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Valuations on Ternary Semirings

Sumana Pal*
Department of Mathematics and Statistics, Aliah University, Kolkata 700160, West Bengal, India
e-mail : sumana.pal@gmail.com

Jayasri Sircar
Department of Mathematics, Lady Brabourne College, Kolkata 700017, West Bengal, India
e-mail : jayasrisircar@gmail.com

Pinki Mondal
Sunity Academy, Cooch Behar, West Bengal 736170, India
e-mail : pinkimondal1992@gmail.com

Abstract. In the present study, we introduce a valuation of ternary semiring on an ordered abelian group. Motivated by the construction of valuation rings, we study some properties of ideals in ternary semiring arising in connection with the valuation map. We also explore ternary valuation semirings for a noncommuative ternary division semiring. We further consider the notion of convexity in a ternary semiring and how it is reflected in the valuation map.

1. Introduction

Introduced by W. Krull [17] on fields in 1932, valuations on various algebraic systems have since been studied by many mathematicians. A valuation is a mapping into a field of real numbers, a linearly ordered group, or a linearly ordered semigroup. There are many notions of valuations on a commutative field or ring. Though valuation theory was initially connected precisely with commutative fields, later it was studied in noncommutative cases also.

Krull introduced valuations for rings in [18], and this has proved to be a very useful tool in ring theory, presenting a nice connection between rings and ordered

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abelian groups. The notion of valuation on a ring was covered by Bourbaki [1] in 1964. Manis [19] defined a valuation on a commutative ring R with the help of an ordered multiplicative group Γ with a zero adjoined. Valente [23] examined the interplay between the orderings of Γ and valuations of R. Valuation rings, defined from valuations, were first studied for fields. While studying them for noncommutative rings, Schilling [21] considered valuations of a division ring to a simply ordered l-group. The author studied the different ideals associated with this valuation, the valuation ring, and the valuation on a quotient ring associated with the valuation. Schilling also initiated the study of invariant valuation rings of division rings. Gubareni [11] continued with the study of valuations of division rings. Nasehpour [20] defined valuation maps from a semiring to a totally ordered commutative monoid.

Definition 1.1.([11]) A totally ordered (or linearly ordered) group is an additive group G along with a binary order relation $\geq$ which satisfies the following axioms:

(i) if $\alpha \geq \beta$ and $\beta \geq \alpha$ then $\alpha = \beta$;

(ii) if $\alpha \geq \beta$ and $\beta \geq \gamma$ then $\alpha \geq \gamma$;

(iii) if $\alpha \geq \beta$ then $\gamma + \alpha \geq \gamma + \beta$ and $\alpha + \gamma \geq \beta + \gamma$; and

(iv) either $\alpha \geq \beta$ or $\beta \geq \alpha$,

for any $\alpha, \beta, \gamma \in G$.

Let us consider a totally ordered (additive) group G. We now add an element ∞ which is larger than any other element of G which satisfies the condition

$$\infty + g = g + \infty = \infty, \forall g \in G \ \text{ and } \ \infty + \infty = \infty;$$

and denote $G \cup \{\infty\}$ by G^*.

Definition 1.2.([24]) Let K' be the multiplicative group of a field K. Let Γ be an additive abelian group which is totally ordered. A valuation of K is a mapping of $v : K' \to \Gamma$ which satisfies the following conditions:

(i) $v(xy) = v(x) + v(y)$,

(ii) $v(x + y) \geq \min\{v(x), v(y)\}$.

Given such a valuation v, the element $v(x) \in \Gamma$, for x in K', is called the value of x. The set of values of elements of K' forms a subgroup of Γ which is said to be the value group of v. It is assumed that the mapping v is onto Γ. Condition (i) signifies that v is a group homomorphism of K' (a multiplicative group) onto Γ (an additive group). Thus $v(1) = 0$. Further $1 = (-1)(-1)$ yields $v(-1) = 0$. We now obtain $v(-x) = v(x)$. The consequences that follow are [24]:

(i) $v(x - y) \geq \min\{v(x), v(y)\}$,

(ii) $v(y/x) = v(y) - v(x), x \neq 0$,

(iii) $v(1/x) = -v(x), x \neq 0$,

(iv) $v(x) < v(y) \Rightarrow v(x+y) = v(x)$.

