KYUNGPOOK Math. J. 62(2022), 1-28
https://doi.org/10.5666/KMJ.2022.62.1.1
pISSN 1225-6951 eISSN 0454-8124

On the Decomposition of Cyclic G-Brauer's Centralizer Algebras

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Abstract. In this paper, we define the G-Brauer algebras $D_f^G(x)$, where G is a cyclic group, called cyclic G-Brauer algebras, as the linear span of r-signed 1-factors and the generalized m, k signed partial 1-factors is to analyse the multiplication of basis elements in the quotient $\overrightarrow{I_f}^G(x, 2k)$. Also, we define certain symmetric matrices $\overrightarrow{T}^{[\lambda]}_{m,k}(x)$ whose entries are indexed by generalized m, k signed partial 1-factor. We analyse the irreducible representations of $D_f^G(x)$ by determining the quotient $\overrightarrow{I_f}^G(x, 2k)$ of $D_f^G(x)$ by its radical. We also find the eigenvalues and eigenspaces of $\overrightarrow{T}^{[\lambda]}_{m,k}(x)$ for some values of m and k using the representation theory of the generalised symmetric group. The matrices $T^{[\lambda]}_{m,k}(x)$ whose entries are indexed by generalised m, k signed partial 1-factors, which helps in determining the non semisimplicity of these cyclic G-Brauer algebras $D_f^G(x)$, where $G = \mathbb{Z}_r$.

1. Introduction

The invariant theory of classical groups, algebraic Lie theory, algebraic number theory, knot theory, integrable models and statistical mechanics, quantum computing are the few areas of diagram algebras arising in different areas of mathematics and physics. In order to characterise invariants of classical groups acting on tensor powers of the vector representations, Brauer [2] introduced a new class of algebras called Brauer algebras. The Brauer algebras used graphs to represent its basis. Hence it can be considered as a class of diagram algebras, that are finite dimensional algebras whose basis consists of diagrams. These basis have interesting combinatorial properties to be studied in their own right.

Parvathi and Kamaraj [10] introduced a new class of algebras called signed Brauer algebras $S_f^{(x)}$ which are a generalization of Brauer algebras. Parvathi and Selvaraj [12] studied signed Brauer algebras as a class of centraliser algebras, which are the direct product of orthogonal groups over the field of real numbers $\mathbb{R}$. Par-

* Corresponding Author.
Received January 20, 2020; accepted February 8, 2021.
2010 Mathematics Subject Classification: 05E10, 20C30.
Key words and phrases: G-Brauer algebras, centraliser algebras, eigenvalues.

vathi and Savithri [11] introduced a new class of algebras called G-Brauer algebras $D_f^G(x)$, where G is abelian, which are a generalization of signed Brauer algebras $S_f^{(x)}$ introduced by [10] and Brauer algebras.

Brown [3, 4], Hanlon and Wales [6, 7] and Wenzl [14] studied the Brauer algebras by using diagrams to represent its basis and Young diagrams to represent its irreducible representations. Brown has not discussed the structure of Brauer algebras when the radical is non zero. This study was carried out by Hanlon and Wales in [6]. They determined the structure of the radical of a non-semisimple Brauer algebras by introducing the notion of 1-factor, m, k-partial 1-factor and the combinatorially defined matrix $T_{m,k}^{\lambda}(x)$. In [7], they used these matrices to find the eigen values and eigen vectors corresponding to Brauer's centraliser algebras.

However for signed Brauer algebras, the eigen values corresponding to a non-semisimple signed Brauer algebras have not been dealt completely. This motivated us to study the eigen values for the signed Brauer algebras [13] and G-Brauer algebras where $G = \mathbb{Z}_r$.

In this paper, we analyse the irreducible representations of $D_f^G(x)$ by determining the quotient $\overrightarrow{I_f}^G(x, 2k)$ of $D_f^G(x)$ by its radical. We also find the eigenvalues and eigenspaces of certain symmetric matrices $\overrightarrow{T}_{m,k}^{[\lambda]}(x)$ for some values of m and k using the representation theory of the generalised symmetric group. The matrices $T_{m,k}^{[\lambda]}(x)$ whose entries are indexed by generalised m, k signed partial 1-factors, which helps in determining the non semisimplicity of these cyclic G-Brauer algebras $D_f^G(x)$, where $G = \mathbb{Z}_r$. In this paper G refers to $\mathbb{Z}_r,\ r > 0$.

2. Preliminaries

We begin by recalling some known results in the representation theory of the generalised symmetric group [1, 5, 8].

Definition 2.1. For each standard multi-tableaux $[s]$ and $[t]$ of shape $[\lambda]$,

$$m_{[s][t]} = \pi \sum_{\sigma \in R_{[t]}} \sigma,\ \pi \in D([t])$$

where $\pi[s] = [t]$, $D([t])$ is the left coset representative of $R_{[t]}$, row stabiliser of $[t]$.

Definition 2.2. For each multi-partition $[\lambda]$, there exists two sided ideals $K^{[\lambda]}$ and $\overline{K^{[\lambda]}}$ of $\widehat{\widehat{S}}_m$, group algebras of the generalised symmetric group.

1. $K^{[\lambda]}$ = Linear span $\{m_{[s][t]}$ / $[s]$ and $[t]$ are standard multi-tableaux of shape $[\mu]$ for $[\mu] \trianglerighteq [\lambda]\}$.

2. $\overline{K^{[\lambda]}}$ = Linear span $\{m_{[s][t]}$ / $[s]$ and $[t]$ are standard multi-tableaux of shape $[\mu]$ for $[\mu] \triangleright [\lambda]\}$.

3. $\widehat{\widehat{S}}_m^{[\lambda]}$ = Linear span$\{m_{[s][t]}$ / $[s]$ and $[t]$ are standard multi-tableaux of shape $[\lambda]\}$.

Remark 2.3.
- For each multi-partition $[\lambda]$ of m, let $\overrightarrow{S}^{[\lambda]}$ denote the Specht module corresponding to $[\lambda]$ and let $d_{[\lambda]}$ denote the dimension of $\overrightarrow{S}^{[\lambda]}$.
- The ideal $\widehat{\widehat{S}}_m^{[\lambda]}$ considered as a vector space of linear transformations of $\overrightarrow{S}^{[\lambda]}$ is the full matrix algebras End$(\overrightarrow{S}^{[\lambda]})$.

Definition 2.4. There exists left ideals $\overrightarrow{I}_1, \overrightarrow{I}_2, \dots, \overrightarrow{I}_{d_{[\lambda]}}$, right ideals $\overrightarrow{J}_1, \overrightarrow{J}_2, \dots, \overrightarrow{J}_{d_{[\lambda]}}$ and the unique minimal two sided ideal $\widehat{\widehat{S}}_m^{[\lambda]}$ of $\widehat{\widehat{S}}_m$, group algebras of the generalised symmetric group. $\widehat{\widehat{S}}_m^{[\lambda]}$ can be written either as direct sum of simple left ideals

$$\widehat{\widehat{S}}_m^{[\lambda]} = \overrightarrow{I}_1 \oplus \cdots \oplus \overrightarrow{I}_{d_{[\lambda]}},$$

or as a direct sum of simple right ideals

$$\widehat{\widehat{S}}_m^{[\lambda]} = \overrightarrow{J}_1 \oplus \cdots \oplus \overrightarrow{J}_{d_{[\lambda]}}$$

where each $\overrightarrow{I}_i$ is a left ideal of $\widehat{\widehat{S}}_m$ for which multiplication on the left gives a representation isomorphic to $\overrightarrow{S}^{[\lambda]}$ and each $\overrightarrow{J}_i$ is a right ideal of $\widehat{\widehat{S}}_m$ for which right multiplication is isomorphic to $\overrightarrow{S}^{[\lambda]}$.

There exists a basis $A_1, \dots, A_{d_{[\lambda]}}$ for $\overrightarrow{S}^{[\lambda]}$ with respect to which the matrices $\Psi_{[\lambda]}(\sigma)$ for $\sigma \in \widehat{\widehat{S}}_m$ acting on $\overrightarrow{S}^{[\lambda]}$ are orthogonal. i.e. $\Psi_{[\lambda]}(\sigma^{-1}) = \Psi_{[\lambda]}(\sigma)^t$. For any elements $x_i, y_j \in \widehat{\widehat{S}}_m^{[\lambda]}, 1 \le i, j \le d_{[\lambda]}$, choose x_i, y_j so that

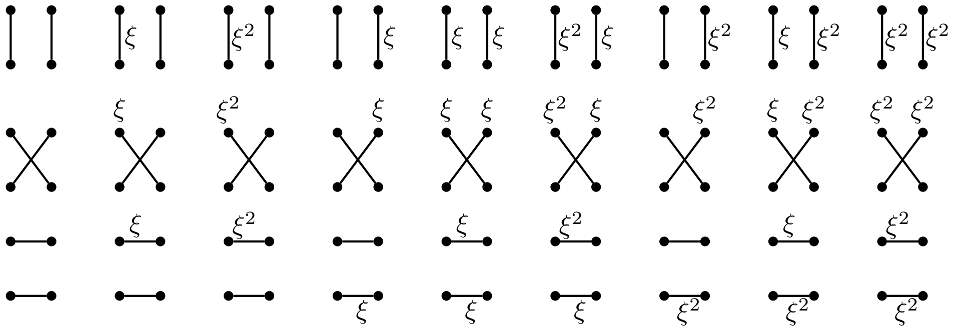
$$x_i y_j x_r y_l = \begin{cases} x_i y_l, & \text{if } j = r; \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

3. The Structure of $D_f^G(x)$

Definition 3.1. A r-signed 1-factor on $2f$ vertices is a signed diagram with f vertices arranged in two rows and f, $r-$signed edges such that each vertex is incident to exactly one $r-$signed edge, labeled by the primitive r^{th}-root of unity $\xi^1, \xi^2, \dots, \xi^r$. A $r-$signed edge is said to be an $i-$edge if it is labeled by ξ^i. The set of all r-signed $1-$factor on $2f$ vertices is denoted by P_f^G, where G is assumed to be a cyclic group.

A r-signed 1-factor $\delta \in P_f^G$ will be represented as a diagram having two rows of f vertices each, the f vertices in the top row are labeled by $1, 2, \ldots, f$ from left to right and the f vertices in the bottom row are labeled by $f+1, f+2, \ldots, 2f$ from left to right. There are $r^f(2f-1)!! = r^f \cdot 1 \cdot 3 \cdots (2f-1)$ ways of joining these $2f$ vertices which is incident to exactly to one $r-$signed edge.

Example 3.2. The 27 r-signed 1-factors in $P_2^{\mathbb{Z}_3}$ is as follows.



An r-signed edge of $\delta \in P_f^G$ is called r-signed horizontal edge if it joins two vertices in the same row of $\delta \in P_f^G$.

An r-signed edge of $\delta \in P_f^G$ is called r-signed vertical edge if it joins two vertices in different rows of $\delta \in P_f^G$.

Definition 3.3. Let $\overrightarrow{V_f}$ be the vector space over a field K with P_f^G as its basis.

Definition 3.4. Let $\overrightarrow{V_f}(2k)$ =Linear span$\{\delta \in P_f^G$/the number of $r-$signed horizontal edges in $\delta \geq 2k\}$, $k = 0, 1, \ldots, \left\lfloor \frac{f}{2} \right\rfloor$ which is a subspace of $\overrightarrow{V_f}$. $\overrightarrow{V_f}(2k)$ can be written as direct sum of vector subspaces $\overrightarrow{V_f^*}(2m)$, $k \leq m \leq \left\lfloor \frac{f}{2} \right\rfloor$ spanned by all $r-$signed $1-$factors having exactly $2m$ $r-$signed horizontal edges.

$$\overrightarrow{V_f}(2k) = \overrightarrow{V_f}^*(2k) \oplus \overrightarrow{V_f}^*(2k+2) \oplus \ldots \oplus \overrightarrow{V_f}^*\left(2\left\lfloor \frac{f}{2} \right\rfloor\right).$$

In the following, we make $\overrightarrow{V_f}$ as an algebra over the field $K(x)$, where K is any field and x is an indeterminate, by defining composition of two elements $\delta_1, \delta_2 \in P_f^G$.

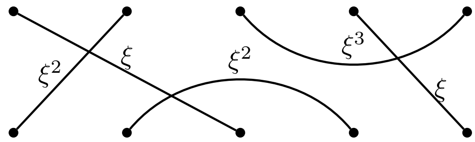
For $\delta_1, \delta_2 \in P_f^G$, the graph $\overrightarrow{U}_{\delta_2}^{\delta_1}$ with $3f$ vertices arranged in three rows with the first row, the top row of δ_1, the second row is obtained by identifying the vertices in the bottom row of δ_1 with the vertices in the top row of δ_2 and the third row, the bottom row of δ_2. The graph $\overrightarrow{U}_{\delta_2}^{\delta_1}$ consists of exactly f, r-signed paths $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_f$, some number $m_i(\delta_1, \delta_2))$ of $i-$cycles for $i = 1, 2, \ldots, r$ such that

1. The $r-$signed path $\mathcal{P}_i$ contains one or more $r-$signed edges. The initial and endpoint of $\mathcal{P}_i$ does not meet each other.

