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AN INNOVATION DIFFUSION MODEL IN PARTIAL COMPETITIVE AND COOPERATIVE MARKET: ANALYSIS WITH TWO INNOVATIONS

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Abstract. An innovation diffusion model is proposed model consists of three classes, namely, a non-adopter class, adopter class innovation-I, and adopter class innovation-II in a partially competitive and cooperative market. The proposed model is analyzed with the help of the qualitative theory of a system of ordinary differential equations. Basic influence numbers associated with first and second innovation R_{0_1} and R_{0_2} respectively in the absence of each other are quantified. Then the overall basic influence number (R_0) of the system is assessed for analyzing stability in the market in different situations. Sensitivity analysis of basic influence numbers associated with first and second innovation in the absence of each other is carried out. Numerical simulation supports our analytical findings.

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Key words and phrases : Basic influence number, word of mouth effect, steady state analysis, sensitivity.

1. Introduction

The present article presents an innovation diffusion model for a partially competitive and cooperative market. The Bass model [1] has been playing a role of an appropriate model in the field of research related to innovation diffusion for the last four decades. The proliferation of a new product can be easily predicted with the help of this model. This model is still the breeding ground for the generation of new assumptions for gaining insight into the diffusion of innovations among potential buyers. In one line, it can be said that the Bass model has paved the way for the research on innovation diffusion. In the Bass model assumed that the value of market potential M is constant. Rogers [5] explained the innovation-decision process as an activity of information seeking and information processing

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with five steps: knowledge (the initial stage of the innovation-decision), persuasion (shaping attitude after knowing about innovation), decision (choosing stage to adopt or reject the innovation), implementation (individual puts innovation into practice), confirmation (individual finds support for their decision). Dhar et al. [6, 17] rationalized the five steps of the innovation diffusion process into a two steps flow process.

Wang et al. [7] proposed mathematical models with stage structure and discussed the stages of awareness, evaluation, and decision making. Kumar et al. [8] discussed a mathematical model consisting of a system of two delayed differential equations and explained the rate of change of adopter and non-adopter population and extended the work of Wang et al. [7]. The concept of logistic growth and the effect of internal and external influence on the adoption process was considered in [8]. Yu et al. [9] discussed the global stability of the innovation diffusion model with three competing products in the market. Jain et al. [10] concentrated on the effect of price on the adoption of durable products and presented diffusion models that include the price effect. Jones et al. [11] considered the effect of the distribution of the product with two parallel adoption processes, first for the retailers and second for the consumers. Lee et al. [12] analyzed the dynamic relationship between two competing markets Korean Stock Exchange (KSE) and Korean Securities Dealers Automated Quotation (KoSDAQ), at the Korean stock market by adopting the Lotka-Volterra system of equations to represent the competitive situation of the market. Tuli et al. [13] modified the model presented in [6] and classified buyers into two categories. The first category is instant buyers, and the second category is thinker buyers.

In the last decade, many realistic models have been proposed and analyzed the adopter dynamics [2, 3, 4, 16, 14, 15]. Dhar et al. [16] represented the innovation diffusion model with two innovations and three classes of the population. They discussed the different market situations under which the innovation would survive with the help of a basic influence number.

In our proposed model, we present modifications to [16] by taking into consideration an innovation diffusion model with two innovations in a partially competitive and cooperative market. The collocation of this is as follows: Section 1 is an introductory section. Section 2 contains assumptions and a mathematical model with three population classes. The study regarding basic influence numbers is presented in section 3. In section 4, stability analysis of the model has been carried out. In section 5, sensitivity analysis of basic influence numbers R_{0_1} and R_{0_2} associated with first and second innovation respectively in the absence of each other is performed. Section 6 deals with numerical validation. At last, in section 7, a brief conclusion of the findings is presented.

2. Mathematical Assumptions and Proposed Model

In this paper, we consider a three-compartment model and divide the population into three classes, namely N(t); $A_1(t)$; $A_2(t)$, where N(t) represents non-adopter population at time t, $A_1(t)$ is the adopter population of first innovation and $A_2(t)$ is the adopter population of second innovation at time t. Our proposed mathematical model is based on the following assumptions:

(1) Adopter will use only one product at a time.
(2) Λ be the constant recruitment rate of population in non-adopter class and δ be the constant death rate of all classes of the population.
(3) Word of mouth plays an important role in the diffusion of a new product in the market and switching from one product to another product. Let α_i, $(i = 1, 2)$ be the valid contact rate of non-adopters with the adopters of first and second innovation respectively.
(4) Let β_i, $(i = 1, 2)$ be the frustration rate of adopters of first and second products respectively.
(5) $A_1(t)$ and $A_2(t)$ are competing for market share with rate γ_{12} and γ_{21} respectively.
(6) $A_1(t)$ and $A_2(t)$ are cooperating for market share with rate γ_{21} and γ_{12} respectively.