The set $\{x \in K : v(x) \geq 0\}$ forms a ring, denoted by $\mathcal{R}_v$ and is said to be the valuation ring of v. For every x from K, either x or $1/x$ is an element of the valuation ring. This happens because either $v(x) \geq 0$ or $v(x) \leq 0$ and so $v(x) \geq 0$ or $v(1/x) \geq 0$. Both $v(x)$ and $-v(x)$ being nonnegative implies they are 0, so the kernel of the homomorphism v gives us the multiplicative group of units in $\mathcal{R}_v$. The set $\mathcal{P}_v = \{y \in K : v(y) > 0\}$ is the set of nonunits in $\mathcal{R}_v$. We find that $\mathcal{P}_v$ is not only a prime ideal but also a maximal ideal of $\mathcal{R}_v$. The divisibility relation $y|x$ in $\mathcal{R}_v$ is the same as the relation $v(x) \geq v(y)$.

The notion of a semiring was introduced by Vandiver in 1934 and since then the theory concerning semirings has evolved in various directions. It is well known that semirings have considerable applications not only in mathematics but also in computer science and operation research ([10], [12]). Nasehpour [20] explored valuation semirings in the realm of commutative semirings. Chang and Kim [2] worked on k-valuation semirings.

Ternary semirings were introduced by Dutta and Kar in [6] and they continued this study in [7], [9], [14], and [15]. Many other authors, as well, have worked on ternary semirings, as one may find in references like [4], [8], [5], [3] and [13]. The concept of a ternary semiring has been applied to soft sets to introduce the notion of soft ternary semiring [16].

Definition 1.3.([6]) A nonempty set S equipped with a binary operation called addition and a ternary multiplication (denoted by juxtaposition) is said to be a ternary semiring if S is an additive commutative semigroup which satisfies for all $a, b, c, d, e \in S$

(i) $(abc)de = a(bcd)e = ab(cde)$ (associativity),

(ii) $ab(c+d) = abc + abd$ (left distributive law),

(iii) $(a+b)cd = acd + bcd$ (right distributive law), and

(iv) $a(b+c)d = abd + acd$ (lateral distributive law).

By a zero of the ternary semiring S, we mean an element 0 in S, provided such an element exists, which satisfies $0 + x = x$ and $0xy = 0 = x0y = xy0$ for all x, y in S. If one exists, then S is called a ternary semiring with zero.

Example 1.4. The set of all non-positive integers, $\mathbb{Z}_0^-$ with respect to usual (binary) addition and ternary multiplication is a ternary semiring.

Example 1.5. Let us consider a topological space X. Then the set S consisting of continuous functions $f : X \to \mathbb{R}^-$, where $\mathbb{R}^-$ stands for the set of negative real numbers, is a ternary semiring in which binary addition and ternary multiplication are defined by

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (fgh)(x) &= f(x)g(x)h(x),\end{aligned}$$

for all $f, g, h \in S$ and $x \in X$.

Two more examples of ternary semiring are $S_1 = \{r\sqrt{2} : r \in \mathbb{Q}\}$ and $S_2 = \{ri : r \in \mathbb{R}\}$ both with respect to usual (binary) addition and ternary multiplication.

A ternary semiring S is called additively cancellative if $a + b = a + c$ implies that $b = c$, for any a, b, c in S. By a ternary subsemiring of S we mean an additive subsemigroup T of S in which $abc \in T$ for any $a, b, c \in T$. We call an additive subsemigroup I of S a left ideal of S if $SSI \subseteq I$, a lateral ideal if $SIS \subseteq I$ and a right ideal if $ISS \subseteq I$. If I is both a left ideal as well as a right ideal of S then it is called a (two sided) ideal of S. If I is further a lateral ideal then it is called an ideal of S.