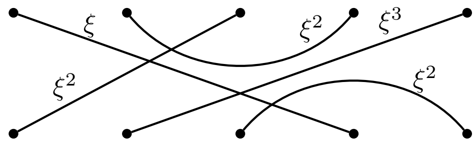
2. Each $i-$cycle is of even length consisting some number k_j of i_j-edges, $1 \leq i_j \leq r$ with $\sum k_j i_j \equiv i(\mod r)$, entirely of vertices lying in the middle row.

Example 3.5. For $\delta_1, \delta_2 \in P_5^{\mathbb{Z}_3}$, the diagram in $\overrightarrow{U}^{\delta_1}_{\delta_2}$ is

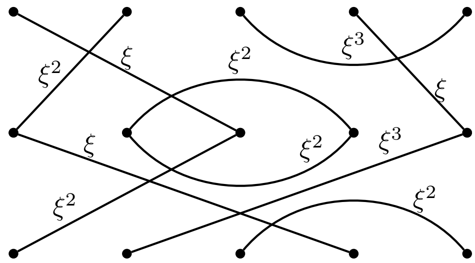
$\delta_1 =$



$\delta_2 =$



$\overrightarrow{U}^{\delta_1}_{\delta_2} =$



Definition 3.6. Let δ_1 and δ_2 be r-signed 1-factors in P_f^G. Define the composition of $r-$signed diagrams $\delta_1 \circ \delta_2$ to be the r-signed $1-$factor in the following way

1. The top (respectively bottom) row of $\delta_1 \circ \delta_2$ have the same r-signed horizontal edges in the top (respectively bottom) row of δ_1 (respectively δ_2).

2. The vertices u and v are adjacent if and only if there is a r-signed path $\mathcal{P}_i$ in $\overrightarrow{U}^{\delta_1}_{\delta_2}$ joining u to v and an edge joining u to v is an $i-$edge if the path contains some number k_j of i_j-edges, $1 \leq i_j \leq r$ with $\sum k_j i_j \equiv i(\mod r)$.

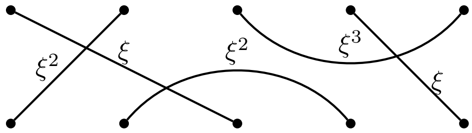
Definition 3.7. The cyclic G-Brauer algebra $D_f^G(x)$ is an associative algebras over the field $K[x]$ with basis P_f^G and the multiplication $*$ of r-signed 1-factors given by

$$\delta_1 * \delta_2 = x^{\sum_{i=1}^{r} i m_i(\delta_1, \delta_2)} (\delta_1 \circ \delta_2), \text{ where } G = \mathbb{Z}_r$$

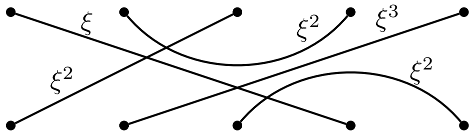
This algebras $D_f^G(x)$ is called the G-Brauer algebras defined in [11], when $G = \mathbb{Z}_r$.

Example 3.8. For $\delta_1, \delta_2 \in P_5^{\mathbb{Z}_3}$, the diagram in $\delta_1 * \delta_2$ is

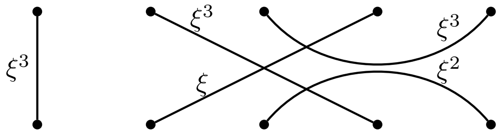
$\delta_1 =$



$\delta_2 =$



$\delta_1 \star \delta_2 = x$



4. The Structure of Ideals $I_f^G(x, 2k)$

Let $I_f^G(x, 2k)$ =Linear span$\{\delta \in P_f^G/$number of $r-$signed horizontal edges in $\delta \geq 2k\}$. Clearly by the multiplication defined above $I_f^G(x, 2k)$ is an ideal of $D_f^G(x)$.

Let $\overrightarrow{I_f}^G(x, 2k)$ =Linear span$\{\delta \in P_f^G/$number of $r-$signed horizontal edges in δ is equal to $2k\}$. $\overrightarrow{I_f}^G(x, 2k)$ denotes the quotient $I_f^G(x, 2k)/I_f^G(x, 2k+2)$.

To describe the structure of the quotients $\overrightarrow{I_f}^G(x, 2k)$ in terms of the eigenvalues and eigenspaces of certain matrices.

Definition 4.1. A generalised m, k signed partial 1-factor on $f = m + 2k$ vertices is a r-signed diagram whose vertices are arranged in a single row with k, r-signed horizontal edges and m free vertices.

Let $P_{m,k}^G$ denotes the set of all generalised m, k signed partial 1-factors and let $V_{m,k}^G$ be the real vector space with basis $P_{m,k}^G$.

The generalised symmetric group on m symbols $\{1, 2, \ldots, m\}$ is denoted by S_m^G.

Let us now define $\pi \in S_m^G$ by $\pi(i) = (\tau(i), \sigma(i))$ where $\sigma \in S_m$ and the function $\tau : \underline{m} \to \underline{r}$, where $\underline{m}$ denotes the set $\{1, 2, \ldots, m\}$ and $\underline{r}$ denotes the set $\{\xi, \xi^2, \ldots, \xi^r\}$, ξ^i's are primitive r^{th} root of unity.

Definition 4.2. Let f_1 (respectively f_2) be generalised m, k signed partial 1-factors with the free vertices of f_1 (respectively f_2) is labeled by $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ (respectively $\beta_1 < \beta_2 < \cdots < \beta_m$).

The union of f_1 and f_2 is a r-signed graph obtained by identifying $i - th$ vertex of f_1 with the $i - th$ vertex of f_2 consists some number $m_i(f_1, f_2)$ of disjoint i-cycles together with m disjoint r-signed paths $\mathcal{P}_1, \ldots, \mathcal{P}_m$ whose endpoints are in the set $\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_m\}$.

Definition 4.3. Let $f_1, f_2 \in P^G_{m,k}$. Define an inner product $< f_1, f_2 >$ on $V^G_{m,k}$ as follows.

1. If any r-signed path $\mathcal{P}_i$ joins a α_j to a α_i (or equivalently a β_j to a β_i) then $< f_1, f_2 >= 0$.
2. If each r-signed path $\mathcal{P}_i$ joins β_i to $\alpha_{\sigma i}$ and some number k_j of i_j edges $1 \leq i_j \leq r$ with $\sum\limits_{i_j=1}^{r} k_j i_j \equiv \tau(i)(\text{mod } r)$, then

$$< f_1, f_2 >= x^{\sum\limits_{i=1}^{r} i m_i(f_1,f_2)} \pi, \text{ where } \pi = (\tau, \sigma) \in S^G_m, \sigma \in S_m.$$

Note. $< f_1, f_2 >=< f_2^{\maltese}, f_1^{\maltese} >$, where $\maltese$ is the anti-isomorphism defined on the algebras KS^G_m by $\sigma \to \sigma^{-1}, \sigma \in S^G_m$.

Proposition 4.4. *Let $f = m + 2k$. Then the quotient $\overrightarrow{I_f}^G(x, 2k)$ is isomorphic as algebras to $(V^G_{m,k} \otimes V^G_{m,k} \otimes KS^G_m, \cdot)$, where*

$$(a \otimes b \otimes \pi_1) \cdot (c \otimes d \otimes \pi_2) = a \otimes d \otimes (\pi_1 < b, c > \pi_2), a, b, c, d \in P^G_{m,k}, \pi_1, \pi_2 \in S^G_m.$$

Proof. The proof follows as in [6]. Instead of m, k partial $1-$factors and symmetric group, the generalised m, k signed partial $1-$factors and the generalised symmetric group are used to prove the theorem, we give it here for the sake of completion.

As a vector space $\overrightarrow{I_f}^G(x, 2k)$ has basis the set of all r-signed 1-factors with exactly $2k$ r-signed horizontal edges.

Consider the linear map $\phi : V^G_{m,k} \otimes V^G_{m,k} \otimes KS^G_m \to \overrightarrow{I_f}^G(x, 2k)$.

Given $f_1, f_2 \in P^G_{m,k}$, we define $\phi(f_1 \otimes f_2 \otimes \sigma)$ to be the r-signed 1-factor on $2f$ vertices in the following way.

Let f_1 be the generalised m, k signed partial $1-$factor with free vertices $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ and let f_2 be the generalised m, k signed partial $1-$factor with free vertices $\beta_1 < \beta_2 < \cdots < \beta_m$ and given $\sigma \in S^G_m$ such that

1. A r-signed horizontal edge joining i to j in the top row if and only if i and j are joined by an $r-$signed horizontal edge in f_1.
2. A r-signed horizontal edge joining $f + i$ to $f + j$ in the bottom row if and only if i and j are joined by an $r-$signed horizontal edge in f_2.
3. A r-signed vertical edge joining α_i to $\beta_{\sigma(i)}$ for $i = 1, 2, \cdots m$.

The linear map ϕ defined in this way is clearly $1 - 1$ and onto. Hence it is a vector space isomorphism of $V^G_{m,k} \otimes V^G_{m,k} \otimes KS^G_m$ onto $\overrightarrow{I_f}^G(x, 2k)$.

It remains to show that ϕ is multiplicative.

$$\text{i.e } \phi(d_1 \cdot d_2) = \phi(d_1) \circ \phi(d_2), \ d_1, d_2 \in V^G_{m,k} \otimes V^G_{m,k} \otimes KS^G_m.$$

Let us now assume that $d_1 = a \otimes b \otimes \pi_1$ and $d_2 = c \otimes d \otimes \pi_2$ where a, b, c, d be the generalised m, k signed partial 1-factors with free vertices $\alpha_1 < \alpha_2 < \cdots < \alpha_m$, $\beta_1 < \beta_2 < \cdots < \beta_m$, $\gamma_1 < \gamma_2 < \cdots < \gamma_m$, $\psi_1 < \psi_2 < \cdots < \psi_m$ respectively and $\pi_1, \pi_2 \in S_m^G$.

Consider the product in $\overrightarrow{I_f}^G(x, 2k)$

$$\phi(d_1) \circ \phi(d_2) = x^{\sum\limits_{i=1}^{r} im_i(d_1,d_2)} d_3,\ d_3 \in D_f^G(x).$$

Case 1. Suppose there is a r-signed path joining α_i to α_j or ψ_i to ψ_j in $\overrightarrow{U}^{\phi(d_1)}_{\phi(d_2)}$ then $\phi(a \otimes b \otimes \pi_1) \circ \phi(c \otimes d \otimes \pi_2) = 0 = \phi((a \otimes b \otimes \pi_1) \cdot (c \otimes d \otimes \pi_2))$. Therefore

$$\phi(d_1 \cdot d_2) = \phi(d_1) \circ \phi(d_2).$$

Case 2. Suppose there is a r-signed path joining α_i to $\psi_{\sigma(i)}$ in $\overrightarrow{U}^{\phi(d_1)}_{\phi(d_2)}$ for $i = 1, 2, \ldots, m$, then $\phi(d_1) \circ \phi(d_2) = x^{\sum\limits_{i=1}^{r} im_i(\delta_1,\delta_2)} d_3 = \phi(d_1 \cdot d_2)$.

Hence ϕ is an algebra isomorphism. □

5. The Structure of Ideals $\overrightarrow{I_f}^G(x, 2k)$

To describe the structure of the ring $\overrightarrow{I_f}^G(x, 2k)$ in terms of the eigenvalues of certain matrices.

Definition 5.1. Let $\overrightarrow{T}^{[\lambda]}_{m,k}(x)$ be the $(pd_{[\lambda]})$-by-$(pd_{[\lambda]})$ matrix which is $d_{[\lambda]}$-by-$d_{[\lambda]}$ blocks of p-by-p matrices where p is the number of generalised m, k signed partial $1-$factors. The matrices in the each block are indexed by pairs of generalised m, k signed partial 1-factors being $\psi_{[\lambda]}(< b, c >)$, for the corresponding r-signed $1-$factor.

Let $\overrightarrow{N}^{[\lambda]}$ denotes the null space of $\overrightarrow{T}^{[\lambda]}_{m,k}(x)$, the matrix corresponding to generalised m, k signed partial $1-$factors and the multi partition $[\lambda]$ and $\overrightarrow{R}^{[\lambda]}$ denotes the range of $\overrightarrow{T}^{[\lambda]}_{m,k}(x)$, the matrix corresponding to generalised m, k signed partial $1-$factors and the multi partition $[\lambda]$.

Note. If $< b, c >= x^{\sum\limits_{i=1}^{r} im_i)(b,c)} \sigma$ then $< c, b >= x^{\sum\limits_{i=1}^{r} im_i(c,b)} \sigma^{-1}$. So the matrix $\overrightarrow{T}^{[\lambda]}_{m,k}(x)$ is symmetric.

Choose a basis $u^{(1)}, \ldots, u^{(n)}$ for $\overrightarrow{N}^{[\lambda]}$ and an orthonormal basis of eigenvectors $v^{(1)}, \ldots, v^{(r)}$ for the nonzero eigenvalues $\mu^{(1)}, \ldots, \mu^{(r)}$.

