that $\beta(I) \subseteq I$. If either $[xy\alpha(z), \beta(z)] = 0$ or $x[y\alpha(z), \beta(z)] = 0$ for all $x, y, z \in I$, then $I[\alpha(z), \beta(z)] = (0)$ for all $z \in I$.*

2. Main results

Let us begin with the following theorem:

Theorem 2.1. *Let R be a prime ring, I be a non zero left ideal of R. Suppose that $F : R \to R$ is multiplicative (generalized) (α, β)-derivation associated with the map f and $T : R \to R$ is a multiplicative left α-centralizer. If F and T satisfy $F(xy) \pm T(x)T(y) = 0$ for all $x, y \in I$, then F is multiplicative left α-centralizer on I, where α and β are automorphisms on R.*

Proof. We have

$$F(xy) \pm T(x)T(y) = 0 \text{ for all } x, y \in I. \tag{1}$$

Replacing y by yz, we get

$$F(xy)\alpha(z) + \beta(x)\beta(y)f(z) \pm T(x)T(y)\alpha(z) = 0$$

for all $x, y, z \in I$. Using equation (1), we get

$$\beta(x)\beta(y)f(z) = 0$$

for all $x, y, z \in I$. Replace y by ry to get

$$\beta(x)\beta(ry)f(z) = 0$$

for all $x, y, z \in I$ and $r \in R$. i.e., $\beta(x)\beta(r)\beta(y)f(z) = 0$ for all $x, y, z \in I$ and $r \in R$. Replacing r by $\beta^{-1}(r)$, we obtain

$$\beta(x)r\beta(y)f(z) = 0$$

for all $x, y, z \in I$ and $r \in R$. Primeness of R implies that either $\beta(x) = 0$ or $\beta(y)f(z) = 0$ for all $x, y, z \in I$. If $\beta(x) = 0$ for all $x \in I$, then $\beta(I) = (0)$. Since I is nonzero left ideal of R, then we get a contradiction. So, $\beta(y)f(z) = 0$ for all $y, z \in I$, then $F(yz) = F(y)\alpha(z)$ for all $y, z \in I$, which is the required result. □

Theorem 2.2. *Let R be a prime ring, I be a non zero left ideal of R. Suppose that $F : R \to R$ is multiplicative (generalized) (α, β)-derivation associated with the map f and $T : R \to R$ is a multiplicative left α-centralizer. If F and T satisfy $F(xy) \pm T(xy) = 0$ for all $x, y \in I$, then F is multiplicative left α-centralizer on I, where α and β are automorphisms on R.*

Proof. We have
$$F(xy) \pm T(xy) = 0 \text{ for all } x, y \in I. \tag{2}$$
Substituting yz for y, we obtain
$$F(xy)\alpha(z) + \beta(x)\beta(y)f(z) \pm T(xy)\alpha(z) = 0 \tag{3}$$
for all $x, y, z \in I$. Making use of equation (2), we find
$$\beta(x)\beta(y)f(z) = 0 \tag{4}$$
for all $x, y, z \in I$. Put ry in place of y to get
$$\beta(x)\beta(r)\beta(y)f(z) = 0 \tag{5}$$
for all $x, y, z \in I$ and $r \in R$. Putting $\beta^{-1}(r)$ in place of r, we obtain $\beta(x)r\beta(y)f(z) = 0$ for all $x, y, z \in I$ and $r \in R$. Primeness of R implies that either $\beta(x) = 0$ or $\beta(y)f(z) = 0$ for all $x, y, z \in I$. If $\beta(x) = 0$ for all $x \in I$, then $\beta(I) = (0)$. Since I is nonzero left ideal of R, then we get a contradiction. So, $\beta(y)f(z) = 0$ for all $y, z \in I$, then $F(yz) = F(y)\alpha(z)$ for all $y, z \in I$, which is the required result. □

Theorem 2.3. *Let R be a prime ring, I be a non zero left ideal of R. Suppose that $F : R \to R$ is multiplicative (generalized) (α, β)-derivation associated with the map f and $T : R \to R$ is a multiplicative left α-centralizer. If F and T satisfy $F(xy) \pm T(xy) \in Z(R)$ for all $x, y \in I$, then $I[f(z), \alpha(z)] = (0)$, where α and β are automorphisms on R with $\beta(I) \subseteq I$.*

