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Locating-Hop Domination in Graphs

Sergio R. Canoy, Jr.*
Department of Mathematics and Statistics, College of Science and Mathematics, Center for Graph Theory, Algebra, and Analysis, Premier Research Institute of Science and Mathematics, MSU-Iligan Institute of Technology, Andres Bonifacio Ave., Iligan City, Philippines
e-mail : sergio.canoy@g.msuiit.edu.ph

Gemma P. Salasalan
Department of Arts and Sciences, Institute of Teacher Education, Arts and Sciences, Davao del Sur State College, Matti, Digos City, Davao del Sur
e-mail : gemma.salasalan@g.msuiit.edu.ph

Abstract. A subset S of $V(G)$, where G is a simple undirected graph, is a hop dominating set if for each $v \in V(G) \setminus S$, there exists $w \in S$ such that $d_G(v,w) = 2$ and it is a locating-hop set if $N_G(v,2) \cap S \neq N_G(v,2) \cap S$ for any two distinct vertices $u, v \in V(G) \setminus S$. A set $S \subseteq V(G)$ is a locating-hop dominating set if it is both a locating-hop and a hop dominating set of G. The minimum cardinality of a locating-hop dominating set of G, denoted by $\gamma_{lh}(G)$, is called the locating-hop domination number of G. In this paper, we investigate some properties of this newly defined parameter. In particular, we characterize the locating-hop dominating sets in graphs under some binary operations.

1. Introduction

Locating-domination in a graph, a variation of the standard domination, was defined by Slater et al. in [13] and [14]. For protection of a certain system or network, the problem of determining the location of monitoring devices (e.g. fire alarms or cameras) so as to identify the exact location of an intruder (e.g. fire, burglar) when a problem at a facility or system arises can be modelled by this concept. This parameter and some of its variations had been studied in [2], [3], [6], [7], and [8].

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In 2015, Natarajan and Ayyaswamy [10] introduced and studied the concept of hop domination in a graph. This concept and some of its variations were also studied in [1], [4], [9], and [12].

Apart from other possible applications of the concept of locating-hop domination similar to those of the standard locating-domination, the concept can also be used for a social network application. For example, consider a certain company with a large number of employees and suppose the company's president decides to form an internal committee of evaluators to do an assessment of the employees' job performance and productivity for management purpose. To minimize possible biases that may occur in the process, it is imposed that an evaluator should neither be a close friend nor an enemy to any employee (an individual outside this committee) under his or her evaluation list and that no two employees are evaluated by exactly the same set of evaluators from the committee. Also, to save time, an employee needs to be evaluated by a nearest non-biased evaluator. Since these criteria may still allow at most one employee (a non-evaluator) to be unevaluated or unassessed, it is further imposed that every employee outside the committee must go through the evaluation process, i.e., nobody is left out unevaluated. To model this evaluation process, a social network can be constructed, where each employee (including members of the committee) is represented by a vertex and an edge between two employees is formed if the they are either close friends or enemies. With respect to this network, the first set of imposed guidelines in the aforementioned evaluation process would actually require that the set of evaluators satisfies the 'locating-hop' condition. The additional imposed guideline (to ensure completeness of the evaluation process) would necessitate that the set of evaluators is a 'hop dominating' set. This social network application is a slight modification of the one given by Desormeaux et al. in [5] for non-dominating sets in graphs.

In this paper, we define and do an initial study of the concept of locating-hop dominating set in a graph. As formally defined in a bit and as implicitly mentioned earlier, a locating-hop set is 'almost' a hop dominating set as it may allow at most a vertex outside the set to be 'hop undominated'. In a portion of this paper, one may find a result (a result that deals with the concept of locating-hop dominating on a disconnected graph) that does not hold when the condition 'locating-hop dominating set' is replaced by just 'locating hop set'. This somehow also makes 'locating-hop dominating set' a bit interesting concept to define and study.

Let G be a simple graph with vertex-set $V(G)$ and edge-set $E(G)$. For any two vertices $u, v \in V(G)$, the *distance* $d_G(u,v)$ between u and v is the length of a shortest u-v path in G. Any u-v path of length equal to $d_G(u,v)$ is called a *geodesic* (u-v *godesic*). Two vertices u and v are *neighbors* (sometimes *adjacent*) if $uv \in E(G)$. The set of neighbors of a vertex u of G, denoted by $N_G(u)$, is called the *open neighborhood* of u in G. The *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. The *open neighborhood* of $X \subseteq V(G)$ is the set $N_G(X) = \bigcup\limits_{u \in X} N_G(u)$ and its closed neighborhood is the set $N_G[X] = N_G(X) \cup X$. The *open hop neighborhood* of u in G, denoted by $N_G(u,2)$, is the set given by $N_G(u,2) = \{w \in V(G) : d_G(u,w) = 2\}$ and its *closed hop neighborhood* is $N_G[u,2] = N_G(u,2) \cup \{u\}$. The *open hop neighborhood* of $S \subseteq V(G)$ is the set $N_G(S,2) = \bigcup_{u \in S} N_G(u,2)$ and its *closed hop neighborhood* is the set $N_G[S,2] = N_G(S,2) \cup S$. The degree of vertex u in G, denoted by $deg_G(u)$, is equal to $|N_G(u)|$.