An ideal I of S is a k-ideal if $x + y \in I$ where $x \in S$ and $y \in I$ yield $x \in I$. A prime ideal is a proper ideal P of S such that if $ABC \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$, where A, B, C are any three ideals of S. It can be proved that if S is commutative, an equivalent condition for a proper ideal P of S to be prime is that $abc \in P$ implies that either $a \in P$ or $b \in P$ or $c \in P$. An element $a \in S$ is said to be invertible in S if one can find an element $b \in S$ (called the ternary semiring inverse of a) satisfying $abx = xab = bax = xba = x, \forall x \in S$. If $|S| \geq 2$ then S is called a ternary division semiring if every non-zero element of S is invertible. If there is an element $e \in S$ such that $eex = exe = xee = x$ for all $x \in S$, then it is called a unital element of S. In this case $exy = xey = xye$ for all x, y. For example, $\mathbb{Z}_0^-$ is a ternary semiring with 0 as zero element and -1 as unital element. Again, let k be a fixed number in $\mathbb{R}$. If we define $a+b = 0$ and $abc = a+b+c+k$ for all $a, b, c \in \mathbb{R}$, then $-k/2$ is a unital element in the ternary semiring $\mathbb{R}$. A ternary semiring may contain more than one unital element. Consider the ternary semiring $\mathbb{Q}$ of rational numbers with respect to ordinary addition and ternary multiplication [ ] defined by $[abc] = abc$, for all $a, b, c \in \mathbb{Q}$. Here, 1 and -1 are both unital elements.

The aim of this paper is to introduce a valuation on a ternary semiring and study the corresponding ideals which arise due to this valuation. We further discuss the basic properties of valuation ring in connection with a ternary division ring.

2. Valuation on Ternary Semiring

We consider a commutative ternary semiring S with a unital element e and a zero element 0, $|S| \geq 2$. Let us introduce the notion of valuation on S as follows.

Definition 2.1. A valuation on S is a function $v : S \to G^*$ fulfilling the following conditions:

(i) $v(abc) = v(a) + v(b) + v(c)$,

(ii) $v(a + b) \geq \min\{v(a), v(b)\}$,

(iii) $v(e) = 0$, and

(iv) $v(0) = \infty$,

for all $a, b, c \in S$.

Suppose S is a ternary semiring with only two elements, a unital element e and a zero element 0. Let G be any totally ordered group. Then $v : S \to G^*$ defined by

$$v(e) = 0, v(0) = \infty$$

is always a valuation on S. We call this function a trivial valuation.

Example 2.2. We consider the ternary semiring $S = \mathbb{Z}_0^-$ of Example 1.4 and let G be the group $(\mathbb{R}, +)$ totally ordered by usual $\leq$, $G^* = G \cup \{\infty\}$. Let $v : S \to G^*$ be a function defined by

$$\begin{array}{rcl} v(x) & = & \log|x|, x \neq 0, \\ v(0) & = & \infty. \end{array}$$

Then v is a valuation on $\mathbb{Z}_0^-$.

Example 2.3. Let S be the ternary semiring of all polynomial functions on X with coefficients from $\mathbb{R}^-$ (we refer to Example 1.5) where $X = [0, \infty)$ and $G = (\mathbb{R}, +)$. We define a function $v : S \to G^*$ by

$$v(f) = -\deg(f) + \log|a_0|,$$

if $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$.
Then v is a valuation on S.

We note that in a ternary semiring S containing the additive inverses of its elements, the following equality holds for any $a, b, c \in S$:

$$(-a)bc = a(-b)c = ab(-c) = -(abc).$$

So we find that $-e$, if present in S, is a unital element of S if e is. In fact, for all $x \in S$, $eex = x$ yields $x = -(-x) = -(-(eex)) = -((-e)ex) = (-e)(-e)x$. We further obtain the following.

Lemma 2.4. *Let S be a ternary semiring, G be a totally ordered group, and $v : S \to G^*$ be a valuation on S. Then*

(i) *if e is a unital element of S, $v(-e) = 0$, provided $-e \in S$.*

(ii) *$v(-a) = v(a)$, for all $a \in S$ with $-a \in S$.*

Proof. (i) As $-e$ is a unital element of S, $v(-e) = 0$. (ii) Let $a \in S$ and e be a unital element of S. Then $a = (-e)(-e)a = (-e)e(-a)$ so that $v(a) = v(-e)+v(e)+v(-a)$. Hence $v(a) = v(-a)$. □

Let $val(S)$ denote the set of valuations of S. For $v \in val(S)$ we define three subsets of S:

$$\begin{aligned} \mathcal{A}_v &= \{a \in S : v(a) \geq 0\}, \\ \mathcal{P}_v &= \{b \in S : v(b) > 0\}, \\ \mathcal{I}_v &= \{c \in S : v(c) = \infty\}. \end{aligned}$$

The sets $\mathcal{A}_v$ and $\mathcal{I}_v$ are nonempty as $v(e) = 0$ and $v(0) = \infty$.

