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The Three-step Intermixed Iteration for Two Finite Families of Nonlinear Mappings in a Hilbert Space

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Abstract. In this work, the three-step intermixed iteration for two finite families of nonlinear mappings is introduced. We prove a strong convergence theorem for approximating a common fixed point of a strict pseudo-contraction and strictly pseudononspreading mapping in a Hilbert space. Some additional results are obtained. Finally, a numerical example in a space of real numbers is also given and illustrated.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. The fixed point problem for the mapping $T : C \to C$ is to find $x \in C$ such that

$$x = Tx.$$

We denote the fixed point set of a mapping T by $Fix(T)$.

Definition 1.1. Let $T : C \to C$ be a mapping. Then

(i) a mapping T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C;$$

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(ii) T is said to be *κ-strictly pseudo-contractive* if there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa \|(I-T)x - (I-T)y\|^2, \forall x, y \in C.$$

Note that the class of κ-strict pseudo-contractions strictly includes the class of nonexpansive mappings, that is, a nonexpansive mapping is a 0-strictly pseudo-contractive mapping.

In 2008, Kohsaka and Takahashi [6] introduced *a nonspreading mapping T* in Hilbert space H as follows:

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|x - Ty\|^2, \forall x, y \in C. \tag{1.1}$$

In 2009, it is shown by Iemoto and Takahashi [2] that (1.1) is equivalent to the following equation.

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \text{ for all } x, y \in C.$$

Later, in 2011, Osilike and Isiogugu [13] proposed a *κ-strictly pseudononspreading mapping*, that is, a mapping $T : C \to C$ is said to be a *κ-strictly pseudononspreading mapping* if there exists $\kappa \in [0,1)$ such that

$$\|Tx-Ty\|^2 \le \|x-y\|^2 + \kappa\|(I-T)x-(I-T)y\|^2 + 2\langle x-Tx, y-Ty\rangle, \text{ for all } x, y \in C.$$

Obviously, every nonspreading mapping is a κ-strictly pseudononspreading mapping, that is, a nonspreading mapping is a 0-stricly pseudononspreading mapping.

Many mathematicians tried to proposed iterative algorithms and proved the strong convergence theorems for a nonspreading mapping and a strictly pseudononspreading mapping in Hilbert space to find their fixed points, see, for instance, [7, 13, 8, 1].

Over the past decades, many others have constructed various types of iterative methods to approximate fixed points. The first one is the Mann iteration introduced by Mann [9] in 1953 and is defined as follows:

$$\begin{cases} x_0 \in H \text{ arbitrary chosen}, \\ x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T x_n, \forall n \ge 0, \end{cases} \tag{1.2}$$

where C is a nonempty closed convex subset of a normed space, $T : C \to C$ is a mapping and the sequence $\{\alpha_n\}$ is in the interval (0,1). But this algorithm has only weak convergence. Thus, many mathematicians have been trying to modify Mann's iteration (1.2) and construct new iterative method to obtain the strong convergence theorem.

By modification of Mann's iteration (1.2), the next iteration process is referred to as Ishikawa's iteration process [3] which is defined recursively as follows:

$$\begin{cases} x_0 \in H \text{ arbitrary chosen}, \\ y_n = \beta_n x_n + (1-\beta_n)\, T x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\, T y_n, \forall n \ge 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. He also obtain the strong convergence theorem for the iterative method (1.3) converging to a fixed point of mapping T. Observe that if $\beta_n = 1$, then the Ishikawa's iteration (1.3) reduces to the Mann's iteration (1.2).

In 2000, Moudafi [11] introduced the viscosity approximation method for nonexpansive mapping S as follows:
Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping such that $Fix(S)$ is nonempty. Let $f : C \to C$ be a contraction, that is, there exists $\alpha \in (0,1)$ such that $\|fx - fy\| \leq \alpha \|x - y\|, \forall x, y \in C$, and let $\{x_n\}$ be a sequence defined by

$$(1.4) \qquad \begin{cases} x_1 \in C \text{ arbitrary chosen,} \\ x_{n+1} = \frac{1}{1+\epsilon_n} S x_n + \frac{\epsilon_n}{1+\epsilon_n} f(x_n), \forall n \in \mathbb{N}, \end{cases}$$

where $\{\varepsilon_n\} \subset (0,1)$ satisfies certain conditions. Then the sequence $\{x_n\}$ converges strongly to $z \in Fix(S)$, where $z = P_{Fix(S)} f(z)$ and $P_{Fix(S)}$ is the metric projection of H onto $Fix(S)$.

In 2006, using the concept of the viscosity approximation method (1.4), Marino and Xu [10] introduced the general iterative method and obtained the strong convergence theorem. Let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $f : H \to H$ be a contractive mapping on H and let $\{x_n\}$ be generated by

$$(1.5) \qquad \begin{cases} x_0 \in H \text{ arbitrary chosen,} \\ x_{n+1} = (I - \alpha_n A) T x_n + \alpha_n \gamma f(x_n), n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the appropriate conditions. Then $\{x_n\}$ converges strongly to a fixed point $\tilde{x}$ of T which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z \rangle \leq 0, z \in Fix(T).$$

In 2015, Yao *et al.* [18] proposed the intermixed algorithm for two strict pseudocontractions S and T as follows:

Algorithm 1.2. For arbitrarily given $x_0 \in C, y_0 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$(1.6) \qquad \begin{aligned} x_{n+1} &= (1 - \beta_n) x_n + \beta_n P_C [\alpha_n f(y_n) + (1 - k - \alpha_n) x_n + kTx_n], n \geq 0, \\ y_{n+1} &= (1 - \beta_n) y_n + \beta_n P_C [\alpha_n g(x_n) + (1 - k - \alpha_n) y_n + kSy_n], n \geq 0, \end{aligned}$$

where $T : C \to C$ is a λ-strict pseudo-contraction, $f : C \to H$ is a ρ_1-contraction and $g : C \to H$ is a ρ_2-contraction, $k \in (0, 1 - \lambda)$ is a constant and $\{\alpha_n\}$, $\{\beta_n\}$ are two real number sequences in $(0,1)$.

Furthermore, under some control conditions, they proved that the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.6) converges independently to $P_{Fix(T)} f(y^*)$ and $P_{Fix(S)} g(x^*)$, respectively, where $x^* \in Fix(T)$ and $y^* \in Fix(S)$.

Motivated by Yao *et al.* [18], in 2018, Suwannaut [15] introduce the S-intermixed iteration for two finite families of nonlinear mappings without considering the constant k as in the following algorithm:

Algorithm 1.3. Starting with $x_1, y_1, z_1 \in C$, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be defined by

$$\begin{aligned} x_{n+1} &= (1-\beta_n)\,x_n + \beta_n\left(\alpha_n f_1(y_n) + (1-\alpha_n)\,Sx_n\right), \\ y_{n+1} &= (1-\beta_n)\,y_n + \beta_n\left(\alpha_n f_2(x_n) + (1-\alpha_n)\,Ty_n\right), n \geq 1, \end{aligned} \tag{1.7}$$

where $S, T : C \to C$, is a nonlinear mapping with $Fix(S) \cap Fix(T) \neq \emptyset$, $f_i : C \to C$ is a contractive mapping with coefficients $\alpha_i; i = 1, 2$ and $\{\beta_n\}$,$\{\alpha_n\}$ are real sequences in $(0,1)$, $\forall n \geq 1$.

Under appropriate conditions, they prove a strong convergence theorem for finding a common solution of two finite families of equilibrium problems.

