KYUNGPOOK Math. J. 62(2022), 133-166
https://doi.org/10.5666/KMJ.2022.62.1.133
pISSN 1225-6951 eISSN 0454-8124

Submanifolds of Codimension 3 in a Complex Space Form with Commuting Structure Jacobi Operator

U-Hang Ki
The National Academy of Sciences, Seoul 06579, Korea
e-mail : uhangki2005@naver.com

Hyunjung Song*
Department of Mathematics Hankuk University of Foreign Studies, Seoul 02450, Korea
e-mail : hsong@hufs.ac.kr

Abstract. Let M be a semi-invariant submanifold with almost contact metric structure (ϕ, ξ, η, g) of codimension 3 in a complex space form $M_{n+1}(c)$ for $c \neq 0$. We denote by S and R_ξ be the Ricci tensor of M and the structure Jacobi operator in the direction of the structure vector ξ, respectively. Suppose that the third fundamental form t satisfies $dt(X, Y) = 2\theta g(\phi X, Y)$ for a certain scalar $\theta \neq 2c$ and any vector fields X and Y on M. In this paper, we prove that if it satisfies $R_\xi \phi = \phi R_\xi$ and at the same time $S\xi = g(S\xi, \xi)\xi$, then M is a real hypersurface in $M_n(c)$ $(\subset M_{n+1}(c))$ provided that $\bar{r} - 2(n-1)c \leq 0$, where $\bar{r}$ denotes the scalar curvature of M.

1. Introduction

A submanifold M is called a *CR submanifold* of a Kaehlerian manifold $\tilde{M}$ with complex structure J if there exists a differentiable distribution $\triangle : p \to \triangle_p \subset M_p$ on M such that $\triangle$ is J-invariant and the complementary orthogonal distribution $\triangle^\perp$ is totally real, where M_p denotes the tangent space at each point p in M ([1], [25]). In particular, M is said to be a *semi-invariant submanifold* provided that $\dim\triangle^\perp = 1$. The unit normal in $J\triangle^\perp$ is called the *distinguished normal* to the semi-invariant submanifold ([4], [23]). In this case, M admits an induced almost contact metric structure (ϕ, ξ, η, g). A typical example of a semi-invariant submanifold is real hypersurfaces. New examples of nontrivial semi-invariant submanifolds in a complex

* Corresponding author.
Received November 12, 2019; revised February 22, 2021; accepted February 24, 2021.
2010 Mathematics Subject Classification: 53B25, 53C40, 53C42.
Key words and phrases: semi-invariant submanifold, distinguished normal, complex space form, structure Jacobi operator, Ricci tensor, Hopf hypersurfaces.

projective space $P_n\mathbb{C}$ are constructed in [13] and [20]. Therefore we may expect to generalize some results which are valid in a real hypersurface to a semi-invariant submanifold.

An n-dimensional complex space form $M_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature $4c$. As is well known, complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbb{C}$, or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0$ or $c < 0$.

For the real hypersurface of a complex space form $M_n(c)$, many results are known. One of them, Takagi([21], [22]) classified all the homogeneous real hypersurfaces of $P_n\mathbb{C}$ as six model spaces which are said to be A_1, A_2, B, C, D and E, and Cecil-Ryan ([5]) and Kimura ([14]) proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds when the structure vector field ξ is principal.

On the other hand, real hypersurfaces in $H_n\mathbb{C}$ have been investigated by Berndt ([2]), Montiel and Romero ([15]) and so on. Berndt ([2]) classified all real hypersurfaces with constant principal curvatures in $H_n\mathbb{C}$ and showed that they are realized as the tubes of constant radius over certain submanifolds. Also such kinds of tubes are said to be real hypersurfaces of type A_0, A_1, A_2 or type B.

Let M be a real hypersurface of type A_1 or type A_2 in a complex projective space $P_n\mathbb{C}$ or that of type A_0, A_1 or A_2 in a complex hyperbolic space $H_n\mathbb{C}$. Now, hereafter unless otherwise stated, such hypersurfaces are said to be of *type* (A) for our convenience sake.

Characterization problems for a real hypersurface of type (A) in a complex space form were studied by many authors ([6], [7], [8], [15], [16], [18], etc.).

Two of them, we introduce the following characterization theorems due to Okumura [18] for $c > 0$ and Montiel and Romero [15] for $c < 0$ respectively.