Based on the modeling assumptions, the schematic flow of the system is shown in the following figure 1:

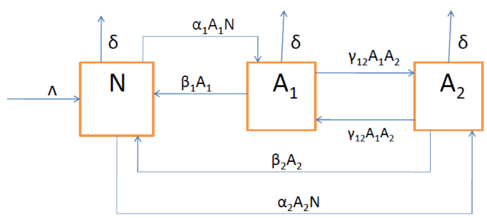

FIGURE 1. Schematic flow competitive product

Keeping in view the schematic scheme shown in figure 1, our proposed mathematical model is governed by a set of simultaneous differential equations:

$$\frac{dN}{dt} = \Lambda - \alpha_1 N A_1 - \alpha_2 N A_2 + \beta_1 A_1 + \beta_2 A_2 - \delta N, \tag{1}$$

$$\frac{dA_1}{dt} = \alpha_1 N A_1 - \beta_1 A_1 - \gamma_{12} A_1 A_2 + \gamma_{21} A_1 A_2 - \delta A_1, \tag{2}$$

$$\frac{dA_2}{dt} = \alpha_2 N A_2 - \beta_2 A_2 + \gamma_{12} A_1 A_2 - \gamma_{21} A_1 A_2 - \delta A_2, \tag{3}$$

with initial populations - $N(0) = N_0$, $A_1(0) = A_{10}$ and $A_2(0) = A_{20}$. It can be easily observed from system (1)-(3) that

$$\frac{d}{dt}(N + A_1 + A_2) \doteq \Lambda - \delta(N + A_1 + A_2),$$

and $limsup_{t\to\infty}(N + A_1 + A_2) \leq \frac{\Lambda}{d}$, which shows the upper boundedness of the system. Thus the feasible region for system is $\Omega = \{(N, A_1, A_2) : 0 \leq$

$N + A_1 + A_2 \leq \frac{\Lambda}{\delta}, N \geq 0, A_1 \geq 0, A_2 \geq 0\}$. Interior of Ω is denoted by IntΩ. Positive invariantness of the system can be verified easily (i.e., the solution satisfying initial conditions in Ω remains in Ω) concerning system (1)-(3). In the next section, we will define and calculate the basic influence number of the system (1)-(3).

3. Steady States and Basic Influence Number

The system (1)-(3) has four feasible steady states, namely:

(1) $E_1(\frac{\Lambda}{\delta}, 0, 0)$: Corresponds to adopter free situation and only non-adopter will outlast in the environment.

(2) $E_2(\widetilde{N}, \widetilde{A_1}, 0)$: Corresponds to a situation where N and A_1 will outlast in the environment and A_2 will fizzle out. Here $\widetilde{N} = \frac{\delta+\beta_1}{\alpha_1}$ and $\widetilde{A_1} = (\frac{\delta+\beta_1}{\alpha_1})(\frac{\Lambda\alpha_1}{\delta(\delta+\beta_1)} - 1)$. E_2 exists for $\frac{\alpha_1\Lambda}{\delta(\beta_1+\delta)} = R_{0_1} > 1$.

(3) $E_3(\widehat{N}, 0, \widehat{A_2})$: Corresponds to a situation where N and A_2 will outlast in the environment and A_1 will fizzle out. Here $\widehat{N} = \frac{\delta+\beta_2}{\alpha_2}$ and $\widehat{A_2} = (\frac{\delta+\beta_2}{\alpha_2})(\frac{\Lambda\alpha_2}{\delta(\delta+\beta_2)} - 1)$. E_3 exists for $\frac{\alpha_2\Lambda}{\delta(\beta_2+\delta)} = R_{0_2} > 1$.