Inspired by the previous work, we introduce the new iterative method called *the three-step intermixed iteration* for two finite families of nonlinear mappings as the following algorithm:

Algorithm 1.4. Starting with $x_1, y_1, z_1 \in C$, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be defined by

$$\begin{aligned} x_{n+1} &= \delta_n x_n + \eta_n S_1 x_n + \mu_n P_C\left[\alpha_n\gamma_1 f_1(y_n) + (I - \alpha_n A_1)T_1 x_n\right], \\ y_{n+1} &= \delta_n y_n + \eta_n S_2 y_n + \mu_n P_C\left[\alpha_n\gamma_2 f_2(z_n) + (I - \alpha_n A_2)T_2 y_n\right], \\ z_{n+1} &= \delta_n z_n + \eta_n S_3 z_n + \mu_n P_C\left[\alpha_n\gamma_3 f_3(x_n) + (I - \alpha_n A_3)T_3 z_n\right], n \geq 1, \end{aligned} \tag{1.8}$$

where $S_i, T_i : C \to C$, where $i = 1, 2, 3$, is nonlinear mappings with $Fix(S_i) \cap Fix(T_i) \neq \emptyset, \forall i = 1, 2, 3$, f_i is a contractive mapping with coefficients ξ_i, $A_i : C \to C$ is a strongly positive linear bounded operator with coefficient $\beta_i > 0$ and $0 < \gamma < \frac{\beta}{\xi}$, where $\gamma = \max_{i\in\{1,2,3\}} \gamma_i$, $\xi = \max_{i\in\{1,2,3\}} \xi_i$ and $\beta = \min_{i\in\{1,2,3\}} \beta_i$, $\{\delta_n\}$, $\{\eta_n\}$, $\{\mu_n\}$ and $\{\alpha_n\}$ are real sequences in $(0,1)$ and $\delta_n + \eta_n + \mu_n = 1, \forall n \geq 1$.

Remark 1.5. From Algorithm 1.2 and 1.4, we observe that Algorithm 1.4 can be seen as a modification and extension of Algorithm 1.2 in senses that we choose to consider the three-step intermixed algorithm for approximating fixed points of two finite families of nonlinear mappings and we study the general iterative method without a constant k.

Remark 1.6. If we take $S_i \equiv I$, $\gamma_i = 1$ and $A_i \equiv I$ for $i = 1, 2, 3$, then the iterative method (1.8) reduces to

$$\begin{aligned} x_{n+1} &= (1-\mu_n)\,x_n + \mu_n\left[\alpha_n f_1(y_n) + (1-\alpha_n)T_1 x_n\right], \\ y_{n+1} &= (1-\mu_n)\,y_n + \mu_n\left[\alpha_n f_2(z_n) + (1-\alpha_n)T_2 y_n\right], \\ z_{n+1} &= (1-\mu_n)\,z_n + \mu_n\left[\alpha_n f_3(x_n) + (1-\alpha_n)T_3 z_n\right]. \end{aligned} \tag{1.9}$$

The iteration (1.9) is a modification and improvement of iteration (1.7) in sense that it extends to three-step iteration for three nonlinear mappings.

Inspired by the previous research, we introduce the three-steps intermixed iteration for two finite families of nonlinear mappings. Under appropriate conditions, we prove a strong convergence theorem for finding a common fixed point of a strictly pseudo-contractive mapping and a strictly pseudononspreading mapping. Finally, we give a numerical example for the main theorem in a space of real numbers.

2. Preliminaries

We denote weak convergence and strong convergence by notations " $\rightharpoonup$" and " $\rightarrow$", respectively. For every $x \in H$, there is a unique nearest point $P_C x$ in C such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C.$$

Such an operator P_C is called the metric projection of H onto C.
We now recall the following definition and well-known lemmas.

Lemma 2.1. ([16]) *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \forall v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \forall x, y \in H.$$

Lemma 2.2. ([12]) *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.3. ([13]) *Let H be a real Hilbert space. Then the following results hold:*

(i) *For all $x, y \in H$ and $\alpha \in [0, 1]$,*

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha(1 - \alpha) \|x - y\|^2,$$

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$, *for each $x, y \in H$.*

Lemma 2.4. ([17]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5. ([10]) *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\beta > 0$ and $0 < \delta < \|A\|^{-1}$. Then $\|I - \delta A\| \le 1 - \delta\beta$.*

Lemma 2.6. ([4, 14]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a κ-strictly pseudo-contractive mapping with $Fix(T) \neq \emptyset$. Then, we there hold the following statement:*

(i) *$Fix(T) = VI(C, I - T)$;*

(ii) *For every $u \in C$ and $v \in Fix(T)$,*

$$\|P_C(I - \lambda(I - T))u - v\| \le \|u - v\|, \text{ for } u \in C, v \in Fix(T) \text{ and } \lambda \in (0, 1 - \kappa).$$

By applying Remark 2.10 in [5], we easily obtain the following result:

Lemma 2.7. *Let $S : C \to C$ be a κ-strictly pseudo nonspreading mapping with $Fix(S) \neq \emptyset$. Define $T : C \to C$ by $Tx := (1 - \lambda)x + \lambda Sx$, where $\lambda \in (0, 1 - \kappa)$. Then there hold the following statement:*

(i) *$Fix(S) = Fix(T)$*

(ii) *T is a quasi-nonexpansive mapping, that is,*

$$\|Tx - y\| \le \|x - y\|, \text{ for every } x \in C \text{ and } y \in Fix(S).$$

Proof. It is clear to prove that (i) holds.
(ii) Let $x \in H$ and $y \in Fix(S)$. Then we derive

$$\begin{aligned}
\|Tx - y\|^2 =& \|(1 - \lambda)(x - y) + \lambda(Sx - y)\|^2 \\
=&(1 - \lambda)\|x - y\|^2 + \lambda\|Sx - y\|^2 - \lambda(1 - \lambda)\|Sx - x\|^2 \\
\le&(1 - \lambda)\|x - y\|^2 + \lambda\left(\|x - y\|^2 + \kappa\|x - Sx\|^2\right) - \lambda(1 - \lambda)\|Sx - x\|^2 \\
=&\|x - y\|^2 + \kappa\lambda\|x - Sx\|^2 - \lambda(1 - \lambda)\|Sx - x\|^2 \\
=&\|x - y\|^2 - \lambda((1 - \kappa) - \lambda)\|x - Sx\|^2 \\
\le&\|x - y\|^2.
\end{aligned}$$

This implies that T is a quasi-nonexpansive mapping. □

3. Strong Convergence Theorem

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, 3$, let $f_i : C \to C$ be a contractive mappings with a coefficient ξ_i and $\xi = \max_{i\in\{1,2,3\}} \xi_i$, let $S_i : C \to C$ be a κ_i-strictly pseudo-contractive mapping and $W_i : C \to C$ be ρ_i-strictly pseudo-nonspreading mapping with $\Omega_i = Fix(S_i) \cap Fix(W_i) \neq \emptyset$. For each $i = 1, 2, 3$, define a mapping $T_n^i : C \to C$ by $T_n^i x = (1-\omega_n)\, x + \omega_n W_i x$, for all $x \in C$, and let $A_i : C \to C$ be a strongly positive linear bounded operator with a coefficient $\beta_i > 0$ and $0 < \gamma < \frac{\beta}{\xi}$, where $\gamma = \max_{i\in\{1,2,3,\}} \gamma_i$ and $\beta = \min_{i\in\{1,2,3\}} \beta_i$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by $x_1, y_1, z_1 \in C$ and*

$$\begin{cases} x_{n+1} & = \delta_n x_n + \eta_n P_C\left(I - \lambda_n\left(I - S_1\right)\right) x_n \\ & \quad + \mu_n P_C\left[\alpha_n \gamma_1 f_1(y_n) + (I - \alpha_n A_1) T_n^1 x_n\right], \\ y_{n+1} & = \delta_n y_n + \eta_n P_C\left(I - \lambda_n\left(I - S_2\right)\right) y_n \\ & \quad + \mu_n P_C\left[\alpha_n \gamma_2 f_2(z_n) + (I - \alpha_n A_2) T_n^2 y_n\right], \\ z_{n+1} & = \delta_n z_n + \eta_n P_C\left(I - \lambda_n\left(I - S_3\right)\right) z_n \\ & \quad + \mu_n P_C\left[\alpha_n \gamma_3 f_3(x_n) + (I - \alpha_n A_3) T_n^3 z_n\right], \end{cases}$$

