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Left Regular and Left Weakly Regular n-ary Semigroups

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Abstract. We study the concept of a quasi-ideal and a generalized bi-ideal of an n-ary semigroup; give a construction of the quasi-ideal of an n-ary semigroup generated by its nonempty subset; and introduce the notions of regularities, namely, a left regularity and a left weakly regularity. Moreover, the notions of a right regularity, a right weak regularity and a complete regularity are given. Finally, characterizations of these regularities are presented.

1. Introduction

The concept of a semigroup has been extensively studied for a long time, as has that of an ordered semigroups, which is a semigroup together with a partially ordered relation. Classical results on ordered semigroups can be found in [1, 2]. Many kinds of regularity of ordered semigroups were studied by Kehayopulu and Tsingelis (see, [12, 13, 14, 15, 16]). In 1991, Kehayopulu [13] introduced the concept of right regular ordered semigroups and considered semigroups in which the right ideals are two-sided. Kehayopulu [14] gave the concept of complete regularity and gave characterizations of completely regular *poe*-semigroups which generalize the characterizations of completely regular semigroups (without order) which is given by Steinfeld [24]. Additionally, the characterizations of right weakly regular semigroups can be found in [10, 17].

The definition and theory of ternary semigroups were first introduced by Lehmer [18] in 1932. Santiago [22] investigated the notion of ideals of ternary semigroups and gave some properties of regular ternary semigroups. In 2012, Daddi and Pawar [3] investigated the notion of ordered quasi-ideals and ordered bi-ideals of ordered

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ternary semigroups and used them to characterize regular ordered ternary semigroups. N. Lekkoksung and P. Jampachon [19] gave the other types of regularities of ordered ternary semigroups, namely, left (right) weakly regular ordered ternary semigroups. Later, several types of regularities of ordered ternary semigroups were characterized in terms of ordered ideals by Pornsurat and Pibaljommee in [20].

The notion of n-ary systems was first introduced by Kasner [11] in 1904. In 1928, Dörnte [5] studied the notion of n-ary groups which is a generalization of groups. Sioson [23] gave the notion of regular n-ary semigroups and used their ideals to characterize regular n-ary semigroups. Later, Dudek and Groździńska [6] gave a new definition of regular n-ary semigroups and investigated some properties of regular n-ary semigroups. Additionally, Dudek proved several results and gave many examples of n-ary groups in [7, 8]. Moreover, Dudek [9] studied a new type of elements of n-ary semigroups where $n \geq 3$, namely, potent elements and investigated properties of ideals in which all elements are potent. Recently, the construction of the j-ideal of an n-ary semihypergroup generated by its nonempty subset was presented by Daengsaen, Leeratanavalee and Davvaz [4]. Later, Pornsurat, Palakawong na Ayutthaya and Pibaljommee [21] gave a construction of the j-ideal of a ordered n-ary semigroup generated by its nonempty subset in a different form.

In this work, we first study the concept of a quasi-ideal and a generalized bi-ideal of an n-ary semigroup and give a construction of the quasi-ideal of an n-ary semigroup generated by its nonempty subset. Then we introduce the notions of regularities, namely, a left regularity, a right regularity, a left weak regularity, a right weak regularity and a complete regularity in an n-ary semigroup. Moreover, we give characterizations of these regularities using properties of their ideals.

2. Preliminaries

Let $\mathbb{N}$ be the set of all natural numbers and $i, j, n \in \mathbb{N}$. A nonempty set S together with an n-ary operation given by $f : S^n \to S$, where $n \geq 2$, is called an *n-ary groupoid* [9]. For $1 \leq i < j \leq n$, the sequence $x_i, x_{i+1}, x_{i+2}, \ldots, x_j$ of elements of S is denoted by x_i^j and if $x_i = x_{i+1} = \ldots = x_j = x$, we write $\overset{j-i+1}{x}$ instead of x_i^j. For $j < i$, we denote x_i^j as an empty symbol and so is $\overset{0}{x}$. Then,

$$f(x_1, \ldots, x_i, \underbrace{x, x, \ldots, x}_{j \text{ terms}}, x_{i+j+1}, \ldots, x_n) = f(x_1^i, \overset{j}{x}, x_{i+j+1}^n).$$

An n-ary groupoid satisfies (i, j)-associative law if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for all $x_1, \ldots, x_{2n-1} \in S$. The n-ary operation f is associative if the above identity holds for all $1 \leq i \leq j \leq n$. An n-ary groupoid (S, f) is called an *n-ary semigroup* if the n-ary operation f satisfies the associative law.