A set $S \subseteq V(G)$ is a *locating set* of G if for any two distinct vertices $v, w \in V(G) \setminus S$, $N_G(v) \cap S \neq N_G(w) \cap S$. A locating set S is a *locating-dominating set* of G if $N_G(v) \cap S \neq \varnothing$ for each $v \in V(G) \setminus S$. The smallest cardinality of a locating (resp. locating-dominating) set of G is denoted by $ln(G)$ (resp. $\gamma_L(G)$). Any locating (resp. locating-dominating) set of G with cardinality $ln(G)$ (resp. $gamma_L(G)$) is called a ln-set (resp. γ_L-set) of G. We also point out that a locating set is 'nearly' or 'almost' a dominating set because it may allow at most a vertex in $V(G) \setminus S$ to be undominated. A study of the concept of locating set can also be found in [11].

A subset S of $V(G)$ is called a *hop dominating set* of G if $N_G[S,2] = V(G)$, that is, for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d_G(u,v) = 2$. The minimum cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set of G with cardinality $\gamma(G)$ is called a γ_h*-set* of G.

A subset S of G is a *locating-hop set* of G if $N_G(u,2) \cap S \neq N_G(v,2) \cap S$ for every two distinct vertices u and v of $V(G) \setminus S$. A locating-hop set of G which is also a hop dominating set is called a *locating-hop dominating* set. The minimum cardinality of a locating-hop dominating set of G, denoted by $\gamma_{lh}(G)$, is called the *locating-hop domination number* of G.

A set $S \subseteq V(G)$ is a *complement-locating set* of G (or a locating set of $\overline{G}$) if for any two distinct vertices $v, w \in V(G) \setminus S$, $N_{\overline{G}}(v) \cap S = [V(G) \setminus N_G(v)] \cap S \neq [V(G) \setminus N_G(w)] \cap S = N_{\overline{G}}(w) \cap S$. A complement-locating set S of G is called a *complement locating-dominating set* of G (or a locating-dominating set of $\overline{G}$) if for each $v \in V(G) \setminus S$, $[V(G) \setminus N_G(v)] \cap S = N_{\overline{G}}(v) \cap S \neq \varnothing$. The smallest cardinality of a complement-locating (resp. complement locating-dominating) set of G is denoted by $cln(G)$ (resp. $cldn(G)$). Any complement-locating (resp. complement locating-dominating) set of G with cardinality $cln(G)$ (resp. $cldn(G)$) is called a cln-set (resp. a $cldn$-set) of G. Clearly, $cln(G) = ln(\overline{G})$ and $cldn(G) = \gamma_L(\overline{G})$.

2. Results

Since any set S of size s has 2^s distinct subsets, it is easily observed from the definition that if S is locating-hop dominating of a graph G on n vertices, then the inequality $2^s > n - s$ is satisfied. We state this formally as a simple result.

Lemma 2.1. *Let G be a connected graph on n vertices. If S is a locating-hop dominating set of G and $s = |S|$, then $n < 2^s + s$.*

Proposition 2.2. *Let G be a connected graph on n vertices. Then $1 \leq \gamma_{lh}(G) \leq n$. Moreover,*

(i) $\gamma_{lh}(G) = 1$ *if and only if G is the trivial graph;*

(ii) *$\gamma_{lh}(G) = n$ if and only if $G = K_n$.*

Proof. Clearly, $1 \leq \gamma_{lh}(G) \leq n$.

(i) Suppose $\gamma_{lh}(G) = 1$, say $S = \{u\}$ is a γ_{lh}-set of G. If G is a nontrivial connected graph, then there exists $v \in V(G) \cap N_G(u)$. This implies that S is not a hop dominating set of G, a contradiction. Therefore, G is the trivial graph. Clearly, if G is the trivial graph, then $\gamma_{lh}(G) = 1$.

(ii) Suppose $\gamma_{lh}(G) = n$ and suppose further that $G \neq K_n$. Then there exist distinct vertices $v, w \in V(G)$ such that $d_G(v, w) = 2$. Let $S = V(G) \setminus \{v\}$. Then S is a locating hop dominating set of G. Hence, $\gamma_{lh}(G) \leq |S| = n-1$, a contradiction. Therefore, $G = K_n$. The converse is clear. □

Proposition 2.3. *Let G be a connected graph of order n. If $\gamma_{lh}(G) = 2$, then $2 \leq n \leq 5$. Moreover,*

(i) *if $n = 2, 3$, then $G = P_2$ and $G = P_3$, respectively;*

(ii) *if $n = 4$, then $G = P_4$ or $G = C_4$; and*

(iii) *if $n = 5$, then $G = C_5$ or G is (isomorphic to) the graph obtained from a cycle $C_5 = [v_1, v_2, v_3, v_4, v_5, v_1]$ by adding the edge v_3v_5, or G is (isomorphic to) the graph obtained from a cycle $C_5 = [v_1, v_2, v_3, v_4, v_5, v_1]$ by adding the edge v_3v_5 and removing the edge v_1v_2.*

Proof. Suppose that $\gamma_{lh}(G) = 2$ and let $S = \{v_1, v_2\}$ be a γ_{lh}-set of G. Clearly, $n \geq 2$. By Lemma 2.1, $n \leq 5$. Hence, $2 \leq n \leq 5$.