(4) $E_4(\breve{N}, \breve{A}_1, \breve{A}_2)$: Corresponds to a situation in which both A_1 and A_2 will survive in the market. Where $\breve{N} = \frac{\delta(\beta_1-\beta_2)+\Lambda(\gamma_{12}-\gamma_{21})}{\delta(\alpha_1-\alpha_2+\gamma_{12}-\gamma_{21})}$,
$\breve{A}_1 = \frac{\delta^2(\alpha_1-\alpha_2+\gamma_{12}-\gamma_{21})+\delta\beta_2(\gamma_{12}-\gamma_{21})-\Lambda\alpha_2(\gamma_{12}-\gamma_{21})-\delta(\alpha_2\beta_1-\alpha_1\beta_2)}{\delta(\alpha_1-\alpha_2+\gamma_{12}-\gamma_{21})(\gamma_{12}-\gamma_{21})}$,
$\breve{A}_2 = \frac{\delta^2(\alpha_2-\alpha_1+\gamma_{21}-\gamma_{12})+\delta\beta_1(\gamma_{21}-\gamma_{12})-\Lambda\alpha_1(\gamma_{21}-\gamma_{12})-\delta(\alpha_1\beta_2-\alpha_2\beta_1)}{\delta(\alpha_1-\alpha_2+\gamma_{12}-\gamma_{21})(\gamma_{21}-\gamma_{12})}$.
Hence E_4 exists only when $\breve{N}, \breve{A}_1, \breve{A}_2 > 0$.

Here, the basic influence number of an innovation can be defined as the number of cases that can be generated on an average over the course of its adoption period in non-adopter population by a member of adopter class. It is helpful in determination whether or not an innovation can spread through population. It is usually denoted by R_0. When $R_0 < 1$, the innovation will run out in future according to the value of R_0. When $R_0 > 1$, the innovation will proliferate in the market according to the value of R_0. In case of equality, the situation will remain same as it is. Adopter free equilibrium of the system (1)-(3) is $E_1(\frac{\Lambda}{\delta}, 0, 0)$. Let F be the vector representing the new adopters from the direct contact (Word of mouth effect) of adopter population with non-adopter population from system equations of adopter compartments [i.e., (2)-(3)] [18, 19]. V is representating the remaining transfer terms of adopter compartments [i.e., (2)-(3)] [18, 19], we have

$$F = \begin{pmatrix} \alpha_1 N A_1 \\ \alpha_2 N A_2 \end{pmatrix}, \quad V = \begin{pmatrix} \beta_1 A_1 + \gamma_{12} A_1 A_2 - \gamma_{21} A_1 A_2 + \delta A_1 \\ \beta_2 A_2 - \gamma_{12} A_1 A_2 + \gamma_{21} A_1 A_2 + \delta A_2 \end{pmatrix}, \tag{4}$$

and jacobian matrices around the user-free equilibrium E_1 are given by

$$F = J(F)_{E_1} = \begin{pmatrix} \frac{\alpha_1\Lambda}{\delta} & 0 \\ 0 & \frac{\alpha_2\Lambda}{\delta} \end{pmatrix}, \quad V = J(V)_{E_1} = \begin{pmatrix} \beta_1 + \delta & 0 \\ 0 & \beta_2 + \delta \end{pmatrix}, \tag{5}$$

where F is non-negative and V is a non-singular matrix, FV^{-1} is non-negative and given by

$$V^{-1} = \begin{pmatrix} \frac{1}{\beta_1+\delta} & 0 \\ 0 & \frac{1}{\beta_2+\delta} \end{pmatrix}, \quad FV^{-1} = \begin{pmatrix} \frac{\alpha_1\Lambda}{\delta(\beta_1+\delta)} & 0 \\ 0 & \frac{\alpha_2\Lambda}{\delta(\beta_2+\delta)} \end{pmatrix}. \quad (6)$$

The next generation matrix for our proposed model (1)-(3) is the matrix FV^{-1}. $\rho(FV^{-1})$ is the spectral radius of matrix FV^{-1}. The basic influence number associated with our proposed model (1)-(3) is spectral radius of matrix FV^{-1} denoted by R_0. Therefore, $R_0 = Max(R_{0_1}, R_{0_2})$, where $R_{0_1} = \frac{\alpha_1\Lambda}{\delta(\beta_1+\delta)}$ and $R_{0_2} = \frac{\alpha_2\Lambda}{\delta(\beta_2+\delta)}$ are the basic influence numbers associated with first and second innovation respectively in absence of each other.