By the associativity of an n-ary semigroup, we define the map f_k where $k \geq 2$ by

$$f_k(x_1^{k(n-1)+1}) = \underbrace{f(f(\ldots,f(f}_{k \text{ terms}}(x_1^n), x_{n+1}^{2n-1}), \ldots), x_{(k-1)(n-1)+2}^{k(n-1)+1})$$

for any $x_1, \ldots, x_{k(n-1)+1} \in S$.

For $1 \leq i < j \leq n$, the sequence $A_i, A_{i+1}, \ldots, A_j$ of nonempty sets of S is denoted by the symbol A_i^j. For nonempty subsets A_1^n of S, we denote

$$f(A_1^n) = \{f(a_1^n) \mid a_i \in A_i \text{ where } 1 \leq i \leq n\}.$$

If $A_1 = A_2 = \ldots = A_n = A$, then we write $f(\overset{n}{A})$ instead of $f(A_1^n)$. If $A_1 = \{a_1\}$, then we write $f(a_1, A_2^n)$ instead of $f(\{a_1\}, A_2^n)$, and similarly in other cases. For $j < i$, we set the notations A_i^j and $\overset{0}{A}$ to be the empty symbols as a similar way of the notations x_i^j and $\overset{0}{x}$, respectively.

Throughout this paper, we write S instead of an n-ary semigroup. A nonempty subset H of S is called an *n-ary subsemigroup* of S if $f(a_1^n) \in H$ for all $a_1, a_2, \ldots, a_n \in H$. Many notations above can be found in [6, 7, 8, 9].

For $1 \leq i \leq n$, a nonempty subset I of S is called an *i-ideal* of S if $f(\overset{i-1}{S}, I, \overset{n-i}{S}) \subseteq I$. A nonempty subset I of S is called an *ideal* of S if I is an i-ideal for all $i = 1, \ldots, n$.

Let A be a nonempty subset of S. The intersection of all i-ideals of S containing A is called the i-ideal of S generated by A. Then, we denote the notation $J_i(A)$ to be the i-ideal of S generated by A. In a particular case $A = \{a\}$, we write $J_i(a)$ instead of $J_i(\{a\})$.

As a special case of n-ary semihypergroups [4] and ordered n-ary semigroups [21], we present the construction of the i-ideal of an n-ary semigroup generated by its nonempty subset as the following theorem.

Theorem 2.1. *Let A be a nonempty subset of S. Then*

$$J_i(A) = \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S}) \cup A.$$

Corollary 2.2. *Let A be a nonempty subset of S. Then the following statements hold:*

(i) $J_1(A) = f(A, \overset{n-1}{S}) \cup A$ *and* $J_n(A) = f(\overset{n-1}{S}, A) \cup A$.

(ii) *If* $1 < i < n$ *and* $i = \frac{n+1}{2} \in \mathbb{N}$, *then*

$$J_i(A) = f(\overset{i-1}{S}, A, \overset{n-i}{S}) \cup f_2(\overset{n-1}{S}, A, \overset{n-1}{S}) \cup A.$$

Next, we define the notion of a quasi-ideal of an n-ary semigroup and give a construction of the quasi-ideal of an n-ary semigroup generated by its nonempty subset as follows.

Definition 2.3. A nonempty subset Q of S is called a *quasi-ideal* of S if

$$\bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, Q, \overset{m(n-i)}{S}) \subseteq Q.$$

Note that for any quasi-ideal Q of S, $f(\overset{n}{Q}) \subseteq \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, Q, \overset{m(n-i)}{S})$ for all $i = 1, \ldots, n$. So, $f(\overset{n}{Q}) \subseteq \bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, Q, \overset{m(n-i)}{S}) \subseteq Q$. So, every quasi-ideal of S is an n-ary subsemigroup of S.

Proposition 2.4. *Let S be an n-ary semigroup. Then the following statements hold.*

(*i*) *Every i-ideal of S is a quasi-ideal of S for each $1 \leq i \leq n$.*

(*ii*) *If $I_1, I_2, \ldots, I_n$ are 1-ideal, 2-ideal,..., n-ideal of S, respectively, then $\bigcap_{i=1}^{n} I_i$ is a quasi-ideal of S.*

Proof. (1) : Let I be an i-ideal of S. By Theorem 2.1,

$$\bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, I, \overset{m(n-i)}{S}) \subseteq \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, I, \overset{m(n-i)}{S}) \subseteq J_i(I) = I.$$

So, every i-ideal of S is a quasi-ideal of S.