Definition 5.2. For given any left ideal $\overrightarrow{I}_t$ and a generalised m, k signed partial 1-factor d, define

$$\overrightarrow{V}_L(\overrightarrow{I}_t, d) = \text{ Linear Span } \{c \otimes d \otimes x / c \in P^G_{m,k}, x \in \overrightarrow{I}_t\}.$$

Lemma 5.3. *$\overrightarrow{V}_L(\overrightarrow{I}_t, d)$ is a left ideal of $\overrightarrow{I_f}^G(x, 2k)$.*

Proof. The proof follows as in [6]. Instead of m, k partial $1-$factors and symmetric group, the generalised m, k signed partial $1-$factors and the generalised symmetric group are used to prove the theorem, we give it here for the sake of completion.

For $c \otimes d \otimes x \in \overrightarrow{V}_L(\overrightarrow{I}_t, d)$, choose $\delta \in \overrightarrow{I_f}^G(x, 2k)$ such that $\delta * (c \otimes d \otimes x)$ not equal to zero, that is $\delta * (c \otimes d \otimes x)$ and $c \otimes d \otimes x$ have same number of $r-$signed horizontal edges.

$$\begin{aligned}
(a \otimes b \otimes \pi) * (c \otimes d \otimes x) &= (a \otimes d \otimes \pi \langle b, c\rangle x), \pi \in S_m^G, x \in \overrightarrow{I}_t \\
&= (a \otimes d \otimes \pi x^{\sum_{i=1}^{r} im_i(b,c)} \pi_1 x), \\
&\qquad \text{where } \pi, \pi_1 \in S_m^G, x \in \overrightarrow{I}_t \\
&= x^{\sum_{i=1}^{r} im_i(b,c)} (a \otimes d \otimes \pi\pi_1 x), \\
&\qquad (\pi\pi_1 \in S_m^G \text{ by the definition of } S_m^G)
\end{aligned}$$

Since $x \in \overrightarrow{I}_t$ and $\overrightarrow{I}_t$ is a left ideal of $\widehat{\widehat{S}}_m$, $y = \pi\pi_1 x \in \overrightarrow{I}_t$. Hence

$$(a \otimes b \otimes \pi) * (c \otimes d \otimes x) = x^{\sum_{i=1}^{r} im_i(b,c)} (a \otimes d \otimes y) \in \overrightarrow{V}_L(\overrightarrow{I}_t, d).$$

Therefore $\overrightarrow{V}_L(\overrightarrow{I}_t, d)$ is a left ideal of $\overrightarrow{I_f}^G(x, 2k)$. □

Define $\overrightarrow{W}_L(\overrightarrow{I}_t, d) \subset \overrightarrow{V}_L(\overrightarrow{I}_t, d)$ to be the linear span of all

$$\sum (u)_{(c,i)} \left(c \otimes d \otimes \overrightarrow{A}_{i,t} \right),$$

where u is in $\overrightarrow{N}^{[\lambda]}$ and $\overrightarrow{A}_{i,t}$ is the basis element of $\overrightarrow{I}_t$ corresponding to the basis element $\overrightarrow{A}_i$ in $\overrightarrow{S}^{[\lambda]}$. i.e. the linear span of the set of all elements mapped to zero in $\overrightarrow{I_f}^G(x, 2k)$.

Proposition 5.4. *Suppose $v = \sum (v)_{(c,i)} \left(c \otimes d \otimes \overrightarrow{A}_{i,t} \right) \in \overrightarrow{V}_L(\overrightarrow{I}_t, d)$. Let a, b be generalised m, k signed partial 1-factors. For any $\sigma \in S_m^G$, $(a \otimes b \otimes \sigma)v = a \otimes d \otimes \sigma \sum \gamma_j \overrightarrow{A}_{j,t}$, where γ_j is the (b, j) entry of $\overrightarrow{T}^{[\lambda]}_{m,k}(x)(v)$.*

Proof. The proof follows as in [6]. Instead of m, k partial $1-$factors and symmetric group, the generalised m, k signed partial $1-$factors and the generalised symmetric group are used to prove the theorem, we give it here for the sake of completion.

$$\begin{aligned}
(a \otimes b \otimes \sigma) * \sum v_{(c,i)} \left(c \otimes d \otimes \overrightarrow{A}_{i,t} \right) &= a \otimes d \otimes \sigma \sum v_{(c,i)} < b, c > \overrightarrow{A}_{i,t} \\
&= a \otimes d \otimes \sigma \sum (v)_{c,i} x^{\sum_{i=1}^{r} im_i(b,c)} \sigma_1 A_{i,t} \\
&= a \otimes d \otimes \sum \gamma_j \overrightarrow{A}_{j,t}
\end{aligned}$$

where $\sigma_1 A_{i,t} = A_{j,t}$ and γ_j is the coefficient of $\overrightarrow{A}_{j,t}$ in $\sum v_{(c,i)} < b, c > \overrightarrow{A}_{j,t}$.

By definition of $\overrightarrow{T}^{[\lambda]}_{m,k}(x)$, the coefficient of $\overrightarrow{A}_{j,t}$ in $< b, c > \overrightarrow{A}_{i,t}$ is the (b, j), (c, i) entry of $\overrightarrow{T}^{[\lambda]}_{m,k}(x)$. Thus γ_j is the (b, j) entry of $\overrightarrow{T}^{[\lambda]}_{m,k}(x)(v)$. □

Proposition 5.5.

1. *$\overrightarrow{I_f}^G(x, 2k)\overrightarrow{W}_L(\overrightarrow{I}_t, d) = 0$.*
2. *$\overrightarrow{I}(v) = \overrightarrow{V}_L(\overrightarrow{I}_t, d)$ for any v in $\overrightarrow{V}_L(\overrightarrow{I}_t, d)$ not in $\overrightarrow{W}_L(\overrightarrow{I}_t, d)$.*
3. *$\overrightarrow{V}_L(\overrightarrow{I}_t, d)/\overrightarrow{W}_L(\overrightarrow{I}_t, d)$ is irreducible as a left $\overrightarrow{I_f}^G(x, 2k)$ module.*

Proof. The proof follows as in [6]. Instead of m, k partial $1-$factors and symmetric group, the generalised m, k signed partial $1-$factors and the generalised symmetric group are used to prove the theorem. We give it here for the sake of completion.

Suppose w is a generating element of $\overrightarrow{W}_L(\overrightarrow{I}_t, d)$ and γ_j is the (b, j) entry of $\overrightarrow{T}^{[\lambda]}_{m,k}(x)(v)$. By the definition of $\overrightarrow{W}_L(\overrightarrow{I}_t, d)$, γ_j are all 0, for any $(a \otimes b \otimes \sigma)$. Hence $\overrightarrow{I_f}^G(x, 2k)\overrightarrow{W}_L(\overrightarrow{I}_t, d) = 0$.

Suppose v is in $\overrightarrow{V}_L(\overrightarrow{I}_t, d)$ but not in $\overrightarrow{W}_L(\overrightarrow{I}_t, d)$. Choose γ_j, the (b, j) entry of $\overrightarrow{T}^{[\lambda]}_{m,k}(x)(v)$ is not 0. Then $a \otimes b \otimes \sigma(v)$ is not 0 Note that a and σ were arbitrary. Since $\overrightarrow{I}_t$ is an irreducible $\widehat{\widehat{S}}_m$ module, the images under $\sigma \in \widehat{\widehat{S}}_m$ of any nonzero vector in $\overrightarrow{I}_t$ generate all of $\overrightarrow{I}_t$. Hence vectors of the form $(a \otimes b \otimes \sigma) * \sum v_{(c,j)} \left(c \otimes d \otimes \overrightarrow{A}_{j,t}\right)$ generate all of $\overrightarrow{V}_L(\overrightarrow{I}_t, d)$. Hence $\overrightarrow{I}(v) = \overrightarrow{V}_L(\overrightarrow{I}_t, d)$.

The last part of the proof from the above two. □

Let $\overrightarrow{W}^{[\lambda]}_L = \bigoplus \overrightarrow{W}_L(\overrightarrow{I}_t, d)$ and . By Proposition 5.4, $\overrightarrow{W}^{[\lambda]}_L$ is a nilpotent left ideal of $\overrightarrow{I_f}^G(x, 2k)$.

Recall that $\widehat{\widehat{S}}^{[\lambda]}_m$ can also be written as a direct sum of right ideals $\overrightarrow{J}_1, \ldots, \overrightarrow{J}_{d_{[\lambda]}}$.

For given any right ideal $\overrightarrow{J}_t$ and a generalised m, k signed partial 1-factor a, define

$$\overrightarrow{V}_R(\overrightarrow{J}_t, a) = \text{ Linear Span } \{a \otimes b \otimes x / b \in P^G_{m,k}, x \in \overrightarrow{J}_t\}.$$

Lemma 5.6. *$\overrightarrow{V}_R(\overrightarrow{J}_t, c)$ is a right ideal of $\overrightarrow{I_f}^G(x, 2k)$.*

Proof. For $c \otimes d \otimes x \in \overrightarrow{V}_R(\overrightarrow{J}_t, c)$, choose $\delta \in \overrightarrow{I_f}^G(x, 2k)$ such that $(c \otimes d \otimes x) * \delta$ not equal to zero, that is $(c \otimes d \otimes x) * \delta$ have same number of signed horizontal edges as in $c \otimes d \otimes x$.

$$(c \otimes d \otimes x) * (a \otimes b \otimes \pi) \quad = \quad (c \otimes b \otimes x \langle d, a \rangle \pi), \pi \in S^G_m, x \in \overrightarrow{J}_t$$

$$= (c\otimes b\otimes x x^{\sum_{i=1}^{r} i m_i(d,a)} \pi_1\pi),$$
$$\text{where } \pi,\pi_1\in S_m^G, x\in \overrightarrow{J}_t$$
$$= x^{\sum_{i=1}^{r} i m_i(d,a)}(c\otimes b\otimes x\pi_1\pi),$$
$$(\pi_1\pi\in S_m^G \text{ by the definition of } S_m^G)$$

Since $x\in\overrightarrow{J}_t$ and $\overrightarrow{J}_t$ is a right ideal of $\widehat{\widehat{S}}_m$, $y=x\pi_1\pi\in\overrightarrow{J}_t$,

$$(c\otimes d\otimes x)*(a\otimes b\otimes\pi)=x^{\sum_{i=1}^{r} i m_i(d,a)}(c\otimes b\otimes y)\in\overrightarrow{V}_R(\overrightarrow{J}_t,c).$$

Therefore $\overrightarrow{V}_R(\overrightarrow{J}_t,c)$ is a right ideal of $\overrightarrow{I_f}^G(x,2k)$. □

Define $\overrightarrow{W}_R(\overrightarrow{J}_t,a)\subset\overrightarrow{V}_R(\overrightarrow{J}_t,a)$ to be the linear span of all

$$\sum(u)_{(c,i)}\left(a\otimes b\otimes\overrightarrow{A}_{j,t}\right)$$

where $u^t\overrightarrow{T}^{[\lambda]}_{m,k}(x)=0$ and $\overrightarrow{A}_{j,t}$ is as before.

The same proofs used in Propositions 5.4 and 5.5 shows that

1. $\overrightarrow{W}_R(\overrightarrow{J}_t,a)\circ(c\otimes d\otimes\sigma)=0$,
2. $\overrightarrow{V}_R(\overrightarrow{J}_t,a)/\overrightarrow{W}_R(\overrightarrow{J}_t,a)$ is an irreducible right $\overrightarrow{I_f}^G(x,2k)$ module.

Define $\overrightarrow{W}^{[\lambda]}_R=\bigoplus\overrightarrow{W}_R(\overrightarrow{J}_t,a)$ and define $\overrightarrow{W}^{[\lambda]}$ to be the nilpotent 2-sided ideal $\overrightarrow{W}^{[\lambda]}=\overrightarrow{W}^{[\lambda]}_L+\overrightarrow{W}^{[\lambda]}_R$.

Definition 5.7. Define $\overrightarrow{D}^{[\lambda]}$ to be the 2-sided ideal of $\overrightarrow{I_f}^G(x,2k)$ given by the linear span of all vectors $a\otimes b\otimes x$, where a and b are arbitrary and $x\in\widehat{\widehat{S}}{}^{[\lambda]}_m$.

Note that $\overrightarrow{I_f}^G(x,2k)$ is the direct sum of the $\overrightarrow{D}^{[\lambda]}$.

Proposition 5.8. *$\overrightarrow{D}^{[\lambda]}/\overrightarrow{W}^{[\lambda]}$ is canonically isomorphic to the full matrix ring $End(\overrightarrow{R}^{[\lambda]})$. Recall that $\overrightarrow{R}^{[\lambda]}$ is the range of $\overrightarrow{T}^{[\lambda]}_{m,k}(x)$.*

Proof. Instead of m,k partial 1−factors and symmetric group, the generalised m,k signed partial 1−factors and the generalised symmetric group are used to prove the theorem. We give it here for the sake of completion.