for $n \geq 1$, where $\{\alpha_n\}$, $\{\delta_n\}$, $\{\eta_n\}$, $\{\mu_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0, 1-\kappa)$, $\kappa = \min_{i\in\{1,2,3\}}$, $\{\omega_n\} \subset (0, 1-\rho)$, where $\rho = \min_{i\in\{1,2,3\}}$ and $\delta_n + \mu_n + \eta_n = 1$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \delta_n, \eta_n, \mu_n, \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\sum_{n=1}^{\infty} \omega_n < \infty$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$,
$\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |\omega_{n+1} - \omega_n| < \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $\tilde{x} = P_{\Omega_1}\left((I - A_1)\tilde{x} + \gamma f_1\left(\tilde{y}\right)\right)$, $\tilde{y} = P_{\Omega_2}\left((I - A_2)\tilde{y} + \gamma f_2\left(\tilde{z}\right)\right)$ and $\tilde{z} = P_{\Omega_3}\left((I - A_3)\tilde{z} + \gamma f_3\left(\tilde{x}\right)\right)$, respectively.

Proof. The proof of this theorem will be divided into five steps.
Step 1. We show that $\{x_n\}$ is bounded.
Since $\alpha_n \to 0$ as $n \to \infty$, without loss of generality, we may assume that $\alpha_n < \frac{1}{\|A_i\|}$, for all $i = 1, 2, 3$ and $n \in \mathbb{N}$.

Let $x^* \in \Omega_1,\ y^* \in \Omega_2,\ z^* \in \Omega_3,\ \beta = \min_{i\in\{1,2,3\}} \beta_i,\ \xi = \max_{i\in\{1,2,3\}} \xi_i$ and $\gamma = \max_{i\in\{1,2,3\}} \gamma_i$. Then we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
\le &\Big\|\delta_n (x_n - x^*) + \eta_n (P_C (I - \lambda_n (I - S_1)) x_n - x^*) \\
&+ \mu_n \left(P_C \left[\alpha_n\gamma_1 f_1(y_n) + (I - \alpha_n A_1)T_n^1 x_n\right] - x^*\right) \Big\| \\
\le &\delta_n \|x_n - x^*\| + \eta_n \|P_C (I - \lambda_n (I - S_1)) x_n - x^*\| \\
&+ \mu_n \left\|P_C \left[\alpha_n\gamma_1 f_1(y_n) + (I - \alpha_n A_1)T_n^1 x_n\right] - x^*\right\| \\
\le &(1 - \mu_n) \|x_n - x^*\| + \mu_n \left[\alpha_n \|\gamma_1 f_1 (y_n) - A_1 x^*\| + \|I - \alpha_n A_1\| \left\|T_n^1 x_n - x^*\right\|\right] \\
\le &(1 - \mu_n) \|x_n - x^*\| \\
&+ \mu_n [\alpha_n\gamma_1\xi_1 \|y_n - y^*\| + \alpha_n \|\gamma_1 f_1 (y^*) - A_1 x^*\| + (1 - \alpha_n\beta) \|x_n - x^*\|] \\
\le &(1 - \mu_n\alpha_n\beta) \|x_n - x^*\| + \mu_n\alpha_n\gamma\xi \|y_n - y^*\| + \mu_n\alpha_n \|\gamma_1 f_1 (y^*) - A_1 x^*\| .
\end{aligned} \tag{3.1}$$

Similarly, we get

$$\begin{aligned}
&\|y_{n+1} - y^*\| \\
\le &(1 - \mu_n\alpha_n\beta) \|y_n - y^*\| + \mu_n\alpha_n\gamma\xi \|z_n - z^*\| + \mu_n\alpha_n \|\gamma_2 f_2 (z^*) - A_2 y^*\|
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\|z_{n+1} - z^*\| \\
\le &(1 - \mu_n\alpha_n\beta) \|z_n - z^*\| + \mu_n\alpha_n\gamma\xi \|x_n - x^*\| + \mu_n\alpha_n \|\gamma_3 f_3 (x^*) - A_3 z^*\| .
\end{aligned} \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| + \|z_{n+1} - z^*\| \\
\le &(1 - \mu_n\alpha_n (\beta - \gamma\xi)) (\|x_n - x^*\| + \|y_n - y^*\| + \|z_n - z^*\|) \\
&+ \mu_n\alpha_n (\|\gamma_1 f_1 (y^*) - A_1 x^*\| + \|\gamma_2 f_2 (z^*) - A_2 y^*\| + \|\gamma_3 f_3 (x^*) - A_3 z^*\|) .
\end{aligned}$$

By induction, we can derive that

$$\begin{aligned}
&\|x_n - x^*\| + \|y_n - y^*\| + \|z_n - z^*\| \\
\le &\max\Big\{ \|x_1 - x^*\| + \|y_1 - y^*\| + \|z_1 - z^*\| , \\
&\frac{\|\gamma_1 f_1 (y^*) - A_1 x^*\| + \|\gamma_2 f_2 (z^*) - A_2 y^*\| + \|\gamma_3 f_3 (x^*) - A_3 z^*\|}{\beta - \gamma\xi}\Big\},
\end{aligned}$$

for every $n \in \mathbb{N}$. This implies that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are bounded.

Step 2. Claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
First, we let

$$u_n = P_C \left[\alpha_n\gamma_1 f_1(y_n) + (I - \alpha_n A_1)T_n^1 x_n\right] ,$$

$$v_n = P_C\left[\alpha_n\gamma_2 f_2(z_n) + (I - \alpha_n A_2)T_n^2 y_n\right]$$

and

$$w_n = P_C\left[\alpha_n\gamma_3 f_3(x_n) + (I - \alpha_n A_3)T_n^3 z_n\right].$$

Then, observe that

$$\begin{aligned}
&\|u_n - u_{n-1}\| \\
=&\| \, P_C\left[\alpha_n\gamma_1 f_1(y_n) + (I - \alpha_n A_1)T_n^1 x_n\right] \\
&- P_C\left[\alpha_{n-1}\gamma_1 f_1(y_{n-1}) + (I - \alpha_{n-1}A_1)T_{n-1}^1 x_{n-1}\right] \| \\
\le& \alpha_n\gamma_1 \|f_1(y_n) - f_1(y_{n-1})\| + \gamma_1|\alpha_n - \alpha_{n-1}|\,\|f_1(y_{n-1})\| \\
&+ \|I - \alpha_n A_1\|\,\left\|T_n^1 x_n - T_{n-1}^1 x_{n-1}\right\| \\
&+ \left\|(I - \alpha_n A_1)T_{n-1}^1 x_{n-1} - (I - \alpha_{n-1}A_1)T_{n-1}^1 x_{n-1}\right\| \\
\le& \alpha_n\gamma_1\xi_1\|y_n - y_{n-1}\| + \gamma_1|\alpha_n - \alpha_{n-1}|\,\|f_1(y_{n-1})\| \\
&+ (1 - \alpha_n\beta_1)\Big((1 - \omega_n)\|x_n - x_{n-1}\| \\
&+ |\omega_n - \omega_{n-1}|\,\|x_{n-1}\| + \omega_n\|W_1x_n - W_1x_{n-1}\| + |\omega_n - \omega_{n-1}|\,\|W_1x_{n-1}\|\Big) \\
&+ |\alpha_n - \alpha_{n-1}|\,\left\|A_1T_n^1x_{n-1}\right\| \\
\le& \alpha_n\gamma\xi\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left(\gamma\|f_1(y_{n-1})\| + \left\|A_1T_n^1x_{n-1}\right\|\right) \\
&+ (1 - \alpha_n\beta)\Big((1 - \omega_n)\|x_n - x_{n-1}\| + |\omega_n - \omega_{n-1}|\left(\|x_{n-1}\| + \|W_1x_{n-1}\|\right) \\
&+ \omega_n\|W_1x_n - W_1x_{n-1}\|\Big).
\end{aligned} \tag{3.4}$$