Clearly, (i) holds. Suppose $n = 4$. Let $a, b \in V(G) \setminus S$. Since S is a hop dominating set, we may assume without loss of generality that $v_1 \in N_G(a, 2) \cap S$. Suppose that $[a, x, v_1]$ is an a-v_1 geodesic. If $x = v_2$, then $v_1v_2 \in E(G)$. It follows that $N_G(a, 2) \cap S = \{v_1\}$. Since S is a locating hop dominating set, $N_G(b, 2) \cap S = \{v_2\}$ or $N_G(b, 2) \cap S = \{v_1, v_2\}$. Suppose $N_G(b, 2) \cap S \neq \{v_2\}$. Then $N_G(b, 2) \cap S = \{v_1, v_2\}$. This implies that $ab \in E(G)$. Since $d_G(a, v_2) = d_G(v_1, v_2) = 1$, it follows that $d_G(b, v_1) = 3$, a contradiction. Therefore, $N_G(b, 2) \cap S = \{v_2\}$. Thus, G is $[b, a, v_2, v_1]$ or $[b, v_1, v_2, a]$ or $[b, a, v_2, v_1, b]$. Suppose now that $x = b$. Then this forces $N_G(b, 2) \cap S = \{v_2\}$ because S is a hop dominating set. If $v_1v_2 \notin E(G)$, then $av_2 \in E(G)$ and $N_G(a, 2) \cap S = \{v_1\}$. It follows that $G = [v_1, b, a, v_2] = P_4$. Suppose $v_1v_2 \in E(G)$. Then G is $[a, b, v_1, v_2]$ or $[a, b, v_1, v_2, a]$. Therefore, $G = P_4$ or $G = C_4$, showing that (ii) holds.

Next, suppose that $n = 5$. Let $v_3, v_4, v_5 \in V(G) \setminus S$. Since S is a locating hop dominating set, we may assume that $N_G(v_3, 2) \cap S = \{v_1\}$, $N_G(v_5, 2) \cap S = \{v_2\}$, and $N_G(v_4, 2) \cap S = \{v_1, v_2\}$. Suppose first that $v_1v_2 \in E(G)$. Since $N_G(v_4, 2) \cap S = \{v_1, v_2\}$, $v_1v_4, v_2v_4 \notin E(G)$. It follows that $v_3v_4, v_4v_5, v_2v_3, v_1v_5 \in E(G)$. If $v_3v_5 \notin E(G)$, then $G = [v_1, v_2, v_3, v_4, v_5, v_1] = C_5$. If $v_3v_5 \in E(G)$, then G is the graph obtained from the cycle $[v_1, v_2, v_3, v_4, v_5, v_1]$ by adding the edge v_3v_5. Suppose that $v_1v_2 \notin E(G)$. Again, it can be shown that $v_3v_4, v_4v_5, v_2v_3, v_1v_5 \in E(G)$. Moreover, since $N_G(v_3, 2) \cap S = \{v_1\}$ and $N_G(v_5, 2) \cap S = \{v_2\}$, it follows that $v_3v_5 \in E(G)$.

Thus, G is the graph obtained from the cycle $C_5 = [v_1, v_2, v_3, v_4, v_5, v_1]$ by adding the edge v_3v_5 and removing the edge v_1v_2. This proves (iii). □

Theorem 2.4. *Let $G_1, G_2, \ldots, G_k$ be the distinct components of G, where $k \geq 2$. Then S is a locating hop dominating set of G if and only if $S_j = S \cap V(G_j)$ is a locating-hop dominating set of G_j for each $j \in \{1, 2, \ldots, k\}$.*

Proof. Suppose S is locating-hop dominating set of G and let $j \in \{1, 2, \ldots, k\}$. Let $v \in V(G_j) \setminus S_j$. Since $v \notin S$ and S is a hop dominating set, there exists $w \in S$ such that $v \in N_G(w, 2)$. This implies that $w \in S_j$ and $v \in N_{G_j}(w, 2)$. This shows that S_j is a hop dominating set of G_j. Next, let $a, b \in V(G_j) \setminus S_j$ where $a \neq b$. Since S is locating-hop set

$$N_{G_j}(a, 2) \cap S_j = N_G(a, 2) \cap S \neq N_G(b, 2) \cap S = N_{G_j}(b, 2) \cap S_j.$$

Thus, S_j is a locating-hop dominating set of G_j for each $j \in \{1, 2, \ldots, k\}$.

For the converse, suppose that $S_j = S \cap V(G_j)$ is a locating-hop dominating set of G_j for each $j \in \{1, 2, \ldots, k\}$. Then clearly, S is a hop dominating set of G. Let $v, w \in V(G) \setminus S$ with $v \neq w$ and let G_i and G_j be the components of G with $v \in V(G_i) \setminus S_i$ and $w \in V(G_j) \setminus S_j$. Since S_i and S_j are locating-hop sets (where it may happen that $i = j$),

$$N_G(a, 2) \cap S = N_{G_i}(v, 2) \cap S_i \neq N_{G_j}(w, 2) \cap S_j = N_G(w, 2) \cap S.$$

Therefore, S is a locating-hop set of G. □

It is worth mentioning that Theorem 2.4 does not hold if 'locating-hop dominating' is replaced by 'locating-hop'. Indeed, if there are two distinct locating sets S_j and S_k which have each a single vertex in $V(G_j) \setminus S_j$ and $V(G_k) \setminus S_k$, respectively, that are not hop-dominated in the respective components, then the set S cannot be a locating hop set of G.