Given eigen vectors $v^{(r)}$ and $v^{(s)}$ define

$$Z(v^{(r)},v^{(s)})=\left(\mu^{(r)}\mu^{(s)}\right)^{-1}\sum\left(v^{(r)}\right)_{a,i}\left(v^{(s)}\right)_{b,j}a\otimes b\otimes x_iy_j.$$

Taking the product of $Z(v^{(r)},v^{(s)})$ and $Z(v^{(t)},v^{(u)})$ we obtain

$$
\begin{aligned}
&Z(v^{(r)},v^{(s)})Z(v^{(t)},v^{(u)})\\
&= \left(\mu^{(r)}\mu^{(s)}\mu^{(t)}\mu^{(u)}\right)^{-1}\\
&\quad \sum \left(v^{(r)}\right)_{a,i}\left(v^{(u)}\right)_{d,l} a\otimes d\otimes \left(v^{(s)}\right)_{b,j}\left(v^{(t)}\right)_{c,k}\{x_iy_j<b,c>x_ky_l\}\\
&= \left(\mu^{(r)}\mu^{(s)}\mu^{(t)}\mu^{(u)}\right)^{-1}\sum \left(v^{(r)}\right)_{a,i}\left(v^{(u)}\right)_{d,l}\\
&\quad a\otimes d\otimes x_iy_j\left(\sum \left(v^{(s)}\right)_{b,j}\left\{\sum \left(v^{(t)}\right)_{c,k}<b,c>x_k\right\}\right)y_l.
\end{aligned}
$$

Now $\sum \left(v^{(t)}\right)_{c,k}<b,c>x_k=\sum \gamma_r x_r$ where γ_r is the (b,r) entry of $\overrightarrow{T}^{[\lambda]}_{m,k}(x)v^{(t)}$. Since $v^{(t)}$ is an eigenvector with eigenvalue $\mu^{(t)}$, $\gamma_r=\mu^{(t)}\left(v^{(t)}\right)_{b,r}$. So

$$
x_iy_j\left(\sum \left(v^{(s)}\right)_{b,j}\left\{\sum \left(v^{(t)}\right)_{c,k}<b,c>x_k\right\}\right)y_l
$$

$$
\begin{aligned}
&= x_iy_j\left(\sum \left(v^{(s)}\right)_{b,j}\left\{\sum \gamma_r x_r\right\}\right)y_l\\
&= \mu^{(t)}\sum \left(v^{(s)}\right)_{b,j}\left(v^{(t)}\right)_{b,r}\{x_iy_jx_ry_l\}\\
&= \mu^{(t)}x_iy_l\left\{\sum \left(v^{(s)}\right)_{b,j}\left(v^{(t)}\right)_{b,j}\right\} \qquad \text{by equation 1.1}\\
&= \mu^{(t)}x_iy_l\delta_{s,t} \qquad \text{by the orthonormality of the } v^{(i)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Z(v^{(r)},v^{(s)})Z(v^{(t)},v^{(u)}) &= \left(\mu^{(r)}\mu^{(s)}\mu^{(t)}\mu^{(u)}\right)^{-1}\sum \left(v^{(r)}\right)_{a,i}\left(v^{(u)}\right)_{d,l} a\otimes d\otimes \mu^{(t)}x_iy_l\delta_{s,t}\\
&= \delta_{s,t}Z(v^{(r)},v^{(u)}).
\end{aligned}
$$

Hence the subspace of $\overrightarrow{D}^{[\lambda]}$ spanned by the $Z(v^{(r)},v^{(s)})$ is isomorphic to $End(\overrightarrow{R}^{[\lambda]})$. □

The ideal $\overrightarrow{D}^{[\lambda]}=V^G_{m,k}\otimes V^G_{m,k}\otimes \widehat{\overrightarrow{S}}^{\lambda}_m$ is isomorphic as a vector space to $(V^G_{m,k}\otimes \overrightarrow{S}^{[\lambda]})\otimes(V^G_{m,k}\otimes\overrightarrow{S}^{[\lambda]})$ via the linear map f sending $(c\otimes\overrightarrow{A}_i)\otimes(d\otimes\overrightarrow{A}_j)$ to $c\otimes d\otimes x_iy_j$.

Writing $V^G_{m,k}\otimes\overrightarrow{S}^{[\lambda]}$ as $\overrightarrow{N}^{[\lambda]}\bigoplus\overrightarrow{R}^{[\lambda]}$ we have, from Propositions 5.4, 5.5 and 5.8, that

A. $f(\overrightarrow{N}^{[\lambda]}\otimes(V^G_{m,k}\otimes\overrightarrow{S}^{[\lambda]})+(V^G_{m,k}\otimes\overrightarrow{S}^{[\lambda]})\otimes\overrightarrow{N}^{[\lambda]})$ is contained in the radical of $D^G_f(x)(2k)$

B. $f(\overrightarrow{R}^{[\lambda]}\otimes\overrightarrow{R}^{[\lambda]})$ is a full matrix ring.

The next theorem follows immediately from **A** and **B**.

Theorem 5.9. *With notation as above:*

1. *Let* $\overrightarrow{W}^{[\lambda]} = f\left(\overrightarrow{N}^{[\lambda]} \otimes (V^G_{m,k} \otimes \overrightarrow{S}^{[\lambda]}) + (V^G_{m,k} \otimes \overrightarrow{S}^{[\lambda]}) \otimes \overrightarrow{N}^{[\lambda]}\right)$. *Then* $\overrightarrow{W}^{[\lambda]}$ *is the intersection of the radical of* $\overrightarrow{I_f}^G(x, 2k)$ *with* $\overrightarrow{D}^{[\lambda]}$.
2. $\overrightarrow{D}^{[\lambda]}/\overrightarrow{W}^{[\lambda]}$ *is a full matrix ring which is canonically isomorphic to* $End(\overrightarrow{R}^{[\lambda]})$.

6. Eigen Values for $\overrightarrow{T}^{[\lambda]}_{m,k}(x)$ when $m = 0$ and f Is Even

In this section, we determine the eigen values of $\overrightarrow{T}_s(x), s = \frac{f}{2}$ and f is even in terms of representation of S^G_f, the generalised symmetric group on f points. Here we deal with the case $s = f/2$, the number of $r-$signed horizontal edges, when f is even.

Let F_s be the set of all r-signed 1-factors on f points arranged in a single row with exactly $s = \frac{f}{2}$ horizontal edges. Let $\overrightarrow{T}_s(x)_{\delta_i\delta_j}$ be the $F_s \times F_s$ matrix whose (δ_i, δ_j) entry is $x^{\left(\sum_{k=1}^{r} km_k(\delta_i,\delta_j)\right)}$ where m_k is the number of k-cycles in $\delta_i \cup \delta_j$.

Example 6.1. When $r = 3$, $f = 2$ and $s = 1$.

$$\overrightarrow{T}_1(x) = \begin{pmatrix} x^3 & x^2 & x \\ x^2 & x & x^3 \\ x & x^3 & x^2 \end{pmatrix}$$

The eigen values are $x(x-1)\sqrt{x^2+x+1}$, $-x(x-1)\sqrt{x^2+x+1}$ and x^3+x^2+x.

A generalised permutation $\sigma \in S^G_f$ induces a signed permutation of F_s by permuting the $i-$edges of r-signed 1-factors. If p and q are joined in δ, then $\sigma(p)$ and $\sigma(q)$ are joined in $\sigma(\delta)$, $\sigma \in S^G_f$.

Suppose $\delta_1, \delta_2 \in F_s$ and if C_1 is a connected component of $\delta_1 \cup \delta_2$, then $\sigma(C_1)$ is a connected component of $\sigma(\delta_1) \cup \sigma(\delta_2)$. In particular the number and size of $\delta_1 \cup \delta_2$ and $\sigma(\delta_1) \cup \sigma(\delta_2)$ are the same.

Let V_s be the vector space with basis F_s. For $\sigma \in S^G_f$, let P_σ be the generalised permutation matrix corresponding to the generalised permutation of F_s induced by $\sigma \in S^G_f$. In particular if $\sigma(\delta_i) = \delta_j$ and $\sigma(i) = (\xi^k, j)$, then P_σ has a ξ^k in the $P_{\delta_i\delta_j}$ and $0'$s elsewhere in the δ_i row and δ_j column. Hence P_σ and $\overrightarrow{T}_s(x)$ commutes to give

$$P_\sigma \overrightarrow{T}_s(x) = \overrightarrow{T}_s(x) P_\sigma.$$

The generalised permutation module has a decomposition as an S^G_f module into irreducible subspaces corresponding to irreducible representations of S^G_f. The irreducible representations of S^G_f are indexed by multi partitions $[\lambda]$ of f. The irreducibles which occur as constituents of the generalised permutation module are indexed by even multi partitions $[\lambda]$ of f. Furthermore the multiplicity of each representation is 1. This means that

$$V = V_1 + V_2 + \dots V_n,$$

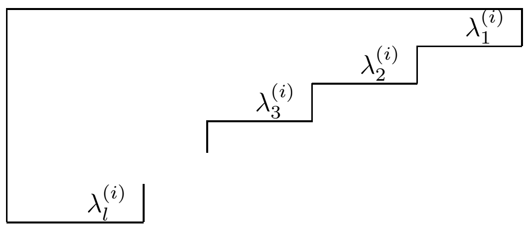
where $V_1, V_2, \dots, V_n$ are invariant subspaces of $P_\sigma, \forall \sigma \in S_f^G$ and n is the number of even multi partitions of f. As the irreducibles are distinct, $\overrightarrow{T}_s(x)V_i \subseteq V_i$. As each V_i is irreducible, $\overrightarrow{T}_s(x)$ restricted to V_i is a scalar, which is denoted by $h_i(x)I$. In order to find the eigen values for $\overrightarrow{T}_s(x)$, it is necessary to determine the scalars for $\overrightarrow{T}_s(x)$ restricted to V_i. The multiplicity will be $dimV_i$.

We determine these scalars $h_i(x)$ in terms of the multi partition associated with the representation and the location of certain integers on a grid. Let Δ be the grid and place the integer $r(2j-i-1)$ in the position i th row and $2j$ th column. It is convenient to place the diagram of the even multi partition $[\lambda]$ on the grid Δ.

$$\begin{array}{lccccccccccc} & & & & & \text{Residue i} & & & & & \\ \text{Column No.} & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \dots \\ & & & 0 & & 2r\xi^i & & 4r\xi^i & & 6r\xi^i & & \\ & & & -r\xi^i & & r\xi^i & & 3r\xi^i & & 5r\xi^i & & \\ \Delta = & & & -2r\xi^i & & 0 & & 2r\xi^i & & 4r\xi^i & & \\ & & & -3r\xi^i & & -r\xi^i & & r\xi^i & & 3r\xi^i & & \\ & & & -4r\xi^i & & -2r\xi^i & & 0 & & 2r\xi^i & & \end{array},$$

Let $[\lambda]$ be a even multi partition of f with every partitions of $[\lambda]$ into even parts. Let $[\lambda] = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$ where $\lambda^{(j)}$ is a partition of m_j with length l_j where each $\lambda_i^{(j)}, i = 1, 2, \dots, l_j, j = 1, 2, \dots, r$ is even, such that $\sum_i m_i = f$. Let $d = [d^{(1)}, d^{(2)}, \dots, d^{(r)}]$. The diagram $d^{(i)}$ corresponding to even partition $\lambda^{(i)}$ on Δ is

Residue i

$d^{(i)} =$ 

There are exactly $s = f/2$ number of integers in Δ contained inside the boundary of d, the diagram of the even multi partition $[\lambda]$.

Theorem 6.2. *Let $[\lambda] = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$ be an even multi partition of f. Let $V^{[\lambda]}$ be the subspace of V_i associated to the multi partition $[\lambda]$ and $h_s(x) = h_{[\lambda]}(x)$. Then*

$$h_{[\lambda]}(x) = \prod_{i=1}^{r} \prod_{d \in d^{(i)}} \prod_{k,l} s_i \left| (x^r + \xi^{i(r-1)}x^{r-1} + \dots + \xi^i x + a_{kl}^{(i)}) \right|,$$

where $|.|$ *denotes modulus,* $a_{ij}^{(i)}$ *are in the diagram* $d^{(i)}$ *of shape* $\lambda^{(i)}$ *for* $i = 1, 2, \ldots, r$ *and*

$$s_i = \begin{cases} -1, & \text{if } 1 \leq i < \lfloor \frac{r}{2} \rfloor \\ 1, & \text{if } \lceil \frac{r}{2} \rceil \leq i \leq r. \end{cases}$$

$h_{[\lambda]}(x)$ *is a polynomial of degree* r *for the multipartition* $[\lambda] = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$.

Proof. The proof follows as in the approach of [6]. Instead of 1−factors, even partition standard tableaux and symmetric group, the r−signed 1−factors, multi partitions multi standard tableaux and generalised symmetric group are used to prove the theorem here. This is proved by induction on $s = f/2$. We give the proof here for the sake of completion.