4. The Stability Analysis of the Model

In this section, we will study the stability of the all possible steady states of our proposed model in term of basic influence number(s).

Theorem 4.1. *The adopter free equilibrium $E_1(\frac{\Lambda}{\delta}, 0, 0)$ is stable for $R_{0_1} < 1$ and $R_{0_2} < 1$.*

Proof. The general variational matrix corresponding to the system is given by

$$J = \begin{pmatrix} K_1 & L_1 & M_1 \\ K_2 & L_2 & M_2 \\ K_3 & L_3 & M_3 \end{pmatrix}.$$

where $K_1 = -\delta - \alpha_1 A_1 - \alpha_2 A_2$, $K_2 = \alpha_1 A_1$, $K_3 = \alpha_2 A_2$, $L_1 = -\alpha_1 N + \beta_1$, $L_2 = \alpha_1 N - \beta_1 - \gamma_{12} A_2 + \gamma_{21} A_2 - \delta$, $L_3 = \gamma_{12} A_2 - \gamma_{21} A_2$, $M_1 = -\alpha_2 N + \beta_2$, $M_2 = -\gamma_{12} A_1 + \gamma_{21} A_1$, $M_3 = \alpha_2 N - \beta_2 + \gamma_{12} A_1 - \gamma_{21} A_1 - \delta$.

The characteristic equation about the equilibrium $E_1(\frac{\Lambda}{\delta}, 0, 0)$ is given by

$$(\delta + \lambda)[(\delta + \beta_1)(R_{0_1} - 1) - \lambda][(\delta + \beta_2)(R_{0_2} - 1) - \lambda] = 0. \quad (7)$$

The roots of the characteristic equation commonly known as characteristic values or eigen values of matrix are $\lambda_1 = -\delta$, $\lambda_2 = (\delta + \beta_1)(R_{0_1} - 1)$ and $\lambda_3 = (\delta + \beta_2)(R_{0_2} - 1)$. All the roots of equation (8) will have negative real parts when both $R_{0_1} < 1$ and $R_{0_2} < 1$. Therefore the equilibrium point E_1 is locally asymptotically stable for $R_{A_1} < 1$ and $R_{A_2} < 1$. A numerical illustration is shown in figure 2. □

Theorem 4.2. *For $R_{0_1} > 1$, the adopter-I dominating equilibrium $E_2(\widetilde{N}, \widetilde{A_1}, 0)$ is conditionally stable.*

Proof. The characteristic equation about the adopter-I dominating equilibrium $E_2(\widetilde{N}, \widetilde{A_1}, 0)$ of general variational matrix J is given by

$$(\delta+\lambda)[(\delta+\beta_1)(R_{0_1}-1)+\lambda][-(\delta+\beta_2)(R_{0_2}-1)+\frac{(\delta+\beta_1)(\alpha_2+\gamma_{21}-\gamma_{12})(R_{0_1}-1)}{\alpha_1}-\lambda] = 0. \quad (8)$$

The roots of the characteristic equation or eigen values of matrix are $\lambda_1 = -\delta$, $\lambda_2 = -(\delta+\beta_1)(R_{0_1}-1)$ and $\lambda_3 = (\delta+\beta_2)(R_{0_2}-1) - \frac{(\delta+\beta_1)(\alpha_2+\gamma_{21}-\gamma_{12})(R_{0_1}-1)}{\alpha_1}$. The real part of these roots must be negative for $R_{0_1} < 1$ and $R_{0_2} < 1+O_1(R_{0_1}-1)$ where $O_1 = \frac{(\delta+\beta_1)(\alpha+\gamma_{21}-\gamma_{12})}{\alpha(\delta+\beta_2)}$. Therefore the equilibrium point E_2 is locally asymptotically stable with conditions $R_{A_1} < 1$ and $R_{0_2} < 1+O_1(R_{0_1}-1)$. An example is shown in figure 3. □

Theorem 4.3. *For $R_{0_2} > 1$, the adopter-I dominating equilibrium $E_3(\widehat{N}, 0, \widehat{A_2})$ is conditionally stable.*

Proof. Proof of this theorem is similar to previous theorem 2. This situation has been graphically shown in figure 4 of numerical simulation section. □

Theorem 4.4. *The positive interior equilibrium E_4 =($\breve{N}$, $\breve{A}_1$, $\breve{A}_2$) is conditionally unstable.*