(2) : Let I_i be an i-ideal for all $1 \leq i \leq n$, respectively. Let $Q = \bigcap_{i=1}^{n} I_i$. Since $I_i \neq \emptyset$, there is $a_i \in I_i$ for all $i = 1, \ldots, n$. So, $f(a_1^n) \in Q \neq \emptyset$. By Theorem 2.1,

$$\begin{aligned}\bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, Q, \overset{m(n-i)}{S}) &\subseteq \bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, I_i, \overset{m(n-i)}{S}) \\ &\subseteq \bigcap_{i=1}^{n} J_i(I_i) \\ &= \bigcap_{i=1}^{n} I_i.\end{aligned}$$

So, Q is a quasi-ideal of S. □

Let S be a semigroup. Then we note that the union of two quasi-ideal need not to be a quai-ideal of S as the following example.

Example 2.5. Let $S = \{a, b, c, d, e, f, g, h\}$. Define the binary operation $\cdot$ on S by the following table:

$\cdot$	a	b	c	d	e	f	g	h
a	a	a	a	a	a	a	a	a
b	a	b	g	a	h	b	g	h
c	a	d	a	a	d	d	a	d
d	a	d	a	a	d	d	a	d
e	a	f	g	a	e	f	g	e
f	a	f	g	a	e	f	g	e
g	a	a	a	a	a	a	a	a
h	a	b	g	a	h	b	g	h

Then $(S, \cdot)$ is a semigroup. It is easy to see that $\{a, c\}$ and $\{a, b\}$ are quasi-ideal of S. However, $\{a, b, c\}$ is not a quasi-ideal of S because $\{a, b, c\}S \cap S\{a, b, c\} = \{a, b, g\} \nsubseteq \{a, b, c\}$.

Proposition 2.6. *A nonempty subset of S is a quasi-ideal of S if and only if it is an intersection of a 1-ideal I_1, a 2-ideal I_2,..., an n-ideal I_n of S.*

Proof. Let Q be a quasi-ideal of S. Define $I_i = \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, Q, \overset{m(n-i)}{S}) \cup Q$. By Theorem 2.1, I_i is an i-ideal of S for all $i = 1, \ldots, n$. It is clear that $Q \subseteq \bigcap_{i=1}^{n} I_i$. We consider $\bigcap_{i=1}^{n} I_i = \bigcap_{i=1}^{n} \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, Q, \overset{m(n-i)}{S}) \subseteq Q$. So, $Q = \bigcap_{i=1}^{n} I_i$. The converse is followed by Proposition 2.4(2). □

For a nonempty subset A of S, we denote the notion $Q(A)$ to be the quasi-ideal of S generated by A. In a particular case $A = \{a\}$, we write $Q(a)$ instead of $Q(\{a\})$.

Theorem 2.7. *The intersection of an arbitrary nonempty family of quasi-ideal of S is either a quasi-ideal of S or an empty set.*

Proof. Let $\{Q_j \mid j \in J\}$ be a family of quasi-ideal of S. Suppose that $Q = \bigcap_{j \in J} Q_j \neq \emptyset$. Since Q_j is a quasi-ideal of S for all $j \in J$, $\bigcap_{i=1}^{n} \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, Q, \overset{m(n-i)}{S}) \subseteq \bigcap_{i=1}^{n} \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, Q_j, \overset{m(n-i)}{S}) \subseteq Q_j$ for all $j \in J$. So, $\bigcap_{i=1}^{n} \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, Q, \overset{m(n-i)}{S}) \subseteq \bigcap_{j \in J} Q_j = Q$. Hence, Q is a quasi-ideal of S. □

Theorem 2.8. *Let S be an n-ary semigroup. Then*

$$Q(A) = \bigcap_{i=1}^{n} \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S}) \cup A.$$

Proof. Let A be a nonempty subset of S. By Theorem 2.1 and Proposition 2.4(2), $\bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S}) \cup A$ is a quasi-ideal of S containing A. So, $Q(A) \subseteq \bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S}) \cup A$. Let Q be a quasi-ideal of S and $A \subseteq Q$. Then

$$\begin{aligned}\bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S}) \cup A &\subseteq \bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, Q, \overset{m(n-i)}{S}) \cup Q\\ &\subseteq Q \cup Q\\ &= Q.\end{aligned}$$

Therefore, $Q(A) = \bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S}) \cup A$. □

Definition 2.9. A nonempty subset B of S is called a *generalized bi-ideal* of S if $f_2(\underbrace{B, S, B, \dots, S, B}_{2n-1 \text{ terms}}) \subseteq B$.