By the definition of x_n, we obtain

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
\le& \delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\,\|x_{n-1}\| \\
&+ \eta_n\|P_C(I - \lambda_n(I - S_1))x_n - P_C(I - \lambda_{n-1}(I - S_1))x_{n-1}\| \\
&+ |\eta_n - \eta_{n-1}|\,\|P_C(I - \lambda_{n-1}(I - S_1))x_{n-1}\| + \mu_n\|u_n - u_{n-1}\| \\
&+ |\mu_n - \mu_{n-1}|\,\|u_{n-1}\| \\
\le& \delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\,\|x_{n-1}\| + \eta_n\|x_n - x_{n-1}\| \\
&+ \eta_n\|\lambda_n(I - S_1)x_n - \lambda_{n-1}(I - S_1)x_{n-1}\| \\
&+ |\eta_n - \eta_{n-1}|\,\|P_C(I - \lambda_{n-1}(I - S_1))x_{n-1}\| \\
&+ \mu_n\Big[\alpha_n\gamma\xi\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left(\gamma\|f_1(y_{n-1})\| + \left\|A_1T_n^1x_{n-1}\right\|\right) \\
&+ (1 - \alpha_n\beta)\Big((1 - \omega_n)\|x_n - x_{n-1}\| + |\omega_n - \omega_{n-1}|\left(\|x_{n-1}\| + \|W_1x_{n-1}\|\right) \\
&+ \omega_n\|W_1x_n - W_1x_{n-1}\|\Big)\Big] + |\mu_n - \mu_{n-1}|\,\|u_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\mu_n)\left\|x_n - x_{n-1}\right\| + \left|\delta_n - \delta_{n-1}\right|\left\|x_{n-1}\right\| \\
&\quad + \eta_n \lambda_n \left\|(I-S_1)x_n - (I-S_1)x_{n-1}\right\| \\
&\quad + \eta_n \left|\lambda_n - \lambda_{n-1}\right|\left\|(I-S_1)x_{n-1}\right\| \\
&\quad + \left|\eta_n - \eta_{n-1}\right|\left\|P_C\left(I - \lambda_{n-1}(I-S_1)\right)x_{n-1}\right\| \\
&\quad + \mu_n\alpha_n\gamma\xi\left\|y_n - y_{n-1}\right\| + \left|\alpha_n - \alpha_{n-1}\right|\left(\gamma\left\|f_1\left(y_{n-1}\right)\right\| + \left\|A_1T_n^1x_{n-1}\right\|\right) \\
&\quad + \mu_n\left(1-\alpha_n\beta\right)\left\|x_n - x_{n-1}\right\| + \left|\omega_n - \omega_{n-1}\right|\left(\left\|x_{n-1}\right\| + \left\|W_1x_{n-1}\right\|\right) \\
&\quad + \omega_n\left\|W_1x_n - W_1x_{n-1}\right\| + \left|\mu_n - \mu_{n-1}\right|\left\|u_{n-1}\right\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\mu_n\alpha_n\beta)\left\|x_n - x_{n-1}\right\| + \mu_n\alpha_n\gamma\xi\left\|y_n - y_{n-1}\right\| + \left|\delta_n - \delta_{n-1}\right|\left\|x_{n-1}\right\| \\
&\quad + \lambda_n\left\|(I-S_1)x_n - (I-S_1)x_{n-1}\right\| + \left|\lambda_n - \lambda_{n-1}\right|\left\|(I-S_1)x_{n-1}\right\| \\
&\quad + \left|\eta_n - \eta_{n-1}\right|\left\|P_C\left(I - \lambda_{n-1}(I-S_1)\right)x_{n-1}\right\| \\
&\quad + \left|\alpha_n - \alpha_{n-1}\right|\left(\gamma\left\|f_1\left(y_{n-1}\right)\right\| + \left\|A_1T_n^1x_{n-1}\right\|\right) \\
&\quad + \left|\omega_n - \omega_{n-1}\right|\left(\left\|x_{n-1}\right\| + \left\|W_1x_{n-1}\right\|\right) + \omega_n\left\|W_1x_n - W_1x_{n-1}\right\| \\
&\quad + \left|\mu_n - \mu_{n-1}\right|\left\|u_{n-1}\right\|. \qquad (3.5)
\end{aligned}
$$

Using the same method as derived in (3.5), we have

$$
\begin{aligned}
&\left\|y_{n+1} - y_n\right\| \\
&\leq (1-\mu_n\alpha_n\beta)\left\|y_n - y_{n-1}\right\| + \mu_n\alpha_n\gamma\xi\left\|z_n - z_{n-1}\right\| + \left|\delta_n - \delta_{n-1}\right|\left\|y_{n-1}\right\| \\
&\quad + \lambda_n\left\|(I-S_2)y_n - (I-S_2)y_{n-1}\right\| + \left|\lambda_n - \lambda_{n-1}\right|\left\|(I-S_2)y_{n-1}\right\| \\
&\quad + \left|\eta_n - \eta_{n-1}\right|\left\|P_C\left(I - \lambda_{n-1}(I-S_2)\right)y_{n-1}\right\| \\
&\quad + \left|\alpha_n - \alpha_{n-1}\right|\left(\gamma\left\|f_2\left(z_{n-1}\right)\right\| + \left\|A_2T_n^2y_{n-1}\right\|\right) \\
&\quad + \left|\omega_n - \omega_{n-1}\right|\left(\left\|y_{n-1}\right\| + \left\|W_2y_{n-1}\right\|\right) + \omega_n\left\|W_2y_n - W_2y_{n-1}\right\| \\
&\quad + \left|\mu_n - \mu_{n-1}\right|\left\|v_{n-1}\right\| \qquad (3.6)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|z_{n+1} - z_n\right\| \\
&\leq (1-\mu_n\alpha_n\beta)\left\|z_n - z_{n-1}\right\| + \mu_n\alpha_n\gamma\xi\left\|x_n - x_{n-1}\right\| + \left|\delta_n - \delta_{n-1}\right|\left\|z_{n-1}\right\| \\
&\quad + \lambda_n\left\|(I-S_3)z_n - (I-S_3)z_{n-1}\right\| + \left|\lambda_n - \lambda_{n-1}\right|\left\|(I-S_3)z_{n-1}\right\| \\
&\quad + \left|\eta_n - \eta_{n-1}\right|\left\|P_C\left(I - \lambda_{n-1}(I-S_3)\right)z_{n-1}\right\| \\
&\quad + \left|\alpha_n - \alpha_{n-1}\right|\left(\gamma\left\|f_3\left(x_{n-1}\right)\right\| + \left\|A_3T_n^3z_{n-1}\right\|\right) \\
&\quad + \left|\omega_n - \omega_{n-1}\right|\left(\left\|z_{n-1}\right\| + \left\|W_3z_{n-1}\right\|\right) + \omega_n\left\|W_3z_n - W_3z_{n-1}\right\| \\
&\quad + \left|\mu_n - \mu_{n-1}\right|\left\|w_{n-1}\right\|. \qquad (3.7)
\end{aligned}
$$