Let

$$d_{[\lambda]}(x) = \prod_{i=1}^{r} \prod_{d \in d^{(i)}} \prod_{k,l} s_i \left| (x^r + \xi^{i(r-1)} x^{r-1} + \ldots + \xi^i x + a_{kl}^{(i)}) \right|,$$

where $a_{kl}^{(i)}$ are in the diagram $d^{(i)}$ of shape $\lambda^{(i)}$. We must show that $h_{[\lambda]}(x) = d_{[\lambda]}(x)$. If $s = 1$, then there are r possible even multi partitions of f, in the r tuple $(2, \emptyset, \emptyset, \ldots, \emptyset), (\emptyset, 2, \emptyset, \ldots, \emptyset), \ldots, (\emptyset, \emptyset, \ldots, \emptyset, 2)$. $\overrightarrow{T}_s(x)$ restricted to V_i is $\left| s_i(x^r + \xi^{i(r-1)} x^{r-1} + \ldots + \xi^i x) \right| (1)$. Since the dimension of V_i's are one, the multiplicity of each $h_s(x)$ will be one. Therefore $h_{[\lambda]}(x) = d_{[\lambda]}(x)$ for $s = 1$. We suppose that the theorem is true for all even multi partitions of f of size smaller than $2s$.

Let $[\lambda]$ be an even multi partition of $f = 2s$. Let $[d]$ be the diagram of shape $[\lambda]$ and t be the standard Young multi-tableau with $1, 2, \ldots, f$ are placed consecutively in each row in the diagram of $[\lambda]$. Let

$$e_t = \sum \varepsilon(\sigma)\sigma\tau, \forall \sigma \in C_t, \tau \in R_t.$$

where $C_{[t]}$ is the column stabiliser of t and R_t is the row stabiliser of t, these are the two subgroups of S_f^G. For any $v \in V, |e_t v| \in V^{[\lambda]}$. Furthermore e_t affords the representation corresponding to $[\lambda]$.

Let δ_0 be the r-signed 1-factor on $\{1, 2, \ldots, f\}$ whose lines joins $2i - 1$ to $2i$, for $i = 1, 2, \ldots, s$. We will show that $|e_t \delta_0|$ has a nonzero δ_0 coefficient u and $|T_r(x) e_t \delta_0|$ has a nonzero δ_0 coefficient, say $p_{[\lambda]}(x)u$. As $T_r(x)$ acts as a scalar on $V^{[\lambda]}$ and $|e_t \delta_0|$ is in $V^{[\lambda]}$, $p_{[\lambda]}(x) = h_{[\lambda]}(x)$.

If δ_i is a r-signed 1-factor in F_s,

$$\overrightarrow{T}_s(x)\delta_i = \sum_j x^{\left(\sum_{k=1}^{r} k m_k(\delta_i, \delta_j)\right)} \delta_j$$

where m_k is the number of k cycles formed in $\delta_i \cup \delta_j$ for which δ_j is in F_s. Therefore,

$$|\overrightarrow{T}_s(x)\sigma\tau\delta_0| = \left| x^{\left(\sum_{k=1}^{r} k m_k(\sigma\tau\delta_0, \delta_0)\right)} \right|.$$

Let $up_{[\lambda]}(x)$ be the δ_0 component of $|\overrightarrow{T}_s(x)e_{[t]}\delta_0|$. Then

$$up_{[\lambda]}(x) \quad = \quad \sum \left| \varepsilon(\sigma) x^{\left(\sum_{k=1}^{r} km_k(\sigma\tau\delta_0,\delta_0)\right)} \right|. \tag{1.1}$$

Some of the terms in $up_{[\lambda]}(x)$ gives the same expression. In particular, let R_{t_0} be the subgroup of R_t which fixes δ_0 and let r_0 be its order. That is, if $\tau_1 \in R_{t_0}$, $\tau_1\delta_0 = \delta_0$. Now for τ in R_t, $m_k(\sigma\tau\tau_1\delta_0,\delta_0) = m_k(\sigma\tau\delta_0,\delta_0), \forall k = 1,2,\ldots,r$. Let R_t/R_{t_0} be the set of right coset representatives for R_{t_0} in R_t. Therefore, equation 1.1 gives

$$up_{[\lambda]}(x) \quad = \quad \left[\sum s_i \left| \varepsilon(\sigma) x^{\left(\sum_{k=1}^{r} km_k(\sigma\tau\delta_0,\delta_0)\right)} \right|\right] r_0, \forall\ \sigma \in C_t,\ \tau \in R_t/R_{t_0}. \tag{1.2}$$

Let C_{t_0} be the subgroup of C_t which fixes δ_0 and c_0 be its order. Furthermore, all $\sigma \in C_{t_0}$ are even as the generalised signed permutation in each odd column is identical to the generalised signed permutation in the column to its immediate right as the line joining $2i-1$ to $2i$ is preserved. Let C_t/C_{t_0} be the set of left coset representatives for C_{t_0} in C_t. Therefore, equation 1.2 gives

$$up_{[\lambda]}(x) \quad = \quad \left[\sum \left| \varepsilon(\sigma) x^{\left(\sum_{k=1}^{r} km_k(\sigma\tau\delta_0,\delta_0)\right)} \right|\right] r_0 c_0, \forall\ \sigma \in C_t/C_{t_0}, \tau \in R_t/R_{t_0}. \tag{1.3}$$

The argument above shows that $\overrightarrow{T}_s(x)$ restricted to $V^{[\lambda]}$ is a scalar $p_{[\lambda]}(x)$. We need only to show that $p_{[\lambda]}(x) = d_{[\lambda]}(x)$. We may also assume that for the diagrams $[\lambda^*]$ of smaller size, we have $d_{[\lambda^*]}(x) = p_{[\lambda^*]}(x) = h_{[\lambda^*]}(x)$. Using equation 1.3, we get

$$p_{[\lambda]}(x) \quad = \quad \left[\sum \left| \varepsilon(\sigma) x^{\left(\sum_{k=1}^{r} km_k(\sigma\tau\delta_0,\delta_0)\right)} \right|\right], \forall\ \sigma \in C_t/C_{t_0}, \tau \in R_t/R_{t_0}. \tag{1.4}$$

It is clear from the definition of C_{t_0} that $C_{t_0} = C_{t_0}^{(1)} \times C_{t_0}^{(2)} \times \ldots \times C_{t_0}^{(r)}$ consists of all generalised signed permutations in C_t, where $C_{t_0}^{(j)}, j = 1,2,\ldots,r-1$ permutes the $2i-1$ column in the same way as $2i$ column in the j th residue with sign changes $\xi^k, k = 1,2,\ldots,r$ for $i = 1,2,\ldots,\lambda_{l_j}^{(j)}/2,\ \ j = 1,2,\ldots,r-1$ and $C_{t_0}^{(r)}$ permutes the $2i-1$ column in the same way as $2i$ column in the r th residue without sign changes for $i = 1,2,\ldots,\lambda_{l_r}^{(r)}/2$.

Coset representatives may be chosen which fix the odd numbered column pointwise in all the residues and permutes the even numbered column with sign change $\xi^k, k = 1,2,\ldots,r$ in $1,2,\ldots,r-1$ residue and with sign change ξ^r in r th residue. The coset representatives in C_t/C_{t_0} are precisely the r-signed permutations acting on even numbered columns. This is a full set as any element of C_t, which is a product of generalised signed permutations in C_{t_0} followed by generalised signed permutation in $C_t^{(1)} \times C_t^{(2)} \times \ldots \times C_t^{(r)}$ moving only elements in even numbered columns.

The choices for coset representatives of R_t/R_{t_0} are not as natural. The group R_t is a direct product of groups $R_i^{(j)}, \forall \;\; j = 1, 2, \ldots, r$, where $R_i^{(j)}$ permutes only elements in the $i-$th row of the j^{th} residue $j = 1, 2, \ldots, r-1$, $R_i^{(r)}$ permutes only elements in the $i-$th row of the r^{th} residue with sign changes $\xi^k, \;\; k = 1, 2, \ldots, r$ and fixes all the other elements. Also R_{t_0} is a direct product of groups $R_{0i}^{(j)}, \forall \;\; j = 1, 2, \ldots, r$ where $R_{0i}^{(j)} = R_i^{(j)} \bigcap R_{t_0}$, for $j = 1, 2, \ldots, r$.

Coset representatives may be chosen as products $\prod_{j=1}^{l_r} r_1 r_2 \ldots r_{l_j}$ where r_i's are coset representatives for $R_i^{(j)}/R_{0i}^{(j)}$, $j = 1, 2, \ldots, r$

There arises the following two cases :

1. The s^{th} horizontal edge lies in the j $^{\text{th}}$ -residue, $j = 1, 2, \ldots, r-1$
2. The s^{th} horizontal edge lies in the r $^{\text{th}}$ residue.

Case 1. If s^{th} horizontal edge lies in the j $^{\text{th}}$ -residue, $j = 1, 2, \ldots, r-1$

In order to prove the theorem for a fixed j we concentrate on $(l_j, \lambda_{l_j}^{(j)} - 1)$ and the $(l_j, \lambda_{l_j}^{(j)})$ position. For convenience we call it the position as a and b. In order to evaluate $m_k(\sigma\tau\delta_0, \delta_0), \; \forall \; k = 1$ to r, it is convenient to place the lines from δ_0 in the diagram t.

Pictured this way $\tau\delta_0$ is a diagram with lines all in the same row. Coset representatives for $R_i^{(j)}/R_{0i}^{(j)}$ may be picked anyway for $i = 1, 2, \ldots, l_j - 1$. We choose coset representatives for $R_{l_j}^{(j)}/R_{l_j0}^{(j)}$ by first restricting a group $R_{l_j}^{(j)}$ to a group $R_{l_j*}^{(j)}$, the subgroup of $R_{l_j}^{(j)}$ fixing a and b. Let $R_{l_j0*}^{(j)}$ be the subgroup of $R_{l_j^*}$ fixing the r-signed $1-$factor δ_0. Let Y be the set of representatives for $R_{l_j^*}^{(j)}/R_{l_j0^*}^{(j)}$. Let τ_i be the r-signed transpositions in $R_{l_j}^{(j)}$ interchanging $2s-1$ and $2s - \lambda_{l_j}^{(j)} + i$ for $i = 1, 2, \ldots, \lambda_{l_j}^{(j)} - 2$ and for $i = 0$, τ_0 is the identity. The elements $\tau_i Y$ are a full set of representatives of $R_{l_j}^{(j)}/R_{l_{j0}}^{(j)}$.

We also wish to choose the coset representatives appropriately for the subgroup of C_t moving elements in the $\lambda_{l_j}^{(j)}$ column only. Denote this subgroup by $C_{l_j}^{(j)}$. Let $C_{l_j^*}^{(j)}$ be the subgroup of $C_{l_j}^{(j)}$ fixing the entry b. Let Z be the set consisting of r-signed transpositions $\sigma_i^{\xi^k}, \forall \;\; k = 1, 2, \ldots, r$ where $\sigma_i^{\xi^k}$ interchanges b with the entry above it in the $i-$ th row with sign change ξ^k in b, for $i = 1, 2, \ldots, l_j - 1$ and $k = 1, 2, \ldots, r$ and for $i = 0$, $\sigma_0^{\xi^k}$ is the identity with sign change $\xi^k, k = 1, 2, \ldots, r$ in b. Coset representatives for $C_{l_j}^{(j)}/C_{l_j0}^{(j)}$ may be taken to be $\sigma' C_{l_j-1}^{(j)}, \sigma' \in Z$.

Now let C_{t^*} be the generalised signed permutations in C_t fixing a and b and R_{t^*} be the generalised signed permutations in R_t fixing a and b. Let $C_{t_0^*}$ and $R_{t_0^*}$ be the corresponding stabilisers of δ_0 fixing a and b. Choose coset representatives L and M for $C_t^*/C_{t_0}^*$ and $R_t^*/R_{t_0}^*$. Now choose coset representatives for C_t/C_{t_0} as $\sigma'\sigma, \sigma' \in Z, \sigma \in L$. Coset representatives for R_t/R_{t_0} can be chosen as $\tau_i\tau, \tau \in M$. The coset representatives appearing in equation 1.4 are

$$m_k(\sigma\tau\delta_0, \delta_0) = m_k(\sigma'\sigma\tau_i\tau\delta_0, \delta_0), \sigma' \in Z, \sigma \in L, \tau \in M, k = 1, 2, \dots, r.$$

Hence the coset representatives appearing in equation 1.4 becomes

$$p_{[\lambda]}(x) \quad = \quad \sum_{n,\sigma' \in Z} \sum \left| \varepsilon(\sigma'\sigma) x^{\left(\sum_{k=1}^{r} k m_k(\sigma'\sigma\tau_n\tau\delta_0, \delta_0)\right)} \right|, \text{ for } \sigma \in L, \tau \in M.$$

Now we concentrate on the inner sum

$$Q = \sum \left| \varepsilon(\sigma'\sigma) x^{\left(\sum_{k=1}^{r} k m_k(\sigma'\sigma\tau_n\tau\delta_0, \delta_0)\right)} \right|, \text{ for } \sigma \in L, \tau \in M.$$

Therefore we can write Q as $Q = \sum_{l=1}^{r} Q_{mn}^{\xi^l}$, where

$$Q_{mn}^{\xi^l} = \left| \sum \varepsilon(\sigma_m^{\xi^l}\sigma) x^{\left(\sum_{k=1}^{r} k m_k(\sigma_m^{\xi^k}\sigma\tau_n\tau\delta_0, \delta_0)\right)} \right|, \text{ for } \sigma \in L, \tau \in M, \tag{1.5}$$

To evaluate the inner sum we deal with the following four cases.