Theorem 2.5. *$\mathcal{A}_v$ is a ternary subsemiring of S.*

Proof. It is clear that $\mathcal{A}_v$ is an additively commutative subsemigroup of S. Let $t_1, t_2, t_3 \in \mathcal{A}_v$, then $v(t_1) \geq 0, v(t_2) \geq 0, v(t_3) \geq 0$. Now

$$v(t_1t_2t_3) = v(t_1) + v(t_2) + v(t_3) \geq 0.$$

This implies that $t_1t_2t_3 \in \mathcal{A}_v$. Therefore $\mathcal{A}_v$ becomes a ternary subsemiring of S.□

Theorem 2.6. *Suppose $\mathcal{P}_v$ is nonempty. Then $\mathcal{P}_v$ is an ideal of $\mathcal{A}_v$. Further, $\mathcal{P}_v$ is a k-ideal of $\mathcal{A}_v$ if $v(x) < v(y)$ implies $v(x + y) = v(x)$.*

Proof. Obviously $\mathcal{P}_v$ is an additive subsemigroup of S. Let $x \in \mathcal{A}_v\mathcal{A}_v\mathcal{P}_v$. Then $x = a_1a_2a_3$ for some $a_1, a_2 \in \mathcal{A}_v$ and $a_3 \in \mathcal{P}_v$. So

$$v(x) = v(a_1a_2a_3) = v(a_1) + v(a_2) + v(a_3) > 0$$

and $x \in \mathcal{P}_v$. Therefore $\mathcal{A}_v\mathcal{A}_v\mathcal{P}_v \subseteq \mathcal{P}_v$. Similarly we can show that $\mathcal{A}_v\mathcal{P}_v\mathcal{A}_v \subseteq \mathcal{P}_v$ and $\mathcal{P}_v\mathcal{A}_v\mathcal{A}_v \subseteq \mathcal{P}_v$. Hence $\mathcal{P}_v$ is an ideal of $\mathcal{A}_v$.

Again, let $x \in \mathcal{A}_v$ and $y, x+y \in \mathcal{P}_v$. Then $v(x) \geq 0, v(y) > 0$, and $v(x+y) > 0$, so we have $v(x + y) \geq \min\{v(x), v(y)\}$. If $v(x)$ equals $v(y)$ then obviously $x \in \mathcal{P}_v$. Let us suppose that $v(x) \neq v(y)$. If $\min\{v(x), v(y)\} = v(x)$, then $v(x) > 0$ otherwise if $v(x) = 0$, then $v(x) < v(y)$ implies that $v(x + y) = v(x) = 0$, a contradiction. So $x \in \mathcal{P}_v$. If $\min\{v(x), v(y)\} = v(y)$, then $v(x) \geq v(y) > 0$ and we obtain $x \in \mathcal{P}_v$. Therefore $\mathcal{P}_v$ is a k-ideal of $\mathcal{A}_v$. □

Theorem 2.7. *$\mathcal{I}_v$ is a prime ideal of S.*

Proof. Undoubtedly $\mathcal{I}_v$ is an additive commutative subsemigroup of S. Let $x = a_1a_2a_3 \in \mathcal{I}_vSS$ for some $a_1 \in \mathcal{I}_v$ and $a_1, a_2 \in S$ then $v(a_1) = \infty$ and

$$v(x) = \infty + v(a_2) + v(a_3) = \infty.$$

So $x \in \mathcal{I}_v$. Thus $\mathcal{I}_vSS \subseteq \mathcal{I}_v$. We can show in a similar way that $S\mathcal{I}_vS \subseteq \mathcal{I}_v$ and $SS\mathcal{I}_v \subseteq \mathcal{I}_v$. Hence $\mathcal{I}_v$ is an ideal of S.

Let a, b, c be elements of S such that $abc \in \mathcal{I}_v$. Then

$$v(abc) = v(a) + v(b) + v(c) = \infty.$$

This gives us at least one of $v(a), v(b), v(c)$ to be ∞, i.e., at least one of a, b, c is in $\mathcal{I}_v$. Therefore $\mathcal{I}_v$ is a prime ideal of S. □

3. Valuation on Noncommutative Ternary Semiring

We now consider a ternary division semiring D containing a unital element e, with a valuation v as defined in Definition 2.1.

Example 3.1. Let

$$D = \left\{ \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

be the set of real quaternions. Then D is a ternary division semiring under binary addition and ternary multiplication defined as follows:

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

and

$$(a_{ij})(b_{ij})(c_{ij}) = (d_{ij}),$$

where $d_{ij} = \sum a_{ip} b_{pq} c_{qj}$.