The next result follows from Theorem 2.4.

Corollary 2.5. *Let $G_1, G_2, \ldots, G_k$ be the distinct components of G. Then $\gamma_{lh}(G) = \sum_{j=1}^{k} \gamma_{lh}(G_j)$.*

As an immediate consequence of Proposition 2.2(ii) and Theorem 2.4, we have the next result.

Corollary 2.6. *Let G be a graph of order n. Then $\gamma_{lh}(G) = n$ if and only if every component of G is complete. In particular, $\gamma_{lh}(K_n) = \gamma_{lh}(\overline{K}_n) = n$.*

The next result is a general version (it includes disconnected graphs) of the one given in [11].

Theorem 2.7. *Let G be graph of order $n \geq 2$. Then the following statements hold.*

(i) $1 \leq ln(G) \leq n-1$ *and* $ln(G) = 1$ *if and only if* $G \in \{K_2, \overline{K_2}, P_3, K_1 \cup K_2\}$.

(ii) $ln(G) = n-1$ *if and only if* $G = K_n$ *or* $G = \overline{K}_n$.

From the preceding result and the fact that $cln(G) = ln(\overline{G})$, we get the next result.

Corollary 2.8. *Let* G *be graph of order* $n \geq 2$. *Then the following statements hold.*

(i) $1 \leq cln(G) \leq n-1$ *and* $cln(G) = 1$ *if and only if* $G \in \{K_2, \overline{K_2}, P_3, K_1 \cup K_2\}$.

(ii) $cln(G) = n-1$ *if and only if* $G = K_n$ *or* $G = \overline{K}_n$.

The following relationship is obtained in [2].

Theorem 2.9. *Let* G *be a connected graph of order* $n \geq 2$. *If* $ln(G) < \gamma_L(G)$, *then* $\gamma_L(G) = ln(G) + 1$.

It should be noted that Theorem 2.9 also holds for disconnected graphs. Hence, the next result is immediate.

Corollary 2.10. *Let* G *be a graph of order* $n \geq 2$. *If* $cln(G) < cldn(G)$, *then* $cldn(G) = \gamma_L(\overline{G}) = ln(\overline{G}) + 1 = cln(G) + 1$.

The *join* of two graphs G and H, denoted by $G + H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 2.11. *Let* G *and* H *be any two graphs. Then* $S \subseteq V(G+H)$ *is a locating-hop dominating set of* $G+H$ *if and only if* $S = S_G \cup S_H$ *where* S_G *and* S_H *are complement locating-dominating sets of* G *and* H (*locating-dominating sets of* $\overline{G}$ *and* $\overline{H}$), *respectively.*

Proof. Suppose that S is a locating-hop dominating set of $G+H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. Since S is a hop dominating set of $G+H$, $S_G \neq \varnothing$ and $S_H \neq \varnothing$. If $S_G = V(G)$, then it is a complement locating-dominating set of G. Suppose $S_G \neq V(G)$ and let $v \in V(G) \setminus S_G$. Since S is a hop dominating set, there exists $w \in S$ such that $d_{G+H}(w, v) = 2$. Hence, $w \in S_G \setminus N_G(v)$. Next, let $x, y \in V(G) \setminus S_G$ where $x \neq y$. Since S is a hop locating set, $[V(G) \setminus N_G(x)] \cap S_G = N_{G+H}(x, 2) \cap S \neq N_{G+H}(y, 2) \cap S = [V(G) \setminus N_G(y)] \cap S_G$, showing that S_G is a complement-locating set of G. Thus, S_G is a complement locating-dominating set of G. Similarly, S_H is a complement locating-dominating set of H.

For the converse, suppose that $S = S_G \cup S_H$ where S_G and S_H are complement locating-dominating sets of G and H, respectively. Let $v \in V(G+H) \setminus S$. Suppose $v \in V(G) \setminus S_G$. Then by assumption, there exists $z \in S_G \setminus N_G(v)$. It follows that $d_{G+H}(z, v) = 2$. Similarly, if $v \in V(H) \setminus S_H$, then there exists $w \in S_H$ such that that $d_{G+H}(w, v) = 2$. This shows that S is a hop dominating set of $G+H$. Now let $a, b \in V(G+H) \setminus S$ where $a \neq b$. Suppose that $a, b \in V(G) \setminus S_G$. Since S_G is a complement-locating set of G, $N_{G+H}(a,2) \cap S = [V(G) \setminus N_G(a)] \cap S_G \neq [V(G) \setminus N_G(b)] \cap S_G = N_{G+H}(b,2) \cap S$. Similarly, $N_{G+H}(a,2) \cap S = [V(H) \setminus N_H(a)] \cap S_H \neq [V(H) \setminus N_H(b)] \cap S_H = N_{G+H}(b,2) \cap S$ if $a, b \in V(H) \setminus S_H$. Suppose now that $a \in V(G) \setminus S_G$ and $b \in V(H) \setminus S_H$. Then $N_{G+H}(a,2) \cap S = [V(G) \setminus N_G(a)] \cap S_G \neq [V(H) \setminus N_G(a)] \cap S_H = N_{G+H}(b,2) \cap S$. Therefore, S is a locating-hop dominating set of $G + H$. $\square$

Corollary 2.12. *Let G be a graph and let n be a positive integer. Then $S \subseteq V(K_n+G)$ is a locating hop dominating set of K_n+G if and only if $S = V(K_n) \cup S_G$ where S_G is a complement locating-dominating set of G.*

Proof. The only complement locating-dominating set of K_n is $V(K_n)$. Thus, by Theorem 2.11, the result follows. $\square$

The next results follow directly from Theorem 2.11 and Corollary 2.12.