Theorem O. *Let M be a real hypersurface of $P_n\mathbb{C}$, $n \geq 2$. If it satisfies*

$$g((A\phi - \phi A)X, Y) = 0 \tag{1.1}$$

for any vector fields X and Y, then M is locally congruent to a tube of radius r over one of the following Kaehlerian submanifolds :

(A_1) *a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \pi/2$,*

(A_2) *a totally geodesic $P_k\mathbb{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$.*

Theorem MR. *Let M be a real hypersurface of $H_n\mathbb{C}$, $n \geq 2$. If it satisfies* (1.1), *then M is locally congruent to one of the following hypersurface :*

(A_0) *a horosphere in $H_n\mathbb{C}$, i.e., a Montiel tube,*

(A_1) *a geodesic hypersphere, or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*

(A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ $(c \leq k \leq n-2)$.*

Denoting by R the curvature tensor of the submanifold, we define the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ with respect to the structure vector ξ. Then R_ξ is a self adjoint endomorphism on the tangent space of a CR submanifold.

Using several conditions on the structure Jacobi operator R_ξ, characterization problems for real hypersurfaces of type (A) have recently studied. In the previous paper ([7]), Cho and one of the present authors gave another characterization of real hypersurface of type (A) in a complex projective space $P_n\mathbb{C}$. Namely they prove the following :

Theorem CK.([7]) *Let M be a connected real hypersurface of $P_n\mathbb{C}$ if it satisfies (1) $R_\xi A\phi = \phi A R_\xi$ or (2) $R_\xi\phi = \phi R_\xi, R_\xi A = AR_\xi$, then M is of type (A), where A denotes the shape operator of M.*

On the other hand, semi-invariant submanifolds of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$ have been studied in [10], [12], [13] and so on by using properties of induced almost contact metric structure and those of the third fundamental form of the submanifold. In the preceding work, Ki, Song and Takagi ([13]) assert the following:

Theorem KST.([13]) *Let M be a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$ with constant holomorphic sectional curvature $4c$. If the structure vector ξ is an eigenvector for the shape operator in the direction of the distinguished normal and the third fundamental form t satisfies $dt = 2\theta\omega$ for a certain scalar $\theta(< 2c)$, where $\omega(X, Y) = g(\phi X, Y)$ for any vectors X and Y on M, then M is a Hopf hypersurface in a complex projective space $P_n\mathbb{C}$.*

In this paper, we consider a semi-invariant submanifold M of codimension 3 in a complex space form $M_{n+1}(c), c \neq 0$ which satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $S\xi = g(S\xi, \xi)\xi$ such that the third fundamental form t satisfies $dt = 2\theta\omega$ for a certain scalar $\theta(\neq 2c)$ and the scalar curvature $\bar{r}$ of M satisfies $\bar{r} - 2c(n-1) \leq 0$, where S denotes the Ricci tensor of M. In the present paper, we prove that M is a real hypersurface of type (A) in $M_n(c)$ mentioned Theorem O and Theorem MR. Our main theorem stated in section 6.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the semi-invariant submanifolds are supposed to be orientable.

2. Preliminaries

Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold with parallel almost complex structure J and a Riemannian metric tensor G. Let M be a real $(2n-1)$-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$. We denote by g the Riemannian metric tensor on M from that of $\tilde{M}$.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation with respect to the metric tensor G on $\tilde{M}$ and by ∇ the one on M. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{3} g(A^{(i)}X, Y)\mathcal{C}^{(i)}, \tag{2.1}$$
$$\tilde{\nabla}_X \mathcal{C}^{(i)} = -A^{(i)}X + \sum_{j=1}^{3} l_j{}^{(i)}(X)\mathcal{C}^{(j)}$$

for any vector fields X and Y tangent to M and any vector field $\mathcal{C}^{(i)}$ normal to M, where $A^{(i)}$ are called the *second fundamental forms* with respect to the normal vector $\mathcal{C}^{(i)}$.

As is well-known, a submanifold of a Kaehlerian manifold is said to be a *CR submanifold* ([1], [25]) if it is endowed with a pair of mutually orthogonal and complementary differentiable distribution $(T, T^{\perp})$ such that for any point $p \in M$ we have $JT_p = T_p$, $JT_p^{\perp} \subset T_p^{\perp}M$, where $T_p^{\perp}M$ denotes the normal space of M at p. In particular, M is said to be *semi-invariant submanifold* provided that $dimT^{\perp} = 1$([4], [23]). In this case the unit vector field in $JT^{\perp}$ is called a *distinguished normal* to the semi-invariant submanifold and denote by C([4], [23]).