From (3.5), (3.6) and (3.7), then we get

$$
\begin{aligned}
&\|x_{n+1}-x_n\|+\|y_{n+1}-y_n\|+\|z_{n+1}-z_n\| \\
\leq &(1-\mu_n\alpha_n(\beta-\gamma\xi))\left[\|x_n-x_{n-1}\|+\|y_n-y_{n-1}\|+\|z_n-z_{n-1}\|\right] \\
&+|\delta_n-\delta_{n-1}|\left(\|x_{n-1}\|+\|y_{n-1}\|+\|z_{n-1}\|\right)+\lambda_n\Big(\|(I-S_1)x_n-(I-S_1)x_{n-1}\| \\
&+\|(I-S_2)y_n-(I-S_2)y_{n-1}\|+\|(I-S_3)z_n-(I-S_3)z_{n-1}\|\Big) \\
&+|\lambda_n-\lambda_{n-1}|\left(\|(I-S_1)x_{n-1}\|+\|(I-S_2)y_{n-1}\|+\|(I-S_3)z_{n-1}\|\right) \\
&+|\eta_n-\eta_{n-1}|\Big(\|P_C(I-\lambda_{n-1}(I-S_1))x_{n-1}\|+\|P_C(I-\lambda_{n-1}(I-S_2))y_{n-1}\| \\
&+\|P_C(I-\lambda_{n-1}(I-S_3))z_{n-1}\|\Big)+|\alpha_n-\alpha_{n-1}|\Big(\gamma\big(\|f_1(x_{n-1})\|+\|f_2(y_{n-1})\| \\
&+\|f_3(z_{n-1})\|\big)+\left(\left\|A_1T_n^1x_{n-1}\right\|+\left\|A_2T_n^2y_{n-1}\right\|+\left\|A_3T_n^3z_{n-1}\right\|\right)\Big) \\
&+|\omega_n-\omega_{n-1}|\Big(\left(\|x_{n-1}\|+\|y_{n-1}\|+\|z_{n-1}\|\right) \\
&+\left(\|W_1x_{n-1}\|+\|W_2y_{n-1}\|+\|W_3z_{n-1}\|\right)\Big) \\
&+\omega_n\left(\|W_1x_n-W_1x_{n-1}\|+\|W_2y_n-W_2y_{n-1}\|+\|W_3z_n-W_3z_{n-1}\|\right) \\
&+|\mu_n-\mu_{n-1}|\left(\|u_{n-1}\|+\|v_{n-1}\|+\|w_{n-1}\|\right).
\end{aligned}
$$

Applying Lemma 2.4 and the condition(iii), (iv), we can conclude that

$$\|x_{n+1}-x_n\|\to 0, \|y_{n+1}-y_n\|\to 0 \text{ and } \|z_{n+1}-z_n\|\to 0 \text{ as } n\to\infty. \tag{3.8}$$

Step 3. Prove that $\lim_{n\to\infty}\|u_n-P_C(I-\lambda_n(I-S_1))u_n\|=\lim_{n\to\infty}\left\|u_n-T_n^1u_n\right\|=0$. To show this, take $\tilde{u}_n=\alpha_n\gamma_1f_1(y_n)+(I-\alpha_nA_1)T_n^1x_n$. Then we derive that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
=&\|\delta_n(x_n-x^*)+\eta_n(P_C(I-\lambda_n(I-S_1))x_n-x^*)+\mu_n(u_n-x^*)\|^2 \\
\leq&\delta_n\|x_n-x^*\|^2+\eta_n\|P_C(I-\lambda_n(I-S_1))x_n-x^*\|^2+\mu_n\|u_n-x^*\|^2 \\
&-\delta_n\eta_n\|x_n-P_C(I-\lambda_n(I-S_1))x_n\|^2 \\
\leq&(1-\mu_n)\|x_n-x^*\|^2+\mu_n\left\|\alpha_n\left(\gamma_1f_1(y_n)-A_1T_n^1x_n\right)+\left(T_n^1x_n-x^*\right)\right\|^2 \\
&-\delta_n\eta_n\|x_n-P_C(I-\lambda_n(I-S_1))x_n\|^2 \\
\leq&(1-\mu_n)\|x_n-x^*\|^2 \\
&+\mu_n\Big[\left\|T_n^1x_n-x^*\right\|^2+2\alpha_n\left\langle\gamma_1f_1(y_n)-A_1T_n^1x_n,\tilde{u}_n-x^*\right\rangle\Big] \\
&-\delta_n\eta_n\|x_n-P_C(I-\lambda_n(I-S_1))x_n\|^2 \\
\leq&\|x_n-x^*\|^2+2\mu_n\alpha_n\left\|\gamma_1f_1(y_n)-A_1T_n^1x_n\right\|\|\tilde{u}_n-x^*\| \\
&-\delta_n\eta_n\|x_n-P_C(I-\lambda_n(I-S_1))x_n\|^2,
\end{aligned}
$$

which implies that

$$\begin{aligned}
&\delta_n\eta_n \left\|x_n - P_C\left(I - \lambda_n(I - S_1)\right)x_n\right\|^2 \\
&\leq \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 + 2\mu_n\alpha_n \left\|\gamma_1 f_1\left(y_n\right) - A_1 T_n^1 x_n\right\| \left\|\tilde{u}_n - x^*\right\| \\
&\leq \left\|x_n - x_{n+1}\right\| \left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right) \\
&\quad + 2\mu_n\alpha_n \left\|\gamma_1 f_1\left(y_n\right) - A_1 T_n^1 x_n\right\| \left\|\tilde{u}_n - x^*\right\|.
\end{aligned}$$

By (3.8), the condition (i) and (ii), thus we get

$$\left\|x_n - P_C\left(I - \lambda_n(I - S_1)\right)x_n\right\| \to 0 \text{ as } n \to \infty. \tag{3.9}$$

Observe that

$$x_{n+1} - x_n = \eta_n\left(P_C\left(I - \lambda_n(I - S_1)\right)x_n - x_n\right) + \mu_n\left(u_n - x_n\right).$$

This follows that

$$\mu_n \left\|u_n - x_n\right\| \leq \eta_n \left\|P_C\left(I - \lambda_n(I - S_1)\right)x_n - x_n\right\| + \left\|x_{n+1} - x_n\right\|.$$

From (3.8) and (3.9), we obtain

$$\left\|u_n - x_n\right\| \to 0 \text{ as } n \to \infty. \tag{3.10}$$

Observe that

$$\begin{aligned}
&\left\|u_n - P_C\left(I - \lambda_n(I - S_1)\right)u_n\right\| \\
&\leq \left\|u_n - x_n\right\| + \left\|x_n - P_C\left(I - \lambda_n(I - S_1)\right)x_n\right\| \\
&\quad + \left\|P_C\left(I - \lambda_n(I - S_1)\right)x_n - P_C\left(I - \lambda_n(I - S_1)\right)u_n\right\| \\
&\leq 2\left\|u_n - x_n\right\| + \left\|x_n - P_C\left(I - \lambda_n(I - S_1)\right)x_n\right\| + \lambda_n \left\|\left(I - S_1\right)x_n - \left(I - S_1\right)u_n\right\|.
\end{aligned}$$

Hence, by (3.9), (3.10) and the condition (iii), we obtain

$$\left\|u_n - P_C\left(I - \lambda_n(I - S_1)\right)u_n\right\| \to 0 \text{ as } n \to \infty. \tag{3.11}$$

Applying the same argument as (3.11), we also obtain

$$\left\|v_n - P_C\left(I - \lambda_n(I - S_2)\right)v_n\right\| \to 0 \text{ and } \left\|w_n - P_C\left(I - \lambda_n(I - S_3)\right)w_n\right\| \to 0 \text{ as } n \to \infty. \tag{3.12}$$

Consider

$$\left\|x_{n+1} - u_n\right\| \leq \left\|x_{n+1} - x_n\right\| + \left\|x_n - u_n\right\|,$$

by (3.8) and (3.10), we have

$$\left\|x_{n+1} - u_n\right\| \to 0 \text{ as } n \to \infty. \tag{3.13}$$

Since

$$\begin{aligned}\left\|x_n - T_n^1 x_n\right\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\| + \left\|u_n - T_n^1 x_n\right\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\| + \left\|\tilde{u}_n - T_n^1 x_n\right\| \\ &= \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\| + \alpha_n \left\|\gamma_1 f_1(y_n) - A_1 T_n^1 x_n\right\|,\end{aligned}$$

from (3.8), (3.13) and the condition (i), we get

$$\left\|x_n - T_n^1 x_n\right\| \to 0 \text{ as } n \to \infty. \tag{3.14}$$

Consider

$$\begin{aligned}\left\|u_n - T_n^1 u_n\right\| &\le \|u_n - x_n\| + \left\|x_n - T_n^1 x_n\right\| + \left\|T_n^1 x_n - T_n^1 u_n\right\| \\ &\le 2\|u_n - x_n\| + \left\|x_n - T_n^1 x_n\right\| + \omega_n \|W_1 x_n - W_1 u_n\|.\end{aligned}$$