Proof. The variational matrix around the positive interior equilibrium E_4 =($\breve{N}$, $\breve{A}_1$, $\breve{A}_2$) is given by

$$J=\begin{pmatrix} -\delta-\alpha_1\breve{A}_1-\alpha_2\breve{A}_2 & -\alpha_1\breve{N}+\beta_1 & -\alpha_2\breve{N}+\beta_2 \\ -\alpha_1\breve{A}_1 & j_{22} & -\gamma_{12}\breve{A}_1+\gamma_{21}\breve{A}_1 \\ -\alpha_2\breve{A}_2 & \gamma_{12}\breve{A}_2-\gamma_{21}\breve{A}_2 & j_{33} \end{pmatrix}.$$

where $j_{22} = -\delta-\beta_1+\alpha_1\breve{N}-\gamma_{12}\breve{A}_2+\gamma_{21}\breve{A}_2$ and
$j_{33} = -\delta-\beta_2+\alpha_2\breve{N}+\gamma_{12}\breve{A}_1-\gamma_{21}\breve{A}_1$.

Then the characteristic equation about the equilibrium $E_4(\breve{N}, \breve{A}_1, \breve{A}_2)$ be given by

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0. \tag{9}$$

where $a_1 = \delta+\alpha_1\breve{A}_1+\alpha_2\breve{A}_2$ and $a_3 = -\delta\breve{A}_1\breve{A}_2(-\alpha_1\gamma_{12}+\alpha_2\gamma_{12}-\gamma_{12}^2+\alpha_1\gamma_{21}-\alpha_2\gamma_{21}+2\gamma_{12}\gamma_{21}-\gamma_{21}^2)$ under the conditions $\alpha_1\breve{N}-\gamma_{12}\breve{A}_2+\gamma_{21}\breve{A}_2 = \delta+\beta_1$ and $\alpha_2\breve{N}+\gamma_{12}\breve{A}_1-\gamma_{21}\breve{A}_1 = \delta+\beta_2$. Clearly $a_1 > 0$. Also $a_3 < 0$ under the conditions $(\gamma_{12}-\gamma_{21})(\alpha_2+\gamma_{21}-\alpha_1-\gamma_{12}) > 0$. Hence from the Routh-Hurwitz Criterion the system is unstable around E_4 with the conditions $\alpha_1\breve{N}-\gamma_{12}\breve{A}_2+\gamma_{21}\breve{A}_2 = \delta+\beta_1$, $\alpha_2\breve{N}+\gamma_{12}\breve{A}_1-\gamma_{21}\breve{A}_1 = \delta+\beta_2$ and $(\gamma_{12}-\gamma_{21})(\alpha_2+\gamma_{21}-\alpha_1-\gamma_{12}) > 0$. □

5. Sensitivity Analysis

In this section, we perform the sensitivity analysis of basic influence numbers R_{0_1} and R_{0_2} associated with first and second innovation respectively in the absence of each other. Sensitivity analysis tells the importance of each parameter in the innovation diffusion dynamics. It is used to discover parameters that have a high impact on basic influence number. The basic influence number R_{0_1} is a function of four parameters. The normalized sensitive indices of basic influence numbers R_{0_1} w.r.t. parameters are shown in the table 1.

From the table 1, it is clear that Λ and α_1 have positive impact on R_{0_1}, while β_1 and δ have negative impact. For example, 10% increase(decrease) in Λ, results in 10% increase(decrease) in R_{0_1}. On the other hand, 10% increase(decrease)

Table 1. The sensitivity indices $\gamma_{y_j}^{R_{0_1}} = \frac{\partial R_{0_1}}{\partial y_j} \times \frac{y_j}{R_{0_1}}$ of the basic influence number R_{0_1} to the values of various parameters y_j, are given in the following table

Parameter(y_j)	Parametric Value	Sensitivity index of R_{0_1} (R_{0_2}) w.r.t. y_j
Λ	0.30	1
α_1 (α_2)	0.20	1
β_1 (β_2)	0.10	-0.33
δ	0.20	-1.66

in δ, results in 16.66% decrease(increase) in R_{0_1}. Similarly Λ and α_2 have positive impact on R_{0_2}, while β_2 and δ have negative impact. Hence, we observe significant change in R_{0_1} (R_{0_2}) by small changes in these parameters.