Remark 2.10. Every quasi-ideal of S is a generalized bi-ideal of S.

Proof. Let B be a quasi-ideal of S. We consider

$$f_2(\underbrace{B, S, B, \dots, S, B}_{2n-1 \text{ terms}}) \subseteq f_2(\overset{2(i-1)}{S}, B, \overset{2(n-i)}{S}) \subseteq \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S})$$

for all $i = 1, \dots, n$.

It follows that, $f_2(\underbrace{B, S, B, \dots, S, B}_{2n-1 \text{ terms}}) \subseteq \bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, B, \overset{m(n-i)}{S}) \subseteq B$.

Now, we recall the notion of a regular n-ary semigroup in sense of Sioson as follows.

An n-ary semigroup S is called *regular*, in sense of Sioson [23], if each $a \in S$, there exist $x_{ij} \in S$ where $i, j = 1, 2, \dots n$ such that

$$f(f(a, x_{12}^{1n}), f(x_{21}, a, x_{23}^{2n}), \dots, f(x_{n1}^{nn-1}, a)) = a. \tag{2.1}$$

Many forms of regular n-ary semigroup which can also be found in [6] are given as follows.

An n-ary semigroup S is called *regular* if each $a \in S$, there exist $x_2, \dots, x_n \in S$ such that

$$f_2(a, x_2, a, x_3, \dots, a, x_n, a) = a. \tag{2.2}$$

An n-ary semigroup S is called *regular* if each $a \in S$, there exist $x_2, \dots, x_{2n-2} \in S$ such that

$$f_2(a, x_2^{2n-2}, a) = a. \tag{2.3}$$

An n-ary semigroup S is called *regular* if each $a \in S$, there exist $x_2, \ldots, x_{n-1} \in S$ such that

$$f(a, x_2^{n-1}, a) = a. \tag{2.4}$$

However, Dudek and Groździńska [6] proved that the regularities conditions (2.1), (2.2), (2.3) and (2.4) are equivalent. □

3. Left Regular and Right Regular n-ary Semigroups

In this section, we define the notion of left regular, right regular, left weakly regular, right weakly regular and completely regular n-ary semigroups and give some their characterizations.

Definition 3.1. An element $a \in S$ is called a *left* (*right*) *regular element* if there exist $x_1, \ldots, x^{n-1} \in S$ such that $a = f_2(x_1^{n-1}, \overset{n}{a})$ $(a = f_2(\overset{n}{a}, x_1^{n-1}))$. An n-ary semigroup S is *left* (*right*) *regular*, if each its element is *left* (*right*) *regular*.

We note that S is a left (right) regular n-ary semigroup where $n \geq 3$ if and only if there exists $x \in S$ such that $a = f(x, \overset{n-1}{a})$ $(a = f(\overset{n-1}{a}, x))$.

Lemma 3.2. *The following statements are equivalent.*

(i) S is left (resp. right) regular.

(ii) $A \subseteq f_2(\overset{n-1}{S}, \overset{n}{A})$ (resp. $A \subseteq f_2(\overset{n}{A}, \overset{n-1}{S})$) for any $\emptyset \neq A \subseteq S$.

(iii) $a \in f_2(\overset{n-1}{S}, \overset{n}{a})$ (resp. $a \in f_2(\overset{n}{a}, \overset{n-1}{S})$) for any $a \in S$.

Definition 3.3. Let P be a nonempty subset of S. Then P is called *semiprime* if for all nonempty subsets A of S, $f(\overset{n}{A}) \subseteq P$ implies $A \subseteq P$.

Remark 3.4. A nonempty subset P of S is semiprime if and only if for all $a \in S$, $f(\overset{n}{a}) \in P$ implies $a \in P$.

Theorem 3.5. *The following statements are equivalent.*

(i) S is left regular.

(ii) Every n-ideal of S is semiprime.

(iii) $J_n(a)$ is a semiprime of S for any $a \in S$.

(iv) $J_n(f(\overset{n}{a}))$ is a semiprime of S for any $a \in S$.

Proof. $(i) \Rightarrow (ii)$. Let I be an n-ideal of S and $\emptyset \neq A \subseteq S$ such that $f(\overset{n}{A}) \subseteq I$. Since S is left regular and By Lemma 3.2, $A \subseteq f_2(\overset{n-1}{S}, \overset{n}{A}) \subseteq f(\overset{n-1}{S}, I) \subseteq I$.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are obvious.