Corollary 2.13. *Let G and H be any two graphs. Then $\gamma_{lh}(G+H) = cldn(G) + cldn(H) = \gamma_L(\overline{G}) + \gamma_L(\overline{H})$.*

Corollary 2.14. *Let G be a graph and let n be a positive integer. Then $\gamma_{lh}(K_n + G) = n + \gamma_L(\overline{G})$.*

The *corona* of graphs G and H, denoted by $G \circ H$, is the graph obtained from G by taking a copy H^v of H and forming the join $\langle v \rangle + H^v = v + H^v$ for each $v \in V(G)$.

Theorem 2.15. *Let G be a non-trivial connected graph and let H be any graph. Then $S \subseteq V(G \circ H)$ is a locating-hop dominating set of $G \circ H$ if and only if $S = A \cup [\cup_{v \in V(G)} D_v]$ where*

(i) *$A \subseteq V(G)$ such that for any two distinct vertices $v, w \in V(G) \setminus A$, $N_G(v) \neq N_G(w)$ or $N_G(v,2) \cap A \neq N_G(w,2) \cap A$;*

(ii) *D_v is a complement-locating set of H^v for each $v \in V(G)$;*

(iii) *D_v is a complement locating-dominating set of H^v for each $v \in V(G) \setminus N_G(A)$;*

(iv) *D_w is a dominating set of H^w for each $w \in V(G)$ such that $N_G(v) = \{w\}$ for some $v \in V(G) \setminus A$ and $N_G(w) \cap A = \varnothing$; and*

(v) *] if D_v is not a complement locating-dominating set of H^v for some $v \in V(G)$, then D_w is a complement locating-dominating set of H^w for each $w \in V(G) \setminus \{v\}$ with $N_G(w) \cap A = N_G(v) \cap A$.*

Proof. Suppose S is a locating-hop dominating set of $G \circ H$. Let $A = S \cap V(G)$ and let $D_v = S \cap V(H^v)$ for each $v \in V(G)$. Then $S = A \cup [\cup_{v \in V(G)} D_v]$. Let $v, w \in V(G) \setminus A$ with $v \neq w$. Since S is a hop-locating set,

$$\begin{aligned} [N_G(v,2) \cap A] \cup [\cup_{x \in N_G(v)} D_x] &= N_{G \circ H}(v,2) \cap S \\ &\neq N_{G \circ H}(w,2) \cap S \\ &= [N_G(w,2) \cap A] \cup [\cup_{y \in N_G(w)} D_y]. \end{aligned}$$

This implies that $N_G(v,2)\cap A \neq N_G(w,2)\cap A$ or $N_G(v)\neq N_G(w)$, showing that (i) holds.

Next, let $v\in V(G)$ and let $a,b\in V(H^v)\setminus D_v$ with $a\neq b$. Since S is a locating-hop set,

$$([V(H^v)\setminus N_{H^v}(a)]\cap D_v)\cup[N_G(v)\cap A] = N_{G\circ H}(a,2)\cap S$$
$$\neq N_{G\circ H}(b,2)\cap S = ([V(H^v)\setminus N_{H^v}(b)]\cap D_v)\cup[N_G(v)\cap A].$$

Hence,

$$[V(H^v)\setminus N_{H^v}(a)]\cap D_v \neq [V(H^v)\setminus N_{H^v}(b)]\cap D_v.$$

This shows that D_v is a complement-locating set of H^v. Hence, (ii) holds. Suppose $v\in V(G)\setminus N_G(A)$. Since S is a hop dominating set, D_v must be a complement locating-dominating set of H^v, showing that (iii) holds. To show (iv), suppose that $w\in V(G)$ such that $N_G(v)=\{w\}$ for some $v\in V(G)\setminus A$ and $N_G(w)\cap A=\varnothing$. If $D_w=V(H^w)$, then we are done. So suppose $D_w\neq V(H^w)$ and let $q\in V(H^w)\setminus D_w$. Then by assumption and the fact that S is a locating-hop set,

$$D_w = N_{G\circ H}(v,2)\cap S \neq N_{G\circ H}(q,2)\cap S = [V(H^w)\setminus N_H(q)]\cap D_w.$$

This implies that $N_H(q)\cap D_w\neq\varnothing$. This shows that D_w is a dominating set of H^w. Again, since S is a locating-hop set $G\circ H$, (v) also holds.