Proof. We have
$$F(xy) \pm T(xy) \in Z(R) \text{ for all } x, y, z \in I. \tag{6}$$
Replacing y by yz, we obtain
$$F(xy)\alpha(z) + \beta(xy)f(z) \pm T(xy)\alpha(z) \in Z(R) \text{ for all } x, y, z \in I,$$
which implies that
$$[F(xy) \pm T(xy)]\alpha(z) + \beta(x)\beta(y)f(z) \in Z(R) \text{ for all } x, y, z \in I.$$
Now, from equation (6), we get
$$[\beta(x)\beta(y)f(z), \alpha(z)] = 0 \text{ for all } x, y, z \in I.$$
$\beta(I) \subseteq I$ implies that
$$[xyf(z), \alpha(z)] = 0 \text{ for all } x, y, z \in I.$$
An application of Lemma 1.3 infer that
$$I[f(z), \alpha(z)] = (0) \text{ for all } z \in I,$$

which is the required result. $\square$

Theorem 2.4. *Let R be a prime ring, I be a non zero left ideal of R. Suppose that $F : R \to R$ is multiplicative (generalized) (α, β)-derivation associated with the map f and $T : R \to R$ is a multiplicative left α-centralizer. If F and T satisfy $F(xy) \pm F(x)T(y) \in Z(R)$ for all $x, y \in I$, then $I[f(z), \alpha(z)] = (0)$, where α and β are automorphisms on R with $\beta(I) \subseteq I$.*

Proof. Given that

$$F(xy) \pm F(x)T(y) \in Z(R) \text{ for all } x, y \in I. \tag{7}$$

Replacing y by yz, we find that

$$F(xyz) \pm F(x)T(yz) \in Z(R) \text{ for all } x, y, z \in I.$$

$$F(xy)\alpha(z) + \beta(xy)f(z) \pm F(x)T(y)\alpha(z) \in Z(R) \text{ for all } x, y, z \in I.$$

Use (7) to get

$$[\beta(x)\beta(y)f(z), \alpha(z)] = 0 \text{ for all } x, y, z \in I.$$

$\beta(I) \subseteq I$ implies that

$$[xyf(z), \alpha(z)] = 0 \text{ for all } x, y, z \in I.$$

An application of Lemma 1.3 yields that $I[f(z), \alpha(z)] = (0)$ for all $z \in I$, as desired. $\square$

Theorem 2.5. *Let R be a prime ring, I be a non zero left ideal of R. Suppose that $F : R \to R$ is multiplicative (generalized) (α, β)-derivation associated with the map f and $T : R \to R$ is a multiplicative left α-centralizer. If F and T satisfy $F(xy) \pm T(x)F(y) = 0$ for all $x, y \in I$, then either $F(xy) = \pm xF(y)$ or $F(xy) = F(x)\alpha(y)$, where α and β are automorphisms on R with $\beta(I) \subseteq I$.*

Proof. Given that

$$F(xy) \pm T(x)F(y) = 0 \text{ for all } x, y \in I. \tag{8}$$

Replacing y by yz, we find that

$$F(xyz) \pm T(x)F(yz) = 0 \text{ for all } x, y, z \in I.$$

$$F(xy)\alpha(z) + \beta(xy)f(z) \pm T(x)F(y)\alpha(z) \pm T(x)\beta(y)f(z) = 0 \text{ for all } x, y, z \in I.$$

Use (8) to get

$$\beta(xy)f(z) \pm T(x)\beta(y)f(z) = 0 \text{ for all } x, y, z \in I.$$

This implies that

$$\beta(x)\beta(y)f(z) \pm T(x)\beta(y)f(z) = 0 \text{ for all } x, y, z \in I.$$

$$[\beta(x) \pm T(x)]\beta(y)f(z) = 0 \text{ for all } x, y, z \in I.$$

Since $\beta(I) \subseteq I$, we can reword the above expression as

$$[x \pm T(x)]yf(z) = 0 \text{ for all } x, y, z \in I.$$

Since R is a prime ring, we conclude that either $T(x) = \mp x$ or $f(z) = 0$ by Brauer's trick. If $T(x) = \mp x$, use (8) to get $F(xy) = \pm xF(y)$ for all $x, y \in I$. Next if $f(z) = 0$, then $F(xz) = F(x)\alpha(z)$ for all $x, z \in I$, as desired. □