1. m and n both zero
2. $m = 0$ and $n \neq 0$
3. $m \neq 0$ and $n = 0$
4. m and n both non-zero

Subcase 1. For m and n both zero, equation 1.5 becomes

$$Q_{00}^{\xi^l} = \left| \sum \varepsilon(\sigma_0^{\xi^l}\sigma) x^{\left(\sum_{k=1}^{r} k m_k(\sigma_0^{\xi^l}\sigma\tau\delta_0, \delta_0)\right)} \right|, \text{ for } \sigma \in L, \tau \in M. \tag{1.6}$$

In this case both σ and τ fixes a and b. Let σ^*, τ^* be the corresponding restrictions of σ and τ to $\overrightarrow{S}_{f-2}$. Let δ_0^* be the restriction of δ_0 with $\{a, b\}$ omitted and let $m_k', k = 1, 2, \dots, r$ be the corresponding inner product on r-signed 1-factors of size $f-2$. The connected component of $\sigma_0^{\xi^l}\sigma\tau\delta_0 \cup$

δ_0 is precisely the orbits of $\sigma^*\tau^*\delta_0^*\cup\delta_0^*$ with $\{a,b\}$ adjoined labeled by ξ^l. Therefore $m_l(\sigma_0^{\xi^l}\sigma\tau\delta_0,\delta_0) = m'_l(\sigma^*\tau^*\delta_0^*,\delta_0^*)+1$ and $m_k(\sigma_0^{\xi^l}\sigma\tau\delta_0,\delta_0) = m'_k(\sigma^*\tau^*\delta_0^*,\delta_0^*)$ for $k\neq l$ and $k=1,2,\ldots,r$. Hence by equation 1.6, we get

$$Q_{00}^{\xi^l} = \sum \varepsilon(\sigma^*)\xi^{lj}x^l x^{\left(\sum\limits_{k=1}^{r} m'_k(\sigma^*\tau^*\delta_0^*,\delta_0^*)\right)}, \text{ for } \sigma^*\in L \text{ and } \tau^*\in M.$$

Since $\varepsilon(\sigma)=\varepsilon(\sigma^*)$ and from the above equation $Q_{00}^{\xi^l}=\xi^{lj}x^l p_{[\lambda^*]}(x)$, where $[\lambda^*]$ is $[\lambda]$ with $\lambda_{l_j}^{(j)}$ is replaced by $\lambda_{l_j}^{(j)}-2$. We know by induction that $p_{[\lambda^*]}(x)=d_{[\lambda^*]}(x)$. Therefore,

$$Q_{00}^{\xi^l} = \xi^{lj}x^l d_{[\lambda^*]}(x). \tag{1.7}$$

Subcase 2. For $m=0$ and $n\neq 0$, equation 1.5 becomes

$$Q_{0n}^{\xi^l} = \sum \varepsilon(\sigma_0^{\xi^l}\sigma)x^{\left(\sum\limits_{k=1}^{r} km_k(\sigma_0^{\xi^l}\sigma\tau_n\tau\delta_0,\delta_0)\right)}, \text{ for } \sigma\in L,\tau\in M. \tag{1.8}$$

Suppose n is fixed between 1 and $\lambda_{l_j}^{(j)}-2$. For $\sigma\in L,\tau\in M$, $\sigma^*,\tau^*,\delta_0^*,m'_k, k=1,2,\ldots,r$ be the corresponding restrictions to the diagrams for $[\lambda^*]$ of size $f-2$.

We will show that $m_k(\sigma_0^{\xi^l}\sigma\tau_n\tau\delta_0,\delta_0) = m'_k(\sigma^*\tau^*\delta_0^*,\delta_0^*)$, for all $k=1,2,\ldots,r$.

Let c be the entry in (l_j,n). Suppose c^{ξ^p} is joined in $\sigma\tau\delta_0$ to d^{ξ^q}. Note that $\sigma\tau_n\tau\delta_0$ is the same as $\sigma^*\tau^*\delta_0^*$ except $\{a,b\}$ has been added and the lines from (c^{ξ^i}) to (d^{ξ^i}) is replaced by two lines one from (c^{ξ^p}) to b and another from (d^{ξ^q}) to a. It is now clear that all orbits of $\sigma^*\tau^*\delta_0^*\cup\delta_0^*$ not containing (c^{ξ^p}) and (d^{ξ^q}) is the orbit of $\sigma_0^{\xi^l}\sigma\tau_n\tau\delta_0\cup\delta_0$.

The orbit containing (c^{ξ^p}) and (d^{ξ^q}) together with $\{a,b\}$ is the orbit of $\sigma_0^{\xi^l}\sigma\tau_n\tau\delta_0\cup\delta_0$. This shows that $m_k(\sigma_0^{\xi^l}\sigma\tau_n\tau\delta_0,\delta_0) = m'_k(\sigma^*\tau^*\delta_0^*,\delta_0^*)$, for all $k=1,2,\ldots,r$.

Therefore, by equation 1.8, we get

$$Q_{0n}^{\xi^l} = \sum \varepsilon(\sigma_0^{\xi^l}\sigma)x^{\left(\sum\limits_{k=1}^{r} km_k(\sigma^*\tau^*\delta_0^*,\delta_0^*)\right)}, \text{ for } \sigma^*\in L \text{ and } \tau^*\in M.$$

Since $\varepsilon(\sigma)=\varepsilon(\sigma^*)$ and from the above equation $Q_{0n}^{\xi^l}=p_{[\lambda^*]}(x)$, where $[\lambda^*]$ is $[\lambda]$ with $\lambda_{l_j}^{(j)}$ is replaced by $\lambda_{l_j}^{(j)}-2$. We know by induction that $p_{[\lambda^*]}(x)=d_{[\lambda^*]}(x)$. Therefore,

$$Q_{0n}^{\xi^l} = d_{[\lambda^*]}(x). \tag{1.9}$$

Subcase 3. For $m \neq 0$ and $n = 0$, equation 1.5 becomes

$$Q_{m0}^{\xi^l} = \sum \varepsilon(\sigma_m^{\xi^l}\sigma) x^{\left(\sum_{k=1}^{r} k m_k(\sigma_m^{\xi^l}\sigma\tau\delta_0, \delta_0)\right)}, \text{ for } \sigma \in L, \tau \in M. \tag{1.10}$$

We will show that $m_k(\sigma_m^{\xi^l}\sigma\tau\delta_0, \delta_0) = m'_k(\sigma^*\tau^*\delta_0^*, \delta_0^*)$, for all $k = 1, 2, \ldots, r$. For $m = 1, 2, \ldots, l_j - 1$, we show that each is $Q_{m0}^{\xi^l} = -d_{[\lambda^*]}(x)$.
We need to consider the orbits of $\sigma_m^{\xi^l}\sigma\tau\delta_0 \cup \delta_0$ where $\sigma_m^{\xi^l}$ is the r-signed transposition interchanging b with entry above it in the m^{th} row and $\lambda_{l_j}^{(j)}$ column, labeled by ξ^l. Again we have to consider the restricted term $\sigma^*\tau^*\delta_0^*$. Let c^{ξ^p} be the entry in $(m, \lambda_{l_j}^{(j)})$ position in $\sigma^*\tau^*\delta_0^*$ joined to entry d^{ξ^q} in $\sigma^*\tau^*\delta_0^*$. The lines in $\sigma_m^{\xi^l}\sigma\tau\delta_0$ are precisely the lines in $\sigma^*\tau^*\delta_0^*$ except line from c^{ξ^p} to d^{ξ^q} is replaced by a line from d^{ξ^q} to b and one from a to c^{ξ^p}.
Again the orbits of $\sigma^*\tau^*\delta_0^* \cup \delta_0^*$ are those of $\sigma_m^{\xi^l}\sigma\tau\delta_0 \cup \delta_0$ except for this one orbit through c^{ξ^p} and d^{ξ^q}. This shows that $m_k(\sigma_m^{\xi^l}\sigma\tau\delta_0, \delta_0) = m'_k(\sigma^*\tau^*\delta_0^*, \delta_0^*)$, for all $k = 1, 2, \ldots, r$.
Note $\varepsilon(\sigma_m^{\xi^l}\sigma) = -\varepsilon(\sigma^*) = -\varepsilon(\sigma)$.
Therefore, by equation 1.10, we get

$$Q_{m0}^{\xi^l} = -\sum \varepsilon(\sigma) x^{\left(\sum_{k=1}^{r} k m_k(\sigma^*\tau^*\delta_0^*, \delta_0^*)\right)}, \text{ for } \sigma^* \in L \text{ and } \tau^* \in M.$$

Hence by the above equation, we get $Q_{m0}^{\xi^l} = -p_{[\lambda^*]}(x)$, where $[\lambda^*]$ is $[\lambda]$ with $\lambda_{l_j}^{(j)}$ is replaced by $\lambda_{l_j}^{(j)} - 2$. We know by induction that $p_{[\lambda^*]}(x) = d_{[\lambda^*]}(x)$. Therefore,

$$Q_{m0}^{\xi^l} = -d_{[\lambda^*]}(x). \tag{1.11}$$

Subcase 4. In this case both $m \neq 0$ and $n \neq 0$, we wish to show that $Q_{mn}^{\xi^l} = 0, \forall l$.
The term in $Q_{mn}^{\xi^l}$, for a fixed m, n, σ, τ with m and n not equal to zero is

$$\varepsilon(\sigma_m^{\xi^l}\sigma) x^{\left(\sum_{k=1}^{r} k m_k(\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0, \delta_0)\right)}.$$

We will show that how to combine the terms for a fixed m into disjoint subsets of size two. The sum over each of these subsets will be zero and so the sum over all these terms in $Q_{mn}^{\xi^l}$ will be zero.
In order to choose the subsets, we suppose m, σ, τ, n are chosen with both m and n both being not zero. Let $(a')^{\xi^p}$ be the endpoint for the

line joined in $\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0$ to a and $(b')^{\xi^q}$ be the endpoint for the line joined in $\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0$ to b. The point $(a')^{\xi^p}$ must be left of a, since $n \neq 0$. Suppose $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in the even numbered column.
Let $\sigma' = ((b')^{\xi^q}, (a')^{\xi^p})\sigma$.
As $\varepsilon((b')^{\xi^q}, (a')^{\xi^p}) = \xi^k$, $\varepsilon(\sigma_m^{\xi^l}\sigma') = \varepsilon(\sigma_m^{\xi^l}((b')^{\xi^q}, (a')^{\xi^p})\sigma) = \xi^k\varepsilon(\sigma_m^{\xi^l}\sigma)$. The orbits of $\sigma_m^{\xi^l}(((b')^{\xi^q}), (a')^{\xi^p})\sigma\tau_n\tau\delta_0 \cup \delta_0$ is the same as the orbits of $\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0 \cup \delta_0$ except $(a')^{\xi^p}$ is the endpoint of b and $(b')^{\xi^q}$ is the endpoint of a. Also the lengths of the orbits and the number of orbits of both were same but their signs were opposite. Therefore they cancel each other, such terms in $Q_{mn}^{\xi^l}$ cancels and the sum of those terms will be zero.
Suppose $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in the odd numbered column. Instead of σ we start with $\sigma' = ((b')^{\xi^q}, (a')^{\xi^p})$, the same result holds using the r-signed transpositions $((b^*)^{\xi^q}, (a^*)^{\xi^p})$ where $(a^*)^{\xi^p}$ is to the immediate right of $(a')^{\xi^p}$ in the even numbered column and $(b^*)^{\xi^q}$ is to the immediate right of $(b')^{\xi^q}$ in the even numbered column.
Suppose $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in different columns. Let c^{ξ^k} be the position $(j, \lambda_{l_j}^{(j)})$. Note that c^{ξ^k} and $(b')^{\xi^q}$ are joined in $\sigma\tau_n\tau\delta_0$. This means $(\sigma\tau_n\tau)^{-1}(c^{\xi^k})$ and $(\sigma\tau_n\tau)^{-1}((b')^{\xi^q})$ are in the same row. Let d' be the entry such that $\sigma\tau_n\tau(d')$ is in the same row as c and $(b')^{\xi^q}$ and in the same column as $(a')^{\xi^p}$. As $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in different columns, $(b')^{\xi^q}$ is not d'. Denote the point joined to d' in $\sigma\tau_n\tau\delta_0$ by (c''). Note that $(\sigma\tau_n\tau)^{-1}(c'')$ is in the same row as $(\sigma\tau_n\tau)^{-1}((b')^{\xi^q})$,$(\sigma\tau_n\tau)^{-1}(c)$ and $(\sigma\tau_n\tau)^{-1}(d')$. Let τ' be the coset representative in M for which $\tau_k\tau'\delta_0 = ((\sigma\tau_n\tau)^{-1}((b')^{\xi^q}), (\sigma\tau_n\tau)^{-1}(c''))\tau_n\tau\delta_0$ for which $\tau_k\tau\delta_0$ is the same as $\tau_n\tau\delta_0$ except that $(\sigma\tau_n\tau)^{-1}(c)$ is joined to $(\sigma\tau_n\tau)^{-1}(c'')$ and $(\sigma\tau_n\tau)^{-1}(d')$ is joined to $(\sigma\tau_n\tau)^{-1}(b')^{\xi^q})$.
Assume now that $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in different even numbered column. We examine the terms in the sum for the r-signed transposition τ_n interchanges $2r-1$ and $f-\lambda_{l_j}^{(j)}+n$ and for the r-signed transposition τ_k interchanges $2r-1$ and $f-\lambda_{l_j}^{(j)}+k$ corresponding to $\sigma'(d', (a')^{\xi^p})\sigma\tau_k\tau$ and to $\sigma'\sigma\tau_n\tau$. Note that $\varepsilon(\sigma_m^{\xi^l}(d', (a')^{\xi^p})\sigma) = \xi^p j\varepsilon(\sigma_m\sigma)$. The orbits of $\sigma_m(d', (a')^{\xi^p})\sigma\tau_k\tau\delta_0 \cup \delta_0$ and the orbits of $\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0 \cup \delta_0$ are the same except the ones through $\{a, b\}, \{(a')^{\xi^p}, (b')^{\xi^q}\}$ and $\{d', c''\}$. $(a')^{\xi^p}$ is joined in $\sigma_m^{\xi^l}(d', (a')^{\xi^p})\sigma\tau_k\tau\delta_0$ to c' and $(a')^{\xi^p}$ is joined in $\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0$ to a and b is joined to $(b')^{\xi^q}$. The orbits of those terms were same and their signs were different, therefore they cancels each other. If $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in odd numbered column use the above result and this proves sum of those terms will be zero. Hence,