$(iv) \Rightarrow (i)$. Let $a \in S$. Since $f(\overset{n}{a}) \in J_n(f(\overset{n}{a}))$ and $J_n(f(\overset{n}{a}))$ is semiprime, we have that $a \in J_n(f(\overset{n}{a}))$. By Corollary 2.2, $a \in J_n(f(\overset{n}{a})) = f(\overset{n-1}{S}, f(\overset{n}{a})) \cup \{f(\overset{n}{a})\}$. Then $a \in f(\overset{n-1}{S}, (f(\overset{n}{a}))$ or $a \in \{f(\overset{n}{a})\}$. If $a \in \{f(\overset{n}{a})\}$, then $a = f(\overset{n}{a}) = f(\overset{n-1}{a}, f(\overset{n}{a})) \in f(\overset{n-1}{S}, f(\overset{n}{a})) = f_2(\overset{n-1}{S}, \overset{n}{a})$. So, S is left regular. □

Next theorem can be similarly proved as Theorem 3.5.

Theorem 3.6. *The following statements are equivalent.*

- (i) *S is right regular.*
- (ii) *Every 1-ideal of S is semiprime.*
- (iii) *$J_1(a)$ is a semiprime of S for any $a \in S$.*
- (iv) *$J_1(f(\overset{n}{a}))$ is a semiprime of S for any $a \in S$.*

Theorem 3.7. *Let S be an n-ary semigroup. Then S is both left regular and right regular if and only if every quasi-ideal of S is semiprime.*

Proof. Assume that S is both left regular and right regular. Let $\emptyset \neq A \subseteq S$ and Q be a quasi-ideal of S such that $f(\overset{n}{A}) \subseteq Q$. By Lemma 3.2, $A \subseteq f_2(\overset{n}{A}, \overset{n-1}{S}) \subseteq f(Q, \overset{n-1}{S})$ and $A \subseteq f_2(\overset{n-1}{S}, \overset{n}{A}) \subseteq f(\overset{n-1}{S}, Q)$. For $i = 2, \ldots, n-1$, we have that

$$\begin{aligned} A &\subseteq f_2(\overset{n}{A}, \overset{n-1}{S}) \\ &\subseteq f_2(f_2(\overset{n-1}{S}, \overset{n}{A}), \overset{n-1}{A}, \overset{n-1}{S}) \\ &= f_4(\overset{n-1}{S}, \overset{2n-1}{A}, \overset{n-1}{S}) \\ &= f(f(\overset{n-1}{S}, A), \overset{i-2}{A}, f(\overset{n}{A}), f(\overset{n-i}{A}, \overset{i}{S}), \overset{n-i-1}{S}) \\ &\subseteq f(\overset{i-1}{S}, Q, \overset{n-i}{S}). \end{aligned}$$

Hence, $A \subseteq f(\overset{i-1}{S}, Q, \overset{n-i}{S})$ for all $i = 1, \ldots, n$. This implies that

$$A \subseteq \bigcap_{i=1}^{n} f(\overset{i-1}{S}, Q, \overset{n-i}{S}) \subseteq \bigcap_{i=1}^{n} \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, Q, \overset{m(n-i)}{S}) \subseteq Q.$$

So, every quasi-ideal of S is semiprime. Conversely, assume that every quasi-ideal of S is semiprime and $\emptyset \neq A \subseteq S$. By Theorem 2.8., $f(\overset{n}{A}) \subseteq Q(f(\overset{n}{A})) = \bigcap_{i=1}^{n} \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, f(\overset{n}{A}), \overset{m(n-i)}{S}) \cup f(\overset{n}{A})$. By assumption,

$$A \subseteq \bigcap_{i=1}^{n} \bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, f(\overset{n}{A}), \overset{m(n-i)}{S}) \cup f(\overset{n}{A}) \subseteq \bigcup_{m=1}^{n-1} f_m(f(\overset{n}{A}), \overset{m(n-1)}{S}) \cup f(\overset{n}{A}).$$