For the converse, suppose that S is as described and satisfies properties (i)-(v). Let $x\in V(G\circ H)\setminus S$ and let $v\in V(G)$ such that $x\in V(v+H^v)$. Suppose $x=v$. Then $v\notin A$. Let $w\in V(G)\cap N_G(v)$. Pick any $y\in D_w$. Then $y\in S\cap N_{G\circ H}(v,2)$. Suppose $x\neq v$. Then $x\in V(H^v)\setminus D_v$. If $N_G(v)\cap A\neq\varnothing$, say $u\in N_G(v)\cap A$, then $u\in S\cap N_{G\circ H}(x,2)$. If $N_G(v)\cap A=\varnothing$, then there exists $z\in[V(H^v)\setminus N_{H^v}(x)]\cap D_v$ by property (iii). Hence, there exists $z\in S\cap N_{G\circ H}(x,2)$. This shows that S is a hop dominating set of $G\circ H$.

Now let $a,b\in V(G\circ H)\setminus S$ with $a\neq b$ and let $v,w\in V(G)$ such that $a\in V(v+H^v)$ and $b\in V(w+H^w)$. Consider the following cases:

Case 1: $v=w$

Suppose $a,b\in V(H^v)\setminus D_v$. By (ii) and (iii), D_v is a complement locating-dominating set of H^v. Consequently, $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$. Suppose $a=v$ and $b\in V(H^v)\setminus D_v$. Pick any $z\in N_G(v)$. Since $D_z\subseteq N_{G\circ H}(a,2)\setminus N_{G\circ H}(b,2)$, it follows that $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$.

Case 2: $v\neq w$

Suppose $a=v$ and $b=w$. Then $v,w\in V(G)\setminus A$. By property (i), $N_G(v)\neq N_G(w)$ or $N_G(v,2)\cap A\neq N_G(w,2)\cap A$. If $N_G(v,2)\cap A\neq N_G(w,2)\cap A$, then $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$. Suppose $N_G(v)\neq N_G(w)$. We may assume that that there exists $p\in N_G(v)\setminus N_G(w)$. Then $D_p\subseteq N_{G\circ H}(a,2)\setminus N_{G\circ H}(b,2)$. Hence, $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$.

Next, suppose that $a=v$ and $b\in V(H^w)\setminus D_w$ (or $b=w$ and $a\in V(H^v)\setminus D_v$). If $|N_G(v)|>1$ or $vw\notin E(G)$, pick any $z\in N_G(v)\setminus\{w\}$. Then $D_z\subseteq N_{G\circ H}(a,2)\setminus N_{G\circ H}(b,2)$. It follows that $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$. Suppose that $N_G(v)=\{w\}$. If $N_G(w)\cap A\neq\varnothing$, then $N_{G\circ H}(a,2)\cap S\neq N_{G\circ H}(b,2)\cap S$

because $N_G(w) \cap A \subseteq N_{G \circ H}(b, 2) \setminus N_{G \circ H}(a, 2)$. If $N_G(w) \cap A = \varnothing$, then D_w is a dominating set of of H^w by (iv). Hence, $N_{G \circ H}(a, 2) \cap S \neq N_{G \circ H}(b, 2) \cap S$.

Finally, suppose that $a \in V(H^v) \setminus D_v$ and $b \in V(H^w) \setminus D_w$. If $[V(H^v) \setminus N_{H^w}(a)] \cap D_v \neq \varnothing$ and $[V(H^w) \setminus N_{H^w}(b)] \cap D_w \neq \varnothing$, then $N_{G \circ H}(a, 2) \cap S \neq N_{G \circ H}(b, 2) \cap S$. Suppose one, say $[V(H^v) \setminus N_{H^w}(a)] \cap D_v = \varnothing$. If $N_G(v) \cap A \neq N_G(w) \cap A$, then $N_{G \circ H}(a, 2) \cap S \neq N_{G \circ H}(b, 2) \cap S$. If $N_G(v) \cap A = N_G(w) \cap A$, then $[V(H^w) \setminus N_{H^w}(b)] \cap D_w \neq \varnothing$ by (v). Thus, $N_{G \circ H}(a, 2) \cap S \neq N_{G \circ H}(b, 2) \cap S$.

Accordingly, S is a locating-hop dominating set of $G \circ H$. □

The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ such that $(v, a)(u, b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$.

Note that every non-empty subset C of $V(G) \times V(H)$ can be expressed as $C = \cup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$.

Theorem 2.16. *Let G and H be non-trivial connected graphs. Then $C = \bigcup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a locating-hop dominating set of $G[H]$ if and only if the following conditions hold:*

(i) $S = V(G)$

(ii) *T_x is a complement-locating set of H for all $x \in S$.*

(iii) *If T_x is not complement locating-dominating for some $x \in S$, then T_y is complement locating-dominating for all $y \in S \setminus \{x\}$ with $N_G(x, 2) = N_G(y, 2)$.*

(iv) *T_x is a complement locating-dominating set of H for each $x \in S \setminus N_G(S, 2)$.*

Proof. Suppose C is a locating-hop dominating set of $G[H]$. Suppose there exists $z \in V(G) \setminus S$. Pick distinct vertices $a, b \in V(H)$. Then $(z, a), (z, b) \in V(G[H]) \setminus C$ and $N_{G[H]}((z, a)) \cap C = \bigcup_{x \in N_G(z,2) \cap S}[\{x\} \times T_x] = N_{G[H]}((z, b)) \cap C$. This implies that C is not a locating-hop set, contrary to our assumption. Thus, $S = V(G)$, showing that (i) holds.