$$Q_{mn}^{\xi^l} = 0. \tag{1.12}$$

Thus, by equations 1.7, 1.9, 1.11 and 1.12, we get

$$\begin{aligned} Q &= \left|\sum_{l=1}^{r} Q_{00}^{\xi^l} + \sum_{l=1}^{r} \sum_{j=1}^{\lambda_{l_j}^{(j)}-2} Q_{0j}^{\xi^l} + \sum_{l=1}^{r} \sum_{i=1}^{l_j-1} Q_{i0}^{\xi^l} + \sum_{l=1}^{r} \sum_{i,j} Q_{ij}^{\xi^l}\right| \\ &= \left|(x^r + \xi^{(r-1)j} x^{r-1} + \ldots + \xi^j x + 2(\lambda_{l_j}^{(j)} - 2) - 2(l_j - 1)) d_{[\lambda^*]}(x)\right| \end{aligned}$$

From the definition of $d_{[\lambda]}(x)$ and $d_{[\lambda^*]}(x)$,

$$\begin{aligned} d_{[\lambda]}(x) &= \left|(x^r + \xi^{j(r-1)} x^{r-1} + \ldots + \xi^j x + 2(\lambda_{l_j}^{(j)} - 2) - 2(l_j - 1))\right| d_{[\lambda^*]}(x) \\ &= h_{[\lambda]}(x). \end{aligned}$$

$$h_{[\lambda]}(x) = d_{[\lambda]}(x) = \prod_{i=1}^{j} \prod_{d \in d^{(i)}} \prod_{k,l} s_i \left|(x^r + \xi^{i(r-1)} x^{r-1} + \ldots + \xi^i x + a_{kl}^{(i)})\right|.$$

Case 2. If s^{th} horizontal edge lies in the r^{th} residue, we may concentrate on the $(l_r, \lambda_{l_r}^{(r)}-1)$ and the $(l_r, \lambda_{l_r}^{(r)})$ positions. For convenience, we call it the position as a and b. In order to evaluate $m_k(\sigma\tau\delta_0, \delta_0)$, for all $k = 1, 2, \ldots, r$, place the lines from δ_0 in the diagram t. Coset representatives for $R_i^{(r)}/R_{0i}^{(r)}$ may be chosen anyway for $i = 1, 2, \ldots, l_r - 1$. We choose coset representatives for $R_{l_r}^{(r)}/R_{0m}^{(r)}$ by restricting to a group $R_{l_r^*}^{(r)}$, the subgroup of $R_{l_r}^{(r)}$ fixing a and b. Let $R_{l_r 0^*}^{(r)}$ be the subgroup of $R_{l_r^*}^{(r)}$ fixing the r-signed 1-factor δ_0. Choose Y a set of representatives for $R_{l_r^*}^{(r)}/R_{l_r 0^*}^{(r)}$. Let Y' be the set consisting of r-signed transpositions $\tau_j^{\xi^k}$,for all $k = 1, 2, \ldots, r$ of $R_{l_r}^{(r)}$ where $\tau_j^{\xi^k}$ interchanges $2r-1$ and $2r - \lambda_{l_r}^{(r)} + j$ with the sign change ξ^k in $2r-1$ for $j = 1, 2, \ldots, \lambda_{l_r}^{(r)} - 2$. For $i = 0, \tau_0^{\xi^k}$ is the identity with the sign change ξ^k in $2r - 1$. The elements $\tau' Y, \tau' \in Y'$ is the set of representatives for $R_{l_r}^{(r)}/R_{l_r 0}^{(r)}$.

We also wish to choose the coset representatives for the subgroup of C_t moving $\lambda_{l_r}^{(r)}$ columns only. Denote this subgroup by $C_{l_r}^{(r)}$. Let $C_{l_r^*}^{(r)}$ be the subgroup of $C_{l_r}^{(r)}$ fixing b. Let σ_i be the r-signed transpositions interchanging b with the entry above it in the i^{th} row for $i = 0, 1, \ldots, l_r - 1$. Coset representatives for $C_{l_r}^{(r)}/C_{l_r 0}^{(r)}$ may be chosen as $\sigma_i C_{l_j-1}^{(r)}$ where $C_{l_j-1}^{(r)}$ interchanges $\lambda_{l_j-1}^{(r)}$ column. Now let C_t^* and R_t^* be the subgroups of C_t and R_t fixing a and b respectively. Let $C_{t_0}^*$ and $R_{t_0}^*$ be the stabilisers of δ_0 fixing a and b. Choose coset representatives L and M for $C_t^*/C_{t_0}^*$ and $R_t^*/R_{t_0}^*$. Now choose coset representatives for C_t/C_{t_0} as $\sigma_i\sigma, \sigma \in L$. Coset representatives for R_t/R_{t_0} can be chosen as $\tau'\tau, \tau' \in Y', \tau \in M$. The coset representatives appearing in equation 1.4 are

$$m_k(\sigma\tau\delta_0, \delta_0) = m_k(\sigma_i\sigma\tau'\tau\delta_0, \delta_0),$$

where $\sigma \in L, \tau' \in Y', \tau \in M$ and for all $k = 1, 2, \ldots, r$.
Hence the coset representatives appearing in equation 1.4 becomes

$$p_{[\lambda]}(x) = \sum\sum \varepsilon(\sigma_i\sigma)x^{\left(\sum_{k=1}^{r} km_k(\sigma_i\sigma\tau'\tau\delta_0,\delta_0)\right)}, \text{ for } \sigma \in L, \tau \in M.$$

Now we concentrate on the inner sum

$$Q = \sum \varepsilon(\sigma_i\sigma)x^{\left(\sum_{k=1}^{r} km_k(\sigma_i\sigma\tau'\tau\delta_0,\delta_0)\right)}, \text{ for } \sigma \in L, \tau \in M.$$

We can also write Q as $Q = \sum_{k=1}^{r} Q_{mn}^{\xi^l}$, where

$$Q_{mn}^{\xi^l} = \sum \varepsilon(\sigma)x^{\left(\sum_{k=1}^{r} km_k(\sigma_m\sigma\tau_n^{\xi^l}\tau\delta_0,\delta_0)\right)}, \text{ for } \sigma \in L, \tau \in M. \tag{1.13}$$

To evaluate the inner sum we deal with the following four cases.

1. m and n both zero
2. $m = 0$ and $n \neq 0$
3. $m \neq 0$ and $n = 0$
4. m and n both non-zero

Subcase 1. For $m = 0$ and $n = 0$, equation 1.13 becomes

$$Q_{00}^{\xi^l} = \sum \varepsilon(\sigma)x^{\left(\sum_{k=1}^{r} km_k(\sigma\tau_0^{\xi^l}\tau\delta_0,\delta_0)\right)}, \text{ for } \sigma \in L, \tau \in M. \tag{1.14}$$

In this case both σ and τ fixes a and b. Let σ^*, τ^* be the corresponding restrictions of σ and τ to $\overrightarrow{S}_{f-2}$. Let δ_0^* be the restriction of δ_0 with $\{a, b\}$ omitted and let m'_k, for all $k = 1, 2, \ldots, r$ be the corresponding inner products on r-signed 1-factors of size $f - 2$. The connected component of $\sigma\tau_0^{\xi^l}\tau\delta_0 \cup \delta_0$ is precisely the orbits of $\sigma^*\tau^*\delta_0^* \cup \delta_0^*$ with $\{a, b\}$ adjoined with sign change ξ^l. Therefore $m_l(\sigma\tau_0^{\xi^l}\tau\delta_0, \delta_0) = m'_l(\sigma^*\tau^*\delta_0^* \cup \delta_0^*) + 1$ and $m_k(\sigma\tau_0^{\xi^l}\tau\delta_0, \delta_0) = m'_k(\sigma^*\tau^*\delta_0^*, \delta_0^*)$ for $k \neq l$ and $k = 1, 2, \ldots, r$. Hence by equation 1.14, we get

$$Q_{00}^{\xi^l} = \sum \varepsilon(\sigma)x^l x^{\left(\sum_{k=1}^{r} km'_k(\sigma^*\tau^*\delta_0^*,\delta_0^*)\right)}, \text{ for } \sigma^* \in L \text{ and } \tau^* \in M.$$

Since $\varepsilon(\sigma) = \varepsilon(\sigma^*)$, $Q_{00}^{\xi^l} = x^l p_{[\lambda^*]}(x)$, where $[\lambda^*]$ is $[\lambda]$ with $\lambda_{l_r}^{(r)}$ replaced by $\lambda_{l_r}^{(r)} - 2$. We know by induction $p_{[\lambda^*]}(x) = d_{[\lambda^*]}(x)$ and so we get

$$Q_{00}^{\xi^l} = x^l d_{[\lambda^*]}(x). \tag{1.15}$$

Subcase 2. For $m = 0$ and $n \neq 0$, equation 1.13 becomes

$$Q_{0n}^{\xi^l} = \sum \varepsilon(\sigma) x^{\left(\sum_{k=1}^{r} km_k(\sigma\tau_n\tau\delta_0,\delta_0)\right)}, \text{ for } \sigma \in L, \tau \in M. \tag{1.16}$$

Suppose n is fixed between 1 and $\lambda_{l_r}^{(r)} - 2$. For $\sigma \in L, \tau \in M$, $\sigma^*, \tau^*, \delta_0^*, m'_k, k = 1, 2, \ldots, r$ be the corresponding restrictions to the diagrams for $[\lambda^*]$ of size $f - 2$.
We will show that $m_k(\sigma\tau_n\tau\delta_0, \delta_0) = m'_k(\sigma^*\tau^*\delta_0^*, \delta_0^*)$, for all $k = 1, 2, \ldots, r$.
Let c be the entry in (l_r, n) and $\tau_n^{\xi^l}$ be the r-signed transpositions in Y' which interchanges $2r - 1$ and $f - \lambda_{l_r}^{(r)} + j$. Suppose c^{ξ^p} is joined in $\sigma^*\tau^*\delta_0$ to d^{ξ^q}. Note that $\sigma\tau_n\tau\delta_0$ are the same as $\sigma^*\tau^*\delta_0^*$ except $\{a, b\}$ has been added and the lines from c^{ξ^p} to (d^{ξ^q}) is replaced by two lines, one from $(c)^{\xi^p}$ to b and another from $(d)^{\xi^q}$ to a.
It is now clear that all orbits of $\sigma^*\tau^*\delta_0^* \cup \delta_0^*$ not containing $(c)^{\xi^p}$ and $(d)^{\xi^q}$ is the orbit of $\sigma\tau_n\tau\delta_0 \cup \delta_0$. The orbit containing $(c)^{\xi^p}$ and $(d)^{\xi^q}$ together with $\{a, b\}$ is an orbit of $\sigma\tau_n\tau\delta_0 \cup \delta_0$. This shows that $m_k(\sigma\tau_n\tau\delta_0, \delta_0) = m'_k(\sigma^*\tau^*\delta_0^*, \delta_0^*)$, for all $k = 1, 2, \ldots, r$.
Therefore, by equation 1.16, we get

$$Q_{0n}^{\xi^l} = \sum \varepsilon(\sigma^*) x^{\left(\sum_{k=1}^{r} km_k(\sigma^*\tau^*\delta_0^*,\delta_0^*)\right)}, \text{ for } \sigma^* \in L \text{ and } \tau^* \in M.$$

Since $\varepsilon(\sigma) = \varepsilon(\sigma^*)$ and from the above equation $Q_{0n}^{\xi^l} = p_{[\lambda^*]}(x)$, where $[\lambda^*]$ is $[\lambda]$ with $\lambda_{l_j}^{(j)}$ is replaced by $\lambda_{l_j}^{(j)} - 2$. We know by induction that $p_{[\lambda^*]}(x) = d_{[\lambda^*]}(x)$. Therefore,

$$Q_{0n}^{\xi^l} = d_{[\lambda^*]}(x). \tag{1.17}$$

Subcase 3. For $m \neq 0$ and $n = 0$, equation 1.13 becomes

$$Q_{i0}^{\xi^l} = \sum \varepsilon(\sigma) x^{\left(\sum_{k=1}^{r} km_k(\sigma_m\sigma\tau_0^{\xi^l}\tau\delta_0,\delta_0)\right)}, \text{ for } \sigma \in L, \tau \in M. \tag{1.18}$$

We will show that $m_k(\sigma_m\sigma\tau_0^{\xi^l}\tau\delta_0, \delta_0) = m'_k(\sigma^*\tau^*\delta_0^*, \delta_0^*)$, for all $k = 1, 2, \ldots, r$. For $m = 1, 2, \ldots, l_r - 1$ and show that each is $-d_{[\lambda^*]}(x)$.