We consider

$$\begin{aligned}
f(\overset{n}{A}) &\subseteq \bigcap_{i=1}^{n}\bigcup_{m=1}^{n-1} f_m(\overset{m(i-1)}{S}, f(\overset{n}{A}), \overset{m(n-i)}{S}) \cup f(\overset{n}{A})\\
&\subseteq \bigcup_{m=1}^{n-1} f_m(f(\overset{n}{A}), \overset{m(n-1)}{S}) \cup f(\overset{n}{A})\\
&\subseteq \bigcup_{m=1}^{n-1} f_m(f(\overset{n}{A}), \overset{m(n-1)}{S}) \cup f(\overset{n-1}{A}, \bigcup_{m=1}^{n-1} f_m(f(\overset{n}{A}), \overset{m(n-1)}{S}) \cup f(\overset{n}{A}))\\
&= \bigcup_{m=1}^{n-1} f_m(f(\overset{n}{A}), \overset{m(n-1)}{S}) \cup \bigcup_{m=1}^{n-1} f_{m+1}(\overset{n-1}{A}, f(\overset{n}{A}), \overset{m(n-1)}{S}) \cup f_2(\overset{2n-1}{A})\\
&= \bigcup_{m=1}^{n-1} f_m(f(\overset{n}{A}), \overset{m(n-1)}{S}) \cup \bigcup_{m=1}^{n-1} f_m(f(\overset{n}{A}), f(\overset{n-1}{A}, S), \overset{m(n-1)-1}{S}) \cup f_2(\overset{2n-1}{A})\\
&\subseteq \bigcup_{m=1}^{n-1} f_m(f(\overset{n}{A}), \overset{m(n-1)}{S})\\
&= f_2(\overset{n}{A}, \overset{n-1}{S}).
\end{aligned}$$

We have that $A \subseteq \bigcup_{m=1}^{n-1} f_m(f(\overset{n}{A}), \overset{m(n-1)}{S}) \cup f(\overset{n}{A}) \subseteq \bigcup_{m=1}^{n-1} f_m(f(\overset{n}{A}), \overset{m(n-1)}{S}) \cup f_2(\overset{n}{A}, \overset{n-1}{S}) = f_2(\overset{n}{A}, \overset{n-1}{S}) \cup f_2(\overset{n}{A}, \overset{n-1}{S}) = f_2(\overset{n}{A}, \overset{n-1}{S})$. Similarly, we can show that $A \subseteq f_2(\overset{n-1}{S}, \overset{n}{A})$. Therefore, S is both left regular and right regular. □

Definition 3.8. *An n-ary semigroup S is called completely regular if it is regular, left regular and right regular.*

Lemma 3.9. *The following statements are equivalent.*

(*i*) *S is completely regular.*

(*ii*) $A \subseteq f_4(\overset{n}{A}, \underbrace{S, A, S, \ldots, A, S}_{2n-3\ terms}, \overset{n}{A})$ *for any $\emptyset \neq A \subseteq S$.*

Proof. Assume that S is completely regular. Let $\emptyset \neq A \subseteq S$. If $n = 2$, then it is easy to see that $A \subseteq f_2(A, S, A) \subseteq f_2(f_2(\overset{2}{A}, S), S, f_2(S, \overset{2}{A})) \subseteq f_4(\overset{2}{A}, S, \overset{2}{A})$. If $n \geq 3$, then

$$\begin{aligned} A &\subseteq f_2(\underbrace{A, S, A, \ldots, S, A}_{2n-1 \text{ terms}}) \\ &\subseteq f_2(f_2(\overset{n}{A}, \overset{n-1}{S}), \underbrace{S, A, S, \ldots, A, S}_{2n-3 \text{ terms}}, f_2(\overset{n-1}{S}, \overset{n}{A})) \\ &= f_2(f(\overset{n}{A}), f(\overset{n-1}{S}, S), \underbrace{A, S, A \ldots, S, A}_{2n-5 \text{ terms}}, f(\overset{n-1}{S}, S), f(\overset{n}{A})) \\ &\subseteq f_2(f(\overset{n}{A}), S, \underbrace{A, S, A \ldots, S, A}_{2n-5 \text{ terms}}, S, f(\overset{n}{A})) \\ &= f_4(\overset{n}{A}, \underbrace{S, A, S, \ldots, A, S}_{2n-3 \text{ terms}}, \overset{n}{A}). \end{aligned}$$

Conversely, assume that $A \subseteq f_4(\overset{n}{A}, \underbrace{S, A, S, \ldots, A, S}_{2n-3 \text{ terms}}, \overset{n}{A})$ for any $\emptyset \neq A \subseteq S$. Let $\emptyset \neq A \subseteq S$. It is not hard to see that $A \subseteq f_4(\overset{n}{A}, \underbrace{S, A, S, \ldots, A, S}_{2n-3 \text{ terms}}, \overset{n}{A}) \subseteq f_2(\overset{n-1}{S}, \overset{n}{A})$, $A \subseteq f_4(\overset{n}{A}, \underbrace{S, A, S, \ldots, A, S}_{2n-3 \text{ terms}}, \overset{n}{A}) \subseteq f_2(\overset{n}{A}, \overset{n-1}{S})$ and $A \subseteq f_4(\overset{n}{A}, \underbrace{S, A, S, \ldots, A, S}_{2n-3 \text{ terms}}, \overset{n}{A}) \subseteq f(A, \overset{n-2}{S}, A)$. So, S is completely regular.