Therefore, by (3.10), (3.14) and the condition (iii), we have

$$\left\|u_n - T_n^1 u_n\right\| \to 0 \text{ as } n \to \infty. \tag{3.15}$$

Applying the same method as (3.15), we also have

$$\left\|v_n - T_n^2 v_n\right\| \to 0 \text{ and } \left\|w_n - T_n^3 w_n\right\| \to 0 \text{ as } n \to \infty. \tag{3.16}$$

Step 4. Claim that

$$\limsup_{n\to\infty} \langle \gamma_1 f_1(\tilde{y}) - A_1 \tilde{x}, u_n - \tilde{x}\rangle \le 0, \text{ where}$$
$$\tilde{x} = P_{\Omega_1}\left((I - A_1)\tilde{x} + \gamma_1 f_1(\tilde{y})\right).$$

First, take a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma_1 f_1(\tilde{y}) - A_1 \tilde{x}, u_n - \tilde{x}\rangle = \lim_{k\to\infty} \langle \gamma_1 f_1(\tilde{y}) - A_1 \tilde{x}, u_{n_k} - \tilde{x}\rangle. \tag{3.17}$$

Since $\{x_n\}$ is bounded, then we can assume that $x_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$. From (3.10), we obtain $u_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$.
Next, assume $\hat{x} \notin Fix(S_1)$. Since $Fix(S_1) = Fix(P_C(I - \lambda_{n_k}(I - S_1)))$, then we get $\hat{x} \ne P_C(I - \lambda_{n_k}(I - S_1))\hat{x}$.
By nonexpansiveness of P_C, (3.11), the condition (iii) and the Opial's condition, we obtain

$$\begin{aligned}\liminf_{k\to\infty} \|u_{n_k} - \hat{x}\| &< \liminf_{k\to\infty} \|u_{n_k} - P_C(I - \lambda_{n_k}(I - S_1))\hat{x}\| \\ &\le \liminf_{k\to\infty} \Big[\|u_{n_k} - P_C(I - \lambda_{n_k}(I - S_1))u_{n_k}\| \\ &\quad + \|P_C(I - \lambda_{n_k}(I - S_1))u_{n_k} - P_C(I - \lambda_{n_k}(I - S_1))\hat{x}\|\Big] \\ &\le \liminf_{k\to\infty} \Big[\|u_{n_k} - P_C(I - \lambda_{n_k}(I - S_1))u_{n_k}\| + \|u_{n_k} - \hat{x}\| \\ &\quad + \lambda_{n_k}\|(I - S_1)u_{n_k} - (I - S_1)\hat{x}\|\Big] \\ &= \liminf_{k\to\infty} \|u_{n_k} - \hat{x}\|.\end{aligned}$$

This is a contradiction. Therefore

$$\hat{x} \in Fix\,(S_1)\,. \tag{3.18}$$

Assume that $\hat{x} \notin Fix\,(W_1)$. Because $Fix\,(W_1) = Fix\left(T^1_{n_k}\right)$, then we have $\hat{x} \neq T^1_{n_k}\hat{x}$.
From (3.15) and the Opial's condition, we deduce that

$$\begin{aligned}
\liminf_{k\to\infty} \|u_{n_k} - \hat{x}\| &< \liminf_{k\to\infty} \left\|u_{n_k} - T^1_{n_k}\hat{x}\right\| \\
&\leq \liminf_{k\to\infty} \left[\left\|u_{n_k} - T^1_{n_k}u_{n_k}\right\| + \left\|T^1_{n_k}u_{n_k} - T^1_{n_k}\hat{x}\right\|\right] \\
&\leq \liminf_{k\to\infty} \Big[\left\|u_{n_k} - T^1_{n_k}u_{n_k}\right\| + \|u_{n_k} - \hat{x}\| \\
&\quad + \omega_{n_k}\left\|(I - W_1)\,u_{n_k} - (I - W_1)\,\hat{x}\right\|\Big] \\
&= \liminf_{k\to\infty} \|u_{n_k} - \hat{x}\|\,.
\end{aligned}$$

This is a contradiction. Thus we obtain

$$\hat{x} \in Fix\,(W_1)\,. \tag{3.19}$$

By (3.18) and (3.19), this yields that

$$\hat{x} \in \Omega_1 = Fix\,(S_1) \cap Fix\,(W_1)\,. \tag{3.20}$$

Since $x_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$, (3.20) and Lemma 2.1, we can derive that

$$\begin{aligned}
\limsup_{n\to\infty} \langle \gamma_1 f_1(\tilde{y}) - A_1\tilde{x}, u_n - \tilde{x}\rangle &= \lim_{k\to\infty} \langle \gamma_1 f_1(\tilde{y}) - A_1\tilde{x}, u_{n_k} - \tilde{x}\rangle \\
&= \langle \gamma_1 f_1(\tilde{y}) - A_1\tilde{x}, \hat{x} - \tilde{x}\rangle \\
&= \langle \gamma_1 f_1(\tilde{y}) - A_1\tilde{x} + \tilde{x} - \tilde{x}, \hat{x} - \tilde{x}\rangle \\
&\leq 0.
\end{aligned} \tag{3.21}$$

Following the same method as (3.21), we easily obtain that

$$\limsup_{n\to\infty} \langle \gamma_2 f_2(\tilde{z}) - A_2\tilde{y}, v_n - \tilde{y}\rangle \leq 0 \text{ and } \limsup_{n\to\infty} \langle \gamma_3 f_3(\tilde{x}) - A_3\tilde{z}, w_n - \tilde{z}\rangle \leq 0. \tag{3.22}$$

Step 5. Finally, Prove that the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $\tilde{x} = P_{\Omega_1}((I - A_1)\tilde{x} + \gamma_1 f_1(\tilde{y}))$, $\tilde{y} = P_{\Omega_2}((I - A_2)\tilde{y} + \gamma_2 f_2(\tilde{z}))$ and $\tilde{z} = P_{\Omega_3}((I - A_3)\tilde{z} + \gamma_3 f_3(\tilde{x}))$, respectively.