We define a mapping $v : D \to \mathbb{R}$ by

$$v(A) = \begin{cases} \log |detA|, & \text{if } \ A \neq 0; \\ \infty, & \text{otherwise.} \end{cases}$$

Then v becomes a valuation on D.

Let us denote by $U(A)$, the set $\{u \in A : v(u) = 0\}$ for any ternary subsemiring A of D. For simplicity, we write U for $U(D)$. Then $0 \notin U$ and U is nonempty as $v(e) = 0$ for a unital element e. Further if $u_1, u_2, u_3 \in U$ then

$$v(u_1 u_2 u_3) = v(u_1) + v(u_2) + v(u_3) = 0$$

and so $u_1 u_2 u_3 \in U$. Thus U is a ternary subsemiring of D.

Now we let $D^* = D \setminus \{0\}$. Then D^* is a ternary multiplicative semigroup [22], i.e., the ternary multiplication satisfies associative property.

Theorem 3.2. *Let D be a ternary division semiring. Then U is a ternary subsemigroup of D^* containing the unital element(s). Further, any ternary inverse of an element of U also belongs to U.*

Proof. The first part is obvious. Let $a \in U$ then $v(a) = 0$ and there exists an element $b \in D$ such that $abx = x = xab = bax = xba$ for all $x \in D$. Now

$$v(x) = v(abx) = v(a) + v(b) + v(x)$$

implies $v(a) + v(b) = 0$, or,$v(a) = -v(b)$, or, $v(b) = 0$. So $b \in U$ and our theorem is proved. □

We call the ternary subsemigroup U the ternary semigroup of valuation units of D. We observe that there may be a unit a in D such that $v(a) \neq 0$.

We now consider the subsets $\mathcal{A}_v, \mathcal{P}_v$ and $\mathcal{I}_v$ for D. Then we note that $U(\mathcal{A}_v)$ is the ternary semigroup of valuation units of D. We obtain the following results in D.

Theorem 3.3. *$\mathcal{P}_v$ is a maximal ideal of $\mathcal{A}_v$.*

Proof. Let us suppose that I is an ideal of $\mathcal{A}_v$ properly containing $\mathcal{P}_v$. Let $a \in \mathcal{A}_v$. Now $\mathcal{P}_v = \mathcal{A}_v \setminus U$. So there exists an element $b \in U \cap I$. Then $b \in U(\mathcal{A}_v)$ and so there is an element $c \in \mathcal{A}_v$ such that $bcx = xbc = cbx = xcb = x$ for all $x \in \mathcal{A}_v$. In particular, $bca = a$. Since $I\mathcal{A}_v\mathcal{A}_v \subseteq I$, $a \in I$. Thus $\mathcal{A}_v = I$ and our contention is justified. □

The notion of valuation ring is now generalized for a ternary division ring.

Definition 3.4. A ternary subsemiring A of a ternary division semiring D is called a ternary valuation semiring of D if for any element $a(\neq 0) \in A$, either $a \in A$ or $b \in A$ for any ternary inverse b of a in D.

Theorem 3.5. *Let v be a valuation of a ternary division semiring D into a totally ordered abelian group G. Further let $a \in D^*$ and b be a ternary inverse of a in D^*. Then*

(i) *$a\mathcal{A}_v b \subseteq \mathcal{A}_v$, and*

(ii) *either $a \in \mathcal{A}_v$ or $b \in \mathcal{A}_v$, and thus $\mathcal{A}_v$ is a ternary valuation semiring of D.*

Proof. (i) For any $x \in D$, $abx = x$ yields $v(a) + v(b) = 0$. Now let $x = ayb \in a\mathcal{A}_v b$ where $y \in \mathcal{A}_v$. Then we find that

$$v(x) = v(a) + v(y) + v(b) = v(y) \geq 0$$

and so $x \in \mathcal{A}_v$.
(ii) Suppose $a \in D^*$ such that $a \notin \mathcal{A}_v$, then $v(a) < 0$. Also $v(a) + v(b) = 0$ gives us $v(b) = -v(a) > 0$. Thus $b \in \mathcal{A}_v$. □

Theorem 3.6. *In a ternary division semiring D, let $a, b \in \mathcal{A}_v$ and we consider the following conditions:*

(i) *$a = bc_1c_2$ where $c_1, c_2 \in \mathcal{A}_v$,*

(ii) $a = c_3c_4b$ *where* $c_3, c_4 \in \mathcal{A}_v$.