6. Numerical Simulations

To verify the analytic results of the model (1)-(3), numerical simulations are performed by applying MATLAB. It provides an impression of completeness to the analytic results. It is observed that adopter free equilibrium $E_1(\frac{\Lambda}{\delta}, 0, 0)$ is stable for the value of parameters of Set-1. It is graphically shown in figure 2. Adopter-I dominating equilibrium $E_2(\widetilde{N}, \widetilde{A_1}, 0)$ is stable for the value of parameters of Set-2. It is graphically shown in figure 3. Adopter-II dominating equilibrium $E_3(\widehat{N}, 0, \widehat{A_2})$ is stable for value of parameters of Set-3. It is graphically shown in figure 4.

Table 2. Values of different parameters

Set No.	Λ	α_1	α_2	β_1	β_2	γ_{12}	γ_{21}	δ
Set-1	0.20	0.15	0.25	0.10	0.15	0.30	0.41	0.20
Set-2	0.75	0.20	0.25	0.15	0.70	0.30	0.41	0.30
Set-3	0.75	0.25	0.20	0.75	0.15	0.30	0.41	0.20

Table 3. Stable of equilibrium w.r.t. R_{0_1}, R_{0_2}

Parameters	R_{A_1}	R_{A_2}	Stable equilibrium	Figure
Set-1	0.50	0.71	E_1=(1, 0, 0)	2
Set-2	1.11	0.625	E_2=(2.25, 0.2475, 0)	3
Set-3	0.59	1.11	E_3=(2.25, 0, 0.2475)	4

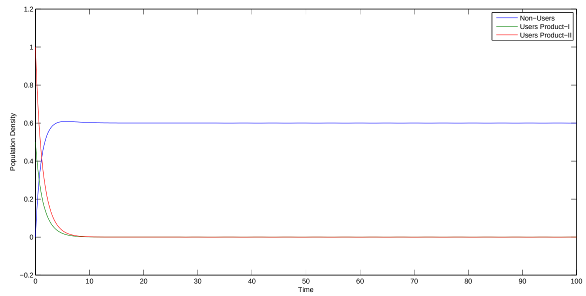

Figure 2. Population densities corresponds to Adopter-free Equilibrium

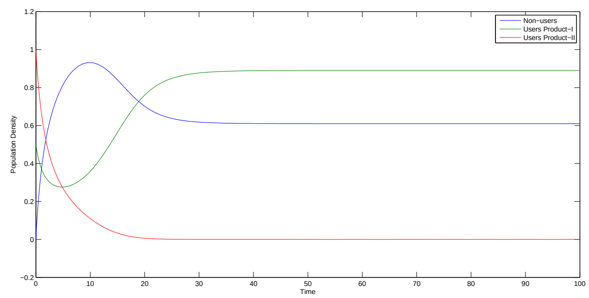

Figure 3. Population densities corresponds to Adopter-I Dominating Equilibrium

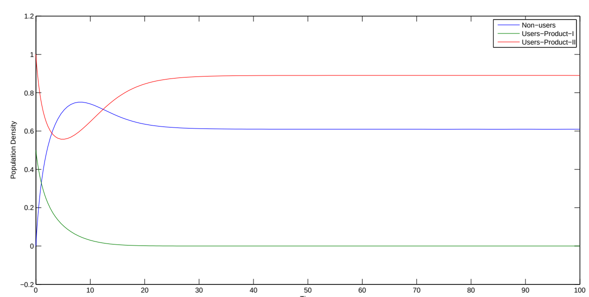

Figure 4. Population densities corresponds to Adopter-II Dominating Equilibrium

7. Conclusions

The present model concluded that the survivability of a product in the market mainly depends upon word of mouth, immigration, emigration, etc. The present article discussed an analytical formulation with two simultaneous competing products to capture the demand of the non-adopter population. The system has four steady states, and the local stability of each equilibrium has been examined. It is shown that the adopter-free steady-state E_1 is conditionally stable in which both products vanished from the market. The other two boundary equilibria E_2 and E_3 are also locally asymptotically stable under certain conditions stated in theorem-2 and theorem-3. Since the interior equilibrium E_4 is conditionally unstable, the co-existence of the users of both products in the market is rarely possible at the steady-state under some conditions. In this scenario, only one product will be served in the market in the long run.

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