By firmly-nonexpansiveness of P_C, we derive that

$$\begin{aligned}
\|u_n-\tilde{x}\|^2 =& \|P_C\tilde{u_n}-\tilde{x}\|^2\\
\leq& \langle \tilde{u_n}-\tilde{x}, u_n-\tilde{x}\rangle\\
=& \left\langle \alpha_n\left(\gamma_1 f_1\left(y_n\right)-A_1\tilde{x}\right)+\left(I-\alpha_n A_1\right)\left(T_n^1 x_n-\tilde{x}\right), u_n-\tilde{x}\right\rangle\\
=& \alpha_n\left\langle \gamma_1 f_1\left(y_n\right)-A_1\tilde{x}, u_n-\tilde{x}\right\rangle+\left\langle\left(I-\alpha_n A_1\right)\left(T_n^1 x_n-\tilde{x}\right), u_n-\tilde{x}\right\rangle\\
\leq& \alpha_n\gamma_1\left\langle f_1\left(y_n\right)-f_1\left(\tilde{y}\right), u_n-\tilde{x}\right\rangle+\alpha_n\left\langle \gamma_1 f_1\left(\tilde{y}\right)-A_1\tilde{x}, u_n-\tilde{x}\right\rangle\\
&+\left\|\left(I-\alpha_n A_1\right)\left(T_n^1 x_n-\tilde{x}\right)\right\|\|u_n-\tilde{x}\|\\
\leq& \alpha_n\gamma_1\xi_1\|y_n-\tilde{y}\|\|u_n-\tilde{x}\|+\alpha_n\left\langle \gamma_1 f_1\left(\tilde{y}\right)-A_1\tilde{x}, u_n-\tilde{x}\right\rangle\\
&+\left(1-\alpha_n\beta_1\right)\|x_n-\tilde{x}\|\|u_n-\tilde{x}\|\\
\leq& \frac{\alpha_n\gamma\xi}{2}\left(\|y_n-\tilde{y}\|^2+\|u_n-\tilde{x}\|^2\right)+\alpha_n\left\langle \gamma_1 f_1\left(\tilde{y}\right)-A_1\tilde{x}, u_n-\tilde{x}\right\rangle\\
&+\frac{1-\alpha_n\beta}{2}\left(\|x_n-\tilde{x}\|^2+\|u_n-\tilde{x}\|^2\right)\\
=& \frac{\alpha_n\gamma\xi}{2}\|y_n-\tilde{y}\|^2+\frac{1-\alpha_n\beta}{2}\|x_n-\tilde{x}\|^2+\frac{1-\alpha_n(\beta-\gamma\xi)}{2}\|u_n-\tilde{x}\|^2\\
&+\alpha_n\left\langle \gamma_1 f_1\left(\tilde{y}\right)-A_1\tilde{x}, u_n-\tilde{x}\right\rangle,
\end{aligned}$$

which yields that

$$\begin{aligned}
\|u_n-\tilde{x}\|^2 \leq& \frac{\alpha_n\gamma\xi}{1+\alpha_n(\beta-\gamma\xi)}\|y_n-\tilde{y}\|^2+\frac{1-\alpha_n\beta}{1+\alpha_n(\beta-\gamma\xi)}\|x_n-\tilde{x}\|^2\\
&+\frac{2\alpha_n}{1+\alpha_n(\beta-\gamma\xi)}\left\langle \gamma_1 f_1\left(\tilde{y}\right)-A_1\tilde{x}, u_n-\tilde{x}\right\rangle.
\end{aligned} \tag{3.23}$$

From the definition of x_n and (3.23), we get

$$\begin{aligned}
&\|x_{n+1}-\tilde{x}\|^2\\
\leq& \delta_n\|x_n-\tilde{x}\|^2+\eta_n\left\|P_C\left(I-\lambda_n(I-S_1)\right)x_n-\tilde{x}\right\|^2+\mu_n\|u_n-\tilde{x}\|^2\\
\leq& \left(1-\mu_n\right)\|x_n-\tilde{x}\|^2+\mu_n\Big[\frac{\alpha_n\gamma\xi}{1+\alpha_n(\beta-\gamma\xi)}\|y_n-\tilde{y}\|^2+\frac{1-\alpha_n\beta}{1+\alpha_n(\beta-\gamma\xi)}\|x_n-\tilde{x}\|^2\\
&+\frac{2\alpha_n}{1+\alpha_n(\beta-\gamma\xi)}\left\langle \gamma_1 f_1\left(\tilde{y}\right)-A_1\tilde{x}, u_n-\tilde{x}\right\rangle\Big]\\
=& \left(1-\mu_n+\frac{\mu_n\left(1-\alpha_n\beta\right)}{1+\alpha_n(\beta-\gamma\xi)}\right)\|x_n-\tilde{x}\|^2+\frac{\mu_n\alpha_n\gamma\xi}{1+\alpha_n(\beta-\gamma\xi)}\|y_n-\tilde{y}\|^2\\
&+\frac{2\mu_n\alpha_n}{1+\alpha_n(\beta-\gamma\xi)}\left\langle \gamma_1 f_1\left(\tilde{y}\right)-A_1\tilde{x}, u_n-\tilde{x}\right\rangle
\end{aligned}$$

$$= \left(1 - \frac{\mu_n\left(1+\alpha_n(\beta-\gamma\xi)\right) - \mu_n\left(1-\alpha_n\beta\right)}{1+\alpha_n(\beta-\gamma\xi)}\right)\|x_n-\tilde{x}\|^2 + \frac{\mu_n\alpha_n\gamma\xi}{1+\alpha_n(\beta-\gamma\xi)}\|y_n-\tilde{y}\|^2$$

$$= \left(1 - \frac{\mu_n\alpha_n(2\beta-\gamma\xi)}{1+\alpha_n(\beta-\gamma\xi)}\right)\|x_n-\tilde{x}\|^2 + \frac{\mu_n\alpha_n\gamma\xi}{1+\alpha_n(\beta-\gamma\xi)}\|y_n-\tilde{y}\|^2$$

$$+ \frac{2\mu_n\alpha_n}{1+\alpha_n(\beta-\gamma\xi)}\left\langle \gamma_1 f_1(\tilde{y}) - A_1\tilde{x}, u_n-\tilde{x}\right\rangle. \tag{3.24}$$

Similarly, as derived above, we also have

$$\|y_{n+1}-\tilde{y}\|^2 \leq \left(1 - \frac{\mu_n\alpha_n(2\beta-\gamma\xi)}{1+\alpha_n(\beta-\gamma\xi)}\right)\|y_n-\tilde{y}\|^2 + \frac{\mu_n\alpha_n\gamma\xi}{1+\alpha_n(\beta-\gamma\xi)}\|z_n-\tilde{z}\|^2$$
$$+ \frac{2\mu_n\alpha_n}{1+\alpha_n(\beta-\gamma\xi)}\left\langle \gamma_2 f_2(\tilde{z}) - A_2\tilde{y}, v_n-\tilde{y}\right\rangle \tag{3.25}$$

and

$$\|z_{n+1}-\tilde{z}\|^2 \leq \left(1 - \frac{\mu_n\alpha_n(2\beta-\gamma\xi)}{1+\alpha_n(\beta-\gamma\xi)}\right)\|z_n-\tilde{z}\|^2 + \frac{\mu_n\alpha_n\gamma\xi}{1+\alpha_n(\beta-\gamma\xi)}\|x_n-\tilde{x}\|^2$$
$$+ \frac{2\mu_n\alpha_n}{1+\alpha_n(\beta-\gamma\xi)}\left\langle \gamma_3 f_3(\tilde{x}) - A_3\tilde{z}, w_n-\tilde{z}\right\rangle. \tag{3.26}$$

From (3.24), (3.25) and (3.26), we deduces that

$$\|x_{n+1}-\tilde{x}\|^2 + \|y_{n+1}-\tilde{y}\|^2 + \|z_{n+1}-\tilde{z}\|^2$$
$$\leq \left(1 - \frac{2\mu_n\alpha_n(\beta-\gamma\xi)}{1+\alpha_n(\beta-\gamma\xi)}\right)\left(\|x_n-\tilde{x}\|^2 + \|y_n-\tilde{y}\|^2 + \|z_n-\tilde{z}\|^2\right)$$
$$+ \frac{2\mu_n\alpha_n}{1+\alpha_n(\beta-\gamma\xi)}\Big(\left\langle \gamma_1 f_1(\tilde{y}) - A_1\tilde{x}, u_n-\tilde{x}\right\rangle + \left\langle \gamma_2 f_2(\tilde{z}) - A_2\tilde{y}, v_n-\tilde{y}\right\rangle$$
$$+ \left\langle \gamma_3 f_3(\tilde{x}) - A_3\tilde{z}, w_n-\tilde{z}\right\rangle\Big). \tag{3.27}$$