If condition (i) or condition (ii) holds, then $v(a) \geq v(b)$. *Further, if we assume that* $v(a) \geq v(b)$ *implies both conditions (i) and (ii), then any left ideal of* $\mathcal{A}_v$ *is also a right ideal of* $\mathcal{A}_v$ *and vice versa.*

Proof. The proof of the first part is immediate. Now let $\mathcal{I}$ be a left ideal of $\mathcal{A}_v$, i.e., $\mathcal{A}_v\mathcal{A}_vI \subseteq I$. Let $x = x_1x_2x_3 \in I\mathcal{A}_v\mathcal{A}_v$ for some $x_1 \in I$ and $x_2, x_3 \in \mathcal{A}_v$. Then

$$v(x) = v(x_1) + v(x_2) + v(x_3) \geq v(x_1).$$

By our assumption $x = c_3c_4x_1$ where $c_3, c_4 \in \mathcal{A}_v$. Then $x \in \mathcal{A}_v\mathcal{A}_vI \subseteq I$. Thus $I\mathcal{A}_v\mathcal{A}_v \subset I$ and so I is a right ideal of $\mathcal{A}_v$. □

4. Convexity in Ternary Semiring

We will now consider a ternary semiring S valued epimorphically upon a totally ordered group G by the valuation function v. We find that a subgroup H of a totally ordered group G is called convex if $\alpha \in H, \beta \leq \alpha$ yields $\beta \in H$. We define convexity in a ternary semiring by employing v.

Definition 4.1. A subset T of a ternary semiring S is called convex if

(i) $a \in T, v(b) \leq v(a)$ implies $b \in T$, and

(ii) $a, b \in T, v(c) = v(a) + v(b)$ implies $c \in T$.

Theorem 4.2. *Let the ternary semiring* S *be valued epimorphically by* v *onto the totally ordered group* G. *If* T *is a convex subsemiring containing* 0 *of* S *then* $v(T)$ *is a convex subsemigroup containing* ∞ *of* $G^\star$.

Proof. Let $\alpha \in v(T)$ and $\beta \leq \alpha$ in G. Then there exists $a \in T, b \in S$ such that $v(a) = \alpha$ and $v(b) = \beta$. As a consequence of T being a convex subset of S, $v(b) \leq v(a)$ gives us $b \in T$ and further $v(b) = \beta \in v(T)$. Next let $\alpha = v(a), \beta = v(b)$ be elements in $v(T)$ where $a, b \in T$. Let $\gamma = \alpha + \beta$. As v is an epimorphism we obtain $c \in S$ such that $v(c) = \gamma = v(a) + v(b)$. Property (ii) of convexity of T yields $c \in T$. So $v(c) = \gamma$ is in $v(T)$. Since $0 \in T, v(0) = \infty \in v(T)$. Also if $\alpha = v(a), \beta = v(b)$ and $\gamma = v(c)$ in $v(T)$ where a, b, c are in T, then $\alpha + (\beta + \gamma) = v(a) + (v(b) + v(c)) = v(abc) = (v(a) + v(b)) + v(c) = (\alpha + \beta) + \gamma$. Hence $v(T)$ is an additive convex semigroup of $G^\star$. □

Theorem 4.3. *Let* v *be defined as in Theorem 4.2. If* H *is a convex subsemigroup containing* ∞ *of* G^* *then* $T = v^{-1}(H)$ *is a ternary convex subsemiring of* S.

Proof. $T \neq \phi$ is obvious. Let $a, b, c \in T$ then $v(a), v(b), v(c) \in H$. We find that $v(a + b) \in H$ for $v(a + b) = \min\{v(a), v(b)\}$, and so $a + b \in T$. Again $v(abc) = v(a) + v(b) + v(c) \in H$ and we obtain $abc \in T$. Thus T is a ternary subsemiring of S. Further let $a \in T, b \in S$ and $v(b) \leq v(a)$. Then $v(a) \in H$. As H is convex, we obtain $v(b) \in H$ which yields $b \in T$. Lastly for $a, b \in T$, we note that $v(a) + v(b) \in H$. Let $v(a) + v(b) = v(c)$ for some $c \in S$. So $v(c) \in H$ and $c \in T$. Therefore T is a convex ternary subsemiring of S. □

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