We need to consider the orbits of $\sigma_m\sigma\tau^{\xi^l}\tau\delta_0 \cup \delta_0$ where σ_i is the r-signed transposition interchanges b with entry above it in the m^{th} row and $\lambda_{l_r}^{(r)}$ column. Again we have to consider the restricted term $\sigma^*\tau^*\delta_0^*$. Let c be the entry in $(i, \lambda_{l_r}^{(r)})$ position, c^{ξ^p} is joined in $\sigma^*\tau^*\delta_0^*$ to entry d^{ξ^q} in $\sigma^*\tau^*\delta_0^*$. The lines in $\sigma_m\sigma\tau_0^{\xi^l}\tau\delta_0$ are precisely the lines in $\sigma^*\tau^*\delta_0^*$ except line from c^{ξ^p} to d^{ξ^q} is replaced by a line from d^{ξ^q} to b and one from a to c^{ξ^p}.

Again the orbits of $\sigma^*\tau^*\delta_0^*\cup\delta_0^*$ are those of $\sigma_m\sigma\tau_0^{\xi^l}\tau\delta_0\cup\delta_0$ except for this one orbit through c^{ξ^p} and d^{ξ^q}. This shows that $m_k(\sigma_m\sigma\tau_0^{\xi^l}\tau\delta_0,\delta_0) = m'_k(\sigma^*\tau^*\delta_0^*,\delta_0^*)$, for all $k=1,2,\ldots,r$.

Note $\varepsilon(\sigma_m\sigma) = -\varepsilon(\sigma^*) = -\varepsilon(\sigma)$.

Therefore, by equation 1.18, we get

$$Q_{m0}^{\xi^l} = -\sum \varepsilon(\sigma)x^{\left(\sum\limits_{k=1}^{r} km_k(\sigma^*\tau^*\delta_0^*,\delta_0^*)\right)}, \text{ for } \sigma^*\in L \text{ and } \tau^*\in M.$$

Hence by the above equation, we get $Q_{m0}^{\xi^l} = -p_{[\lambda^*]}(x)$, where $[\lambda^*]$ is $[\lambda]$ with $\lambda_{l_j}^{(j)}$ is replaced by $\lambda_{l_j}^{(j)}-2$. We know by induction that $p_{[\lambda^*]}(x) = d_{[\lambda^*]}(x)$. Therefore,

$$Q_{m0}^{\xi^l} = -d_{[\lambda^*]}(x). \tag{1.19}$$

Subcase 4. In this case both $m\neq 0$ and $n\neq 0$, we wish to show that $Q_{mn}^{\xi^l} = 0, \forall l$.

The term in $Q_{mn}^{\xi^l}$, for a fixed m,n,σ,τ with m and n not equal to zero is $\varepsilon(\sigma_m^{\xi^l}\sigma)x^{\left(\sum\limits_{k=1}^{r} km_k(\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0,\delta_0)\right)}$. We will show that how to combine the terms for a fixed m into disjoint subsets of size two. The sum over each of these subsets will be zero and so the sum over all these terms in $Q_{mn}^{\xi^l}$ will be zero.

In order to choose the subsets, we suppose m,σ,τ,n are chosen with both m and n both being not zero. Let $(a')^{\xi^p}$ be the endpoint for the line joined in $\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0$ to a and $(b')^{\xi^q}$ be the endpoint for the line joined in $\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0$ to b. The point $(a')^{\xi^p}$ must be left of a, since $n\neq 0$. Suppose $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in the even numbered column.

Let $\sigma' = ((b')^{\xi^q},(a')^{\xi^p})\sigma$.

As $\varepsilon((b')^{\xi^q},(a')^{\xi^p}) = \xi^k$, $\varepsilon(\sigma_m^{\xi^l}\sigma') = \varepsilon(\sigma_m^{\xi^l}((b')^{\xi^q},(a')^{\xi^p})\sigma) = \xi^k\varepsilon(\sigma_m^{\xi^l}\sigma)$. The orbits of $\sigma_m^{\xi^l}(((b')^{\xi^q}),(a')^{\xi^p})\sigma\tau_n\tau\delta_0\cup\delta_0$ is the same as the orbits of $\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0\cup\delta_0$ except $(a')^{\xi^p}$ is the endpoint of b and $(b')^{\xi^q}$ is the endpoint of a. Also the lengths of the orbits and the number of orbits of both were same but their signs were opposite. Therefore they cancel each other, such terms in $Q_{mn}^{\xi^l}$ cancels and the sum of those terms will be zero.

Suppose $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in the odd numbered column. Instead of σ we start with $\sigma' = ((b')^{\xi^q},(a')^{\xi^p})$, the same result holds using the

r-signed transpositions $((b^*)^{\xi^q}, (a^*)^{\xi^p})$ where $(a^*)^{\xi^p}$ is to the immediate right of $(a')^{\xi^p}$ in the even numbered column and $(b^*)^{\xi^q}$ is to the immediate right of $(b')^{\xi^q}$ in the even numbered column.

Suppose $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in different columns. Let c^{ξ^k} be the position $(j, \lambda_{l_j}^{(j)})$. Note that c^{ξ^k} and $(b')^{\xi^q}$ are joined in $\sigma\tau_n\tau\delta_0$. This means $(\sigma\tau_n\tau)^{-1}(c^{\xi^k})$ and $(\sigma\tau_n\tau)^{-1}((b')^{\xi^q})$ are in the same row. Let d' be the entry such that $\sigma\tau_n\tau(d')$ is in the same row as c and $(b')^{\xi^q}$ and in the same column as $(a')^{\xi^p}$. As $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in different columns, $(b')^{\xi^q}$ is not d'. Denote the point joined to d' in $\sigma\tau_n\tau\delta_0$ by (c''). Note that $(\sigma\tau_n\tau)^{-1}(c'')$ is in the same row as $(\sigma\tau_n\tau)^{-1}((b')^{\xi^q})$,$(\sigma\tau_n\tau)^{-1}(c)$ and $(\sigma\tau_n\tau)^{-1}(d')$. Let τ' be the coset representative in M for which $\tau_k\tau'\delta_0 = ((\sigma\tau_n\tau)^{-1}((b')^{\xi^q}), (\sigma\tau_n\tau)^{-1}(c''))\tau_n\tau\delta_0$ for which $\tau_k\tau\delta_0$ is the same as $\tau_n\tau\delta_0$ except that $(\sigma\tau_n\tau)^{-1}(c)$ is joined to $(\sigma\tau_n\tau)^{-1}(c'')$ and $(\sigma\tau_n\tau)^{-1}(d')$ is joined to $(\sigma\tau_n\tau)^{-1}(b')^{\xi^q})$.

Assume now that $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in different even numbered column. We examine the terms in the sum for the r-signed transposition τ_n interchanges $2r-1$ and $f-\lambda_{l_j}^{(j)}+n$ and for the r-signed transposition τ_k interchanges $2r-1$ and $f-\lambda_{l_j}^{(j)}+k$ corresponding to $\sigma'(d', (a')^{\xi^p})\sigma\tau_k\tau$ and to $\sigma'\sigma\tau_n\tau$. Note that $\varepsilon(\sigma_m^{\xi^l}(d', (a')^{\xi^p})\sigma) = \xi^p j\varepsilon(\sigma_m\sigma)$. The orbits of $\sigma_m(d', (a')^{\xi^p})\sigma\tau_k\tau\delta_0 \cup \delta_0$ and the orbits of $\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0 \cup \delta_0$ are the same except the ones through $\{a, b\}, \{(a')^{\xi^p}, (b')^{\xi^q}\}$ and $\{d', c''\}$. $(a')^{\xi^p}$ is joined in $\sigma_m^{\xi^l}(d', (a')^{\xi^p})\sigma\tau_k\tau\delta_0$ to c' and $(a')^{\xi^p}$ is joined in $\sigma_m^{\xi^l}\sigma\tau_n\tau\delta_0$ to a and b is joined to $(b')^{\xi^q}$. The orbits of those terms were same and their signs were different, therefore they cancels each other. If $(a')^{\xi^p}$ and $(b')^{\xi^q}$ are in odd numbered column use the above result and this proves sum of those terms will be zero. Hence,

$$Q_{mn}^{\xi^l} = 0. \tag{1.20}$$

Thus, by equations 1.15, 1.17, 1.19 and 1.20, we get

$$\begin{aligned} Q &= \sum_{l=1}^{r} \left| Q_{00}^{\xi^l} + \sum_{l=1}^{r} \sum_{j=1}^{\lambda_{l_r}-2} Q_{0j}^{\xi^l} + \sum_{l=1}^{r} \sum_{i=1}^{l_r-1} Q_{i0}^{\xi^l} + \sum_{i,j} Q_{ij}^{\xi^l} \right| \\ &= \left| (x^r + x^{r-1} + \ldots + x + 2(\lambda_{l_r}^{(r)} - 2) - 2(l_r - 1)) \right| d_{[\lambda^*]}(x) \end{aligned}$$

From the definition of $d_{[\lambda]}(x)$ and $d_{[\lambda^*]}(x)$,

$$d_{[\lambda]}(x) = \left| (x^r + x^{r-1} + \ldots + x + 2(\lambda_{l_r}^{(r)} - 2) - 2(l_r - 1)) \right| d_{[\lambda^*]}(x) = h_{[\lambda]}(x).$$

$$h_{[\lambda]}(x) = d_{[\lambda]}(x) = \prod_{i=1}^{r} \prod_{d \in d^{(i)}} \prod_{k,l} \left| (x^r + \xi^{i(r-1)}x + \ldots + \xi^i x + a_{kl}^{(i)}) \right|.$$

Therefore,

$$h_{[\lambda]}(x) = \prod_{i=1}^{r} \prod_{d \in d^{(i)}} \prod_{k,l} \Big| (x^r + \xi^{i(r-1)} x^{r-1} + \ldots + \xi^i x + a_{kl}^{(i)}) \Big| .$$

Hence the proof. □

Corollary 6.3. *[6] Let* $[\lambda] = (\lambda_1, \lambda_2, \ldots, \lambda_m$ *be a partition of* f *with all* λ_j *even. Let* $V^{[\lambda]}$ *be the subspace of* V_i *associated to the partition* λ *and* $h_\lambda(x) = h_i(x)$*. Then*

$$h_\lambda(x) = \prod (x + a_{ij}),$$

where a_{ij} *are in the diagram* d *of shape* λ*.*

Proof. The proof follows from the above theorem for $r = 1$. □

Corollary 6.4. *Let* $[\lambda] = (\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_l^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_m^{(2)})$ *be an even bi-partition of* f*. Let* $V^{[\lambda]}$ *be the subspace of* V_i *associated to the bi-partition* $[\lambda]$ *and* $h_r(x) = h_{[\lambda]}(x)$*. Then*

$$h_{[\lambda]}(x) = \prod (x^2 - x + a_{ij}^{(1)}) \prod (x^2 + x + a_{ij}^{(2)}),$$

where $a_{ij}^{(1)}$ *are in the diagram* $d^{(1)}$ *of shape* $\lambda^{(1)}$ *and* $a_{ij}^{(2)}$ *are in the diagram* $d^{(2)}$ *of shape* $\lambda^{(2)}$*.*

Proof. The proof follows from the above theorem for $r = 2$. □

For $r = 3$ and $f = 2$, there are three even multipartitions of 2. They are $\lambda_1 = \big(\square\square, \emptyset, \emptyset\big)$, $\lambda_2 = \big(\emptyset, \square\square, \emptyset\big)$ and $\lambda_3 = \big(\emptyset, \emptyset, \square\square\big)$.

$$\begin{aligned}
h_{\lambda_1} &= |x^3 + \xi^2 x^2 + \xi x| = \sqrt{(x^3 + \xi^2 x^2 + \xi x)\,\overline{(x^3 + \xi^2 x^2 + \xi x)}} \\
&= x(x-1)\sqrt{x^2 + x + 1} \\
h_{\lambda_2} &= |x^3 + \xi x^2 + \xi^2 x| = \sqrt{(x^3 + \xi x^2 + \xi^2 x)\,\overline{(x^3 + \xi x^2 + \xi^2 x)}} \\
&= -x(x-1)\sqrt{x^2 + x + 1} \\
h_{\lambda_3} &= |x^3 + x^2 + x| = \sqrt{(x^3 + x^2 + x)\,\overline{(x^3 + x^2 + x)}} \\
&= \big(x^3 + x^2 + x\big)
\end{aligned}$$

which is same as in example 6.1.

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