For convenience, we write $f_4(\overset{n}{A}, (\overset{n-2}{S, A}), S, \overset{n}{A})$ instead of $f_4(\overset{n}{A}, \underbrace{S, A, S, \ldots, A, S}_{2n-3 \text{ terms}}, \overset{n}{A})$. For $n = 2$, we set $f_4(\overset{n}{A}, (\overset{n-2}{S, A}), S, \overset{n}{A}) = f_4(\overset{2}{A}, S, \overset{2}{A})$. □

Theorem 3.10. *Let S be an n-ary semigroup. Then S is completely regular if and only if every generalized bi-ideal of S is semiprime.*

Proof. Assume that S is completely regular. Let $\emptyset \neq A \subseteq S$ and B be a generalized bi-ideal of S such that $f(\overset{n}{A}) \subseteq B$. By Lemma 3.9,

$$\begin{aligned} A &\subseteq f_4(\overset{n}{A}, \underbrace{S, A, S, \ldots, A, S}_{2n-3 \text{ terms}}, \overset{n}{A}) \\ &= f_2(f(\overset{n}{A}), \underbrace{S, A, S, \ldots, A}_{2n-4 \text{ terms}}, S, f(\overset{n}{A})) \end{aligned}$$

$$
\begin{aligned}
&\subseteq f_2(f(\overset{n}{A}), \underbrace{S, f_2(\overset{n-1}{S}, \overset{n}{A}), S, \ldots, f_2(\overset{n-1}{S}, \overset{n}{A})}_{2n-4 \text{ terms}}, S, f(\overset{n}{A}))\\
&= f_2(f(\overset{n}{A}), \underbrace{f(S, \overset{n-1}{S}), f(\overset{n}{A}), \ldots, f(S, \overset{n-1}{S}), f(\overset{n}{A})}_{2n-4 \text{ terms}}, S, f(\overset{n}{A})\\
&\subseteq f_2(\underbrace{B, S, B, \ldots, S, B}_{2n-1 \text{ terms}})\\
&\subseteq B.
\end{aligned}
$$

So, B is semiprime. Conversely, assume that every generalized bi-ideal of S is semiprime. Let $\emptyset \neq A \subseteq S$ and $B = f_4(\overset{n}{A}, (\overset{n-2}{S}, A), S, \overset{n}{A})$. We will show that B is a generalized bi-ideal of S. We consider

$$
\begin{aligned}
f_2(\underbrace{B, S, B, \ldots, S, B}_{2n-1 \text{ terms}}) &= f_2(f_4(\overset{n}{A}, (\overset{n-2}{S}, A), S, \overset{n}{A}), \underbrace{S, B, \ldots, S, B, S}_{2n-3 \text{ terms}}, f_4(\overset{n}{A}, (\overset{n-2}{S}, A), S, \overset{n}{A}))\\
&= f_4(\overset{n}{A}, (\overset{n-2}{S}, A), f_6(S, \overset{n}{A}, \underbrace{S, B, \ldots, S, B, S}_{2n-3 \text{ terms}}, \overset{n}{A}, (\overset{n-2}{S}, A), S), \overset{n}{A})\\
&\subseteq f_4(\overset{n}{A}, (\overset{n-2}{S}, A), S, \overset{n}{A})\\
&= B.
\end{aligned}
$$

So, B is a generalized bi-ideal of S. Next, we show that S is completely regular.

- If $n = 2$, then it easy to see that $f_7(\overset{8}{A}) \subseteq f_4(\overset{2}{A}, S, \overset{2}{A})$. By assumption, $f_3(\overset{4}{A}) \subseteq f_4(\overset{2}{A}, S, \overset{2}{A}), f(\overset{2}{A}) \subseteq f_4(\overset{2}{A}, S, \overset{2}{A})$ and so $A \subseteq f_4(\overset{2}{A}, S, \overset{2}{A})$. So, S is completely regular.

- If $n = 3$, then it easy to see that $f_4(\overset{9}{A}) \subseteq f_4(\overset{3}{A}, S, A, S, \overset{3}{A})$. By assumption, $f(\overset{3}{A}) \subseteq f_4(\overset{3}{A}, S, A, S, \overset{3}{A})$ and so $A \subseteq f_4(\overset{3}{A}, S, A, S, \overset{3}{A})$. So, S is completely regular.