By (3.21), (3.22), the condition (i) and Lemma 2.4, this implies by (3.27) that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $\tilde{x} = P_{\Omega_1}\left((I-A_1)\tilde{x} + \gamma_1 f_1(\tilde{y})\right)$, $\tilde{y} = P_{\Omega_2}\left((I-A_2)\tilde{y} + \gamma_2 f_2(\tilde{z})\right)$ and $\tilde{z} = P_{\Omega_3}\left((I-A_3)\tilde{z} + \gamma_3 f_3(\tilde{x})\right)$, respectively. This completes the proof. □

The following Corollary is a direct consequence of Theorem 3.1.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1,2,3$, let $f_i : C \to C$ be a contractive mappings with a coefficient ξ_i and $\xi = \max_{i\in 1,2,3}\xi_i$ and let $W_i : C \to C$ be ρ_i-strictly pseudo-nonspreading mapping with $Fix(W_i) \neq \emptyset$. Define a mapping $T_n^i : C \to C$ by $T_n^i x = (1-\omega_n)x + \omega_n W_i x$,* *for all* $x \in C$ *and* $i = 1,2,3$. *Let* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be sequences generated by* $x_1, y_1, z_1 \in C$ *and*

$$\begin{cases} x_{n+1} &= (1-\mu_n)\, x_n + \mu_n \left[\alpha_n f_1(y_n) + (1-\alpha_n) T_n^1 x_n\right], \\ y_{n+1} &= (1-\mu_n)\, y_n + \mu_n \left[\alpha_n f_2(z_n) + (1-\alpha_n) T_n^2 y_n\right], \\ z_{n+1} &= (1-\mu_n)\, z_n + \mu_n \left[\alpha_n f_3(x_n) + (1-\alpha_n) T_n^3 z_n\right], \end{cases} \tag{3.28}$$

for $n \geq 1$, *where* $\{\alpha_n\}$, $\{\mu_n\} \subset (0,1)$ *and* $\{\omega_n\} \subset (0, 1-\rho)$, *where* $\rho = \min_{i\in\{1,2,3\}} \rho_i$, *satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \mu_n, \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $\sum\limits_{n=1}^{\infty} \omega_n < \infty$;

(iv) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum\limits_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$, $\sum\limits_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum\limits_{n=1}^{\infty} |\omega_{n+1} - \omega_n| < \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $\tilde{x} = P_{Fix(W_1)} f_1(\tilde{y})$, $\tilde{y} = P_{Fix(W_2)} f_2(\tilde{z})$ *and* $\tilde{z} = P_{Fix(W_3)} f_3(\tilde{x})$, *respectively.*

Proof. For each $i = 1,2,3$, put $S_i \equiv I$, $\gamma_i = 1$ and $A_i \equiv I$. Then, by Theorem 3.1, we obtain the desired result. $\square$

4. A Numerical Example

In this section, we give a numerical example to support our main theorem.

Example 4.1. For $i = 1,2,3$, let $\gamma_1 = 3$, $\gamma_2 = 0.0001$, $\gamma_3 = 7$ and the mappings $A_i : [-5,5] \to [-5,5]$, $f_i : [-5,5] \to [-5,5]$, $S_i : [-5,5] \to [-5,5]$ and $W_i : [-5,5] \to [-5,5]$ be defined by

$$A_1 x = \frac{2x}{5},\ A_2 x = \frac{3x}{5},\ A_3 x = \frac{4x}{5},$$
$$f_1 x = \frac{x-100}{50},\ f_2 x = \frac{x+95}{20},\ f_3 x = \frac{x+50}{35},$$
$$S_1 x = \frac{x-10}{3},\ S_2 x = \frac{x}{10},\ S_3 x = \frac{x+5}{2},$$
$$W_1 x = \frac{x-25}{6},\ W_2 x = \begin{cases} \frac{-7x}{8}, \text{ if } 0 \leq x \leq 5, \\ x, \text{ if } -5 \leq x < 0, \end{cases} \quad W_3 x = \frac{x+15}{4}, \text{ for all } x \in [-5,5]$$

Let $\alpha_n = \frac{1}{n^{0.2}+1}$, $\delta_n = \frac{n+2}{6n+5}$, $\eta_n = \frac{3n+2}{6n+5}$, $\mu_n = \frac{2n+1}{6n+5}$, $\lambda_n = \frac{1}{n^2+100}$ and $\omega_n = \frac{1}{n^2+100}$ for every $n \in \mathbb{N}$. Then, the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to -5, 0, 5, respectively.

Solution. For every $i = 1, 2, 3$, it is obvious to check that S_i is a 0-strictly pseudo-contractive mapping, where $Fix\,(S_1) = \{-5\}$, $Fix\,(S_2) = \{0\}$, $Fix\,(S_3) = \{5\}$. Moreover, W_i is a κ_i-strictly pseudononspreading mapping with

$$Fix\,(W_1) = \{-5\},\ Fix\,(W_2) = \begin{cases} \{0\}, & \text{if } 0 \leq x \leq 5, \\ \{x\}, & \text{if } -5 \leq x < 0, \end{cases} \qquad Fix\,(W_3) = \{5\}.$$

Thus, we get

$$\begin{aligned} \Omega_1 &= Fix\,(S_1) \cap Fix\,(W_1) = \{-5\} \\ \Omega_2 &= Fix\,(S_2) \cap Fix\,(W_2) = \{0\} \\ \Omega_3 &= Fix\,(S_3) \cap Fix\,(W_3) = \{5\}. \end{aligned}$$

Clearly, all sequences and parameters are satisfied all conditions of Theorem 3.1. Hence, by Theorem 3.1, we can conclude that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to -5, 0, 5, respectively.

Table 1 and Figure 1 show the numerical results of sequences $\{x_n\}$, $\{y_n\}$ and $\{x_n\}$ with $x_1 = 0$, $y_1 = 5$, $z_1 = 0$ and $n = 100$.

n	x_n	y_n	z_n
1	0.000000	5.000000	0.000000
2	-0.796797	4.561817	1.374887
3	-1.553784	4.159760	2.449299
4	-2.262385	3.796830	3.231195
5	-2.923918	3.470288	3.783815
$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	-5.000000	0.132362	5.000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$
96	-5.000000	0.008590	5.000000
97	-5.000000	0.008146	5.000000
98	-5.000000	0.007729	5.000000
99	-5.000000	0.007335	5.000000
100	-5.000000	0.006965	5.000000

Table 1: The values of $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ with initial values $x_1 = 0$, $y_1 = 5$, $z_1 = 0$ and $n = 100$.

Remark 4.2. From the above numerical results, we can conclude that Table 1 and Figure 1 show that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge independently to $-5 \in \Omega_1$, $0 \in \Omega_2$ and $5 \in \Omega_3$, respectively, and the convergence of $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ can be guaranteed by Theorem 3.1.

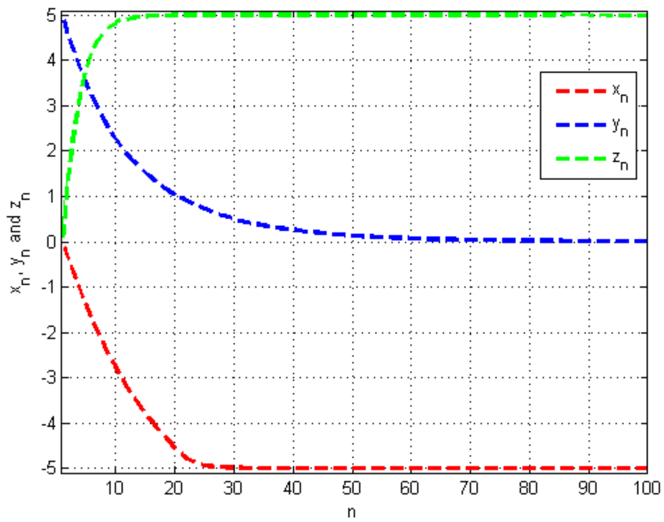

Figure 1: An independent convergence of $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ with initial values $x_1 = 0$, $y_1 = 5$, $z_1 = 0$ and $n = 100$.

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