Now let $x \in S$. Let $p, q \in V(H) \setminus T_x$ with $p \neq q$. Then $(x, p), (x, q) \in V(G[H]) \setminus C$. Now, $N_{G[H]}((x, p), 2) \cap C = [\{x\} \times (V(H) \setminus N_H(p))] \cap (\{x\} \times T_x) \cup [\cup_{w \in N_G(x,2)}(\{w\} \times T_w)]$ and $N_{G[H]}((x, q), 2) \cap C = [\{x\} \times (V(H) \setminus N_H(q))] \cap (\{x\} \times T_x) \cup [\cup_{w \in N_G(x,2)}(\{w\} \times T_w)]$. Since C is a locating-hop set, $[\{x\} \times (V(H) \setminus N_H(p))] \cap (\{x\} \times T_x) \neq [\{x\} \times (V(H) \setminus N_H(q))] \cap (\{x\} \times T_x)$. This implies that $(V(H) \setminus N_H(p)) \cap T_x \neq (V(H) \setminus N_H(q)) \cap T_x$. Hence, T_x is a complement-locating set of H, showing that (ii) holds. Suppose there exists x such that T_x is not complement locating-dominating and let $y \in S \setminus \{x\}$ with $N_G(x, 2) = N_G(y, 2)$. Let $p \in V(H) \setminus T_x$ be such that $[V(H) \setminus N_H(p)] \cap T_x = \varnothing$ and let $q \in V(H) \setminus T_y$. Since C is a locating-hop dominating set, $N_G(x, 2) = N_G(y, 2)$, and $[V(H) \setminus N_H(p)] \cap T_x = \varnothing$, it follows that $[V(H) \setminus N_H(q)] \cap T_y \neq \varnothing$. This implies that T_y is a complement locating-dominating set of H, showing that (iii) holds.

Next, let $x \in S \setminus N_G(S, 2)$. If $T_x = V(H)$, then T_x is a complement locating-dominating set of H. So suppose that $T_x \neq V(H)$ and let $t \in V(H) \setminus T_x$.

Since C is hop dominating and $(x,t) \notin C$, there exists $(w,s) \in C$ such that $d_{G[H]}((x,t),(w,s)) = 2$. The condition $x \in S \setminus N_G(S,2)$ would imply that $w = x$ and $s \in [V(H) \setminus N_H(t)] \cap T_x$. Hence, T_x is a complement locating-dominating set of H, showing that (iv) holds.

For the converse, suppose that C satisfies properties (i)-(iv). Let $(x,a) \in V(G[H]) \setminus C$. Then $a \in V(H) \setminus T_x$. If $x \in N_G(S,2)$, then there exists $z \in N_G(x,2)$. Let $b \in T_z$. Then $(z,b) \in C \cap N_{G[H]}((x,a),2)$. Suppose $x \in S \setminus N_G(S,2)$. By (iv), T_x is a complement locating-dominating set of H. Hence, there exists $p \in [V(H) \setminus N_H(a)] \cap T_x$. This implies that $(x,p) \in C \cap N_{G[H]}((x,a),2)$. Therefore, C is a hop dominating set of $G[H]$.

Next, let $(v,q),(w,s) \in V(G[H]) \setminus C$ with $(v,q) \neq (w,s)$. Then

$$N_{G[H]}((v,q),2) \cap C = [\{v\} \times (V(H) \setminus N_H(q))] \cap (\{v\} \times T_v) \cup [\bigcup_{z \in N_G(v,2)} [\{z\} \times T_z],$$

and

$$N_{G[H]}((w,s),2) \cap C = [\{w\} \times (V(H) \setminus N_H(s))] \cap (\{w\} \times T_w) \cup [\bigcup_{y \in N_G(w,2)} [\{y\} \times T_y].$$

Consider the following cases:
Case 1: $v = w$

Then $q, s \in V(H) \setminus T_v$ with $q \neq s$. By (ii), T_v is a complement-locating set; hence, $[V(H) \setminus N_H(q)] \cap T_v \neq [V(H) \setminus N_H(s)] \cap T_v$. It follows that $N_{G[H]}((v,q),2) \cap C \neq N_{G[H]}((v,s),2) \cap C$.
Case 2: $v \neq w$

Then $q \in V(H) \setminus T_v$ and $s \in V(H) \setminus T_w$. If $N_G(v,2) \neq N_G(w,2)$, then clearly, $N_{G[H]}((v,q),2) \cap C \neq N_{G[H]}((w,s),2) \cap C$. Suppose $N_G(v,2) = N_G(w,2)$. If T_v and T_w are both complement locating-dominating sets, then $N_{G[H]}((v,q),2) \cap C \neq N_{G[H]}((w,s),2) \cap C$. Suppose T_v is not complement locating-dominating. Then T_w is complement locating-dominating by (iii). It follows that $N_{G[H]}((v,q),2) \cap C \neq N_{G[H]}((w,s),2) \cap C$.

Accordingly, C is a locating-hop dominating set of $G[H]$. □

A connected graph G is *distance-two point determining* if $N_G(x,2) \neq N_G(y,2)$ for any distinct vertices $x, y \in V(G)$.