3. Discussion

The present manuscript is primarily addressed to algebraists whose research is connected with maps of rings (algebras) having some additional properties for example Lie and Jordan derivations, generalized derivations, automorphisms and linear preservers etc. Specifically, the paper is addressed to ring theorists dealing with polynomial (algebraic) identities and their generalizations. These identities have turned out to be applicable to certain problems in some other mathematical areas such as in operator theory, functional analysis and Lattice theory. Perhaps one might find some further connections elsewhere. We remark that at least at the level of basic definitions the theory of identities admits some parallels with that of algebraic functions. Therefore, some parts of the paper may be of some interest not only to ring theorists but analysts. In the present paper our conclusion includes the characterization of additive mappings F, T related to identities with multiplicative (generalized)(α, β)-derivations on a nonzero left ideal I of a prime ring R.

When we talked about identities we refer to an expression holding for all elements in R or all elements from a certain subset of R, which involves maps on R. The usual goal in the study of identities is to find the form of the maps involved, or, when this is not possible, to determine the structure of the ring. More precisely the commutative structure of ring.

When dealing with such identities we consider either completely arbitrary maps or sometimes we assume that they are additive (as in our case). One has to find all the trivial solutions of the proposed problems, that is, the solutions which do not depend on some structural properties of the ring but are merely consequences of a formal calculation. We call them standard solutions. The eventual existence of a nonstandard solution implies that the ring has a very special structure. So, as a limitation of our research idea, we conclude our discussion with the following example:

Example 3.1. Consider the arbitrary maps $\gamma_1, \gamma_2 : R \longrightarrow R$ and think of the identity $\gamma_1(x)y + \gamma_2(y)x = 0$ for all $x, y \in R$. A trivial possibility when this holds is that both γ_1 and γ_2 are zero. Are there any nontrivial possibilities? If R is commutative, then there certainly are (for example, take $\gamma_1(x) = -\gamma_2(x) = x$). In general, the identity $\gamma_1(x)y + \gamma_2(y)x = 0$ implies $(\gamma_1(x)yz)w = -\gamma_2(yz)xw = (\gamma_1(xw)y)z = -(\gamma_2(y)x)wz = \gamma_1(x)ywz$ for all $x, y, z, w \in R$. That is, $\gamma_1(R)R[R, R] = (0)$ and so, since R is prime, either it is commutative or $\gamma_1 = 0$. Clearly, $\gamma_1 = 0$ yields $\gamma_2 = 0$. Thus, this identity has nonstandard solutions if and only if R is commutative.

References

1. M. Ashraf and N. Rehman, *On derivations and commutativity in prime rings*, East-West J. Math. **3** (2001), 87-91.
2. M.N. Daif, *When is a multiplicative derivation additive?*, Internat. J. Math. Math. Sci. **14** (1991), 615-618.
3. M.N. Daif and M.S. Tammam El-Sayiad, *Multiplicative generalized derivations which are additive*, East-West J. Math. **9** (2007), 31-37.
4. V.D. Filippis, N. Rehman and A.Z. Ansari, *Generalized Derivations on power values of Lie ideals in prime and semiprime rings*, Internat. J. Math. Math. Sci. **4** (2014), 1-8.
5. H. Goldman and P. Šemrl, *Multiplicative derivations on* $C(X)$, Montash. Math. **121** (1996), 189-197.
6. W.S. Martindale, *When are multiplicative mappins additive?*, Proc. Amer. Math. Soc. **21** (1969), 695-698.
7. N. Muthana, and Z. Alkhamisi, *On centrally-extended multiplicative (generalized)-*(α, β)*-derivations in semiprime rings*, Hacet. J. Math. Stat. **49** (2020), 578-585.
8. E.C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093-1100.
9. N. Rehman, R.M. Al-Omary and N.H. Muthana, *A note on multiplicative (generalized)* (α, β)*-derivations in prime rings*, Annales Math. Silesianae **33** (2019), 266-275.
10. G. Scudo and A.Z. Ansari, *Generalized derivations on Lie ideals and power values on prime rings*, Math. Slovaca **65** (2015), 975-980.

Faiza Shujat received M.Sc., M.Phi. and Ph.D. at Aligarh Muslim University, India. She is currently working as associate professor at Taibah University, Madinah since 2014. Her research interests are Ring theory, Graph theory, algebra and its applications.

Department of Mathematics, Faculty of Science, Taibah University, Madinah, KSA.
e-mail: fullahkhan@taibahu.edu.sa, faiza.shujat@gmail.com