- If $n \geq 4$, then $f(\underbrace{f(\overset{n}{A}), \ldots, f(\overset{n}{A})}_{n \text{ terms}}) = f_5(\overset{n}{A}, \overset{n}{A}, \overset{n}{A}, \underbrace{f(\overset{n}{A}), \ldots, f(\overset{n}{A})}_{n-4 \text{ terms}}, \overset{n}{A})$ $= f_5(\overset{n}{A}, \overset{2n-4}{A}, \overset{4}{A}, \underbrace{f(\overset{n}{A}), \ldots, f(\overset{n}{A})}_{n-4 \text{ terms}}, \overset{n}{A}) = f_4(\overset{n}{A}, \overset{2n-4}{A}, f(\overset{4}{A}, \underbrace{f(\overset{n}{A}), \ldots, f(\overset{n}{A})}_{n-4 \text{ terms}}), \overset{n}{A})$ $\subseteq f_4(\overset{n}{A}, (\overset{n-2}{S}, A), S, \overset{n}{A})$. By assumption, $f(\overset{n}{A}) \subseteq f_4(\overset{n}{A}, (\overset{n-2}{S}, A), S, \overset{n}{A})$ and so $A \subseteq f_4(\overset{n}{A}, (\overset{n-2}{S}, A), S, \overset{n}{A})$. By Lemma 3.9., S is completely regular. □

Definition 3.11. *An n-ary semigroup S is called left (right) weakly regular if* $a \in f(\underbrace{f(\overset{n-1}{S}, a), \ldots, f(\overset{n-1}{S}, a)}_{n \text{ terms}})$ $(a \in f(\underbrace{f(a, \overset{n-1}{S}), \ldots, f(a, \overset{n-1}{S})}_{n \text{ terms}}))$ *for all* $a \in S$.

Remark 3.12. Let S be an n-ary semigroup.

(i) If S is left regular, then it is left weakly regular.

(ii) If S is right regular, then it is right weakly regular.

Theorem 3.13. *The following statements are equivalent.*

(i) *S is left weakly regular.*

(ii) $L = f(\overset{n}{L})$ *for every n-ideal L of S.*

Proof. Assume that S is left weakly regular. Let L be an n-ideal of S and $a \in L$. Then $f(\overset{n-1}{S}, a) \subseteq L$. Since S is left weakly regular, we have $a \in f(\underbrace{f(\overset{n-1}{S}, a), \ldots, f(\overset{n-1}{S}, a)}_{n \text{ terms}}) \subseteq f(\overset{n}{L})$. So, $L \subseteq f(\overset{n}{L})$ It is clear that $f(\overset{n}{L}) \subseteq L$. Hence, $L = f(\overset{n}{L})$ for every n-ideal L of S. Conversely, assume that $L = f(\overset{n}{L})$ for every n-ideal L of S. Let $a \in S$. Then $a \in J_n(a) = f(\underbrace{J_n(a), \ldots, J_n(a)}_{n \text{ terms}})$, i.e., $a = f(b_1, \ldots, b_n)$ where $b_1, \ldots, b_n \in J_n(a)$. By Corollary 2.2, $J_n(a) = f(\overset{n-1}{S}, a) \cup \{a\}$. We have $b_n = a$ or $b_n \in f(\overset{n-1}{S}, a)$ for $i = 1, \ldots, n$. If $b_n = a$, then $a = f(b_1, \ldots, b_n) \in f(\overset{n-1}{S}, a)$. If $b_n \in f(\overset{n-1}{S}, a)$, then $a = f(b_1, \ldots, b_n) \in f(b_1, \ldots, b_{n-1}, f(\overset{n-1}{S}, a)) \subseteq f(\overset{n-1}{S}, a)$. This implies that $a \in f(\overset{n-1}{S}, a)$. We consider $J_n(a) = f(\overset{n-1}{S}, a) \cup \{a\} \subseteq f(\overset{n-1}{S}, a)$. Thus, $b_i \in J_n(a) = f(\overset{n-1}{S}, a)$ for all $i = 1, \ldots, n$. We consider $a = f(b_1, \ldots, b_n) \in f(\underbrace{f(\overset{n-1}{S}, a), \ldots, f(\overset{n-1}{S}, a)}_{n \text{ terms}})$. Therefore, S is left weakly regular. □

The next theorem can be similarly proved as Theorem 3.13.

Theorem 3.14. *The following statements are equivalent.*

(i) S is right weakly regular.

(ii) $R = f(\overset{n}{R})$ for every 1-ideal R of S.

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