Note that P_4, C_4, and the star $K_{1,n}$, where $n \geq 2$, are distance-two point determining.

Corollary 2.17. *Let G and H be non-trivial connected graphs. Then $\gamma_{lh}(G[H]) \leq |V(G)|cldn(H) = |V(G)|\gamma_L(\overline{H})$. If G is distance-two point determining and $\gamma(G) \neq 1$, then $\gamma_{lh}(G[H]) = |V(G).|cln(H) = |V(G)|.ln(\overline{H})$.*

Proof. Let $S = V(G)$ and let T_x be a $cldn$-set of H for each $x \in V(G)$. By Theorem 2.16, $C = \bigcup_{x \in S}[\{x\} \times T_x]$ is a locating-hop dominating set of $G[H]$. It follows that $\gamma_{lh}(G[H]) \leq |C| = |V(G)|cldn(H)$.

Next, suppose that G is distance-two point determining and $\gamma(G) \neq 1$. Let $S' = V(G)$ and let R_x be a cln-set of H for each $x \in S$. Since $\gamma(G) \neq 1$, $x \in N_G(S, 2)$ for each $x \in S$. Thus, by Theorem 2.16, $C = \bigcup_{x \in S'}[\{x\} \times R_x]$ is a locating-hop dominating set of $G[H]$. It follows that $\gamma_{lh}(G[H]) \leq |C| = |V(G)|.cln(H)$. Now, if $C_0 = \bigcup_{x \in S_0}[\{x\} \times T_x]$ is a γ_{lh}-set of $G[H]$, then $S_0 = V(G)$ and T_x is a complement-locating set of H for each $x \in V(G)$ by Theorem 2.16. Hence, $\gamma_{lh}(G[H]) = |C_0| = \sum_{x \in S_0} |T_x| \geq |V(G)|.cln(H)$. Therefore, $\gamma_{lh}(G[H]) = |V(G)|.cln(H)$. □

Corollary 2.18. *Let G and H be non-trivial connected graphs. If G is distance-two point determining and $\gamma(G) = 1$, then $\gamma_{lh}(G[H]) = cldn(H) + (|V(G)| - 1)cln(H)$.*

Proof. Let $D_G = \{v \in V(G) : \{v\}$ is a dominating set of $G\}$. Since G is distance-two point determining, it follows that $|D_G| = 1$. Set $S = V(G)$. Let T_v be a $cldn$-set of H for $v \in D_G$ and let T_x be a cln-set of H for each $x \in V(G) \setminus \{v\}$. Then, by Theorem 2.16, $C = [\bigcup_{x \in S\setminus\{v\}}(\{x\} \times T_x)] \cup (\{v\} \times T_v)$ is a locating-hop dominating set of $G[H]$. Hence, $\gamma_{lh}(G[H]) \leq |C| = cldn(H) + (|V(G)| - 1)cln(H)$.

Suppose now that $C^* = [\bigcup_{x \in S^*}(\{x\} \times R_x)]$ is a γ_{lh}-set of $G[H]$. Again, there exists a unique vertex v such that $\{v\}$ is a dominating set of G. By Theorem 2.16, $S^* = V(G)$, R_v is a complement locating-dominating set and R_x is a complement-locating set of H for each $x \in V(G) \setminus \{v\}$. Thus, $\gamma_{lh}(G[H]) = |C^*| = |R_v| + \sum_{x \in S^*\setminus\{v\}} |R_x| \geq cldn(H) + (|V(G)| - 1)cln(H)$. Therefore, $\gamma_{lh}(G[H]) = cldn(H) + (|V(G)| - 1)cln(H)$ as asserted. □

Corollary 2.19. *Let G be a non-trivial connected distance-two point determining graph and let $p \geq 2$ be a positive integer.*

$$\gamma_{lh}(G[K_p]) = \begin{cases} |V(G)|(p-1) & if\ \gamma(G) \neq 1 \\ (p-1)|V(G)| + 1 & if\ \gamma(G) = 1, \end{cases}$$

Proof. Suppose first that $\gamma(G) \neq 1$. By Corollary 2.17 and the fact that $cln(K_p) = ln(\overline{K}_p = p - 1$, it follows that $\gamma_{lh}(G[K_p]) = |V(G)|(p-1)$.

Next, suppose that $\gamma(G) = 1$. By Corollary 2.18 and the fact that $cldn(K_p) = \gamma_L \overline{K}_p = p$, we have $\gamma_{lh}(G[K_p]) = p + (p-1)(|V(G)| - 1) = (p-1)|V(G)| + 1$. □

Corollary 2.20. *Let H be a non-trivial connected graph and let $p \geq 2$ be a positive integer. Then $\gamma_{lh}(K_p[H]) = p.cldn(H)$.*

Proof. Let $G = K_p$. Then v is a dominating vertex of G for each $v \in V(G)$. Thus, if $C_0 = \bigcup_{z \in S_0}[\{z\} \times T_z]$ is a γ_{lh}-set of $G[H]$, then $S_0 = V(G)$ and each T_z is a $cldn$-set of H by Theorem 2.16. Consequently, $\gamma_{lh}(K_p[H]) = p.cldn(H)$. □

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