More precisely, we choose an orthonormal basis $e_1, \cdots, e_{2n-2}, \xi$ of M_p in such a way that $e_1, e_2, \cdots, e_{2n-2} \in T$, where M_p denotes the tangent space to M at each point p in M. Then we see that

$$G(J\xi, e_i) = -G(\xi, Je_i) = 0$$

for $i = 1, \cdots, 2n-2$.

From now on we consider M is a real $(2n-1)$-dimensional semi-invariant submanifold of a Kaehlerian manifold $\tilde{M}$ of real dimension $2(n+1)$. Then we can write ([4], [24])

$$JX = \phi X + \eta(X)C, \quad JC = -\xi, \quad JD = -E, \quad JE = D, \tag{2.2}$$

where we have put $g(\phi X, Y) = G(JX, Y), \eta(X) = G(JX, \mathcal{C})$ for any vector fields X and Y tangent to M, and put $\mathcal{C}^{(1)} = C$, $\mathcal{C}^{(2)} = D$ and $\mathcal{C}^{(3)} = E$.

By the Hermitian property of J, we see, using (2.2), that the aggregate (ϕ, ξ, η, g) is an *almost contact metric structure* on M, that is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(X) = g(\xi, X),$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vectors X and Y on M.

We can also write the second equation of (2.1) as

$$\tilde{\nabla}_X C = -AX + l(X)D + m(X)E, \tag{2.3}$$
$$\tilde{\nabla}_X D = -KX - l(X)C + t(X)E,$$
$$\tilde{\nabla}_X E = -LX - m(X)C - t(X)D$$

because C, D and E are mutually orthogonal, where we have put

$$A^{(1)} = A, \quad A^{(2)} = K, \quad A^{(3)} = L, \qquad l = l_2{}^{(1)} = -l_1{}^{(2)}, \quad m = l_3{}^{(1)} = -l_1{}^{(3)}, \quad t = l_3{}^{(2)} = -l_2{}^{(3)}, \tag{2.4}$$

In the sequel, we denote the normal components of $\tilde{\nabla}_X C$ by $\nabla^{\perp} C$. The distinguished normal C is said to be *parallel* in the normal bundle if we have $\nabla^{\perp} C = 0$, that is, l and m vanish identically.

From the Kaehler condition $\tilde{\nabla} J = 0$ and take account of the Gauss and Weingarten formulas,we obtain from (2.2)

$$\nabla_X \xi = \phi AX, \tag{2.5}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.6}$$

$$KX = \phi LX - m(X)\xi, \quad K\phi X = LX - \eta(X)L\xi, \tag{2.7}$$

$$LX = -\phi KX + l(X)\xi, \quad L\phi X = -KX + \eta(X)K\xi \tag{2.8}$$

for any vectors X and Y on M. The last two relationships give

$$l(X) = g(L\xi, X), \quad m(X) = -g(K\xi, X), \tag{2.9}$$

$$m(\xi) = -k, \quad l(\xi) = TrA^{(3)}, \tag{2.10}$$

where, we have put $k = TrA^{(2)}$.

We notice here that there is no loss of generality such that we may assume $T_r A^{(3)} = 0$. In fact, a normal vector v of M we denote by Av the second fundamental tensor of M in the direction of v. Then we have $A_{-v} = -Av$. Hence there is a unit normal vector D' of M in the plane spanned by two vectors D and E such that $TrA_{D'} = 0$, which proves our assertion. Therefore we have by (2.10)

$$l(\xi) = 0. \tag{2.11}$$

Applying (2.8) by ϕ and using (2.7), we find

$$-g(KX, Y) - m(X)\eta(Y) = g(\phi KX, \phi Y) - \eta(X)l(\phi Y).$$

If we take the skew-symmetric part of this with respect to X and Y, then we obtain

$$-m(X)\eta(Y) + m(Y)\eta(X) = \eta(X)l(\phi Y) - \eta(Y)l(\phi X),$$

which together with (2.10) gives

$$l(\phi X) = m(X) + k\eta(X). \tag{2.12}$$

Similarly we have

$$m(\phi X) = -l(X) \tag{2.13}$$

because of (2.10).

Transforming (2.7) by L and using (2.8) and (2.9), we obtain

$$g(KLX, Y) + g(LKX, Y) = -l(X)m(Y) - l(Y)m(X). \tag{2.14}$$

In the rest of this paper we shall suppose that $\tilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature $4c$, which is called a *complex space form* and denote by $M_{n+1}(c)$. Then equations of the Gauss and Codazzi are given by

$$\begin{aligned} R(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &- 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY \\ &+ g(KY,Z)KX - g(KX,Z)KY + g(LY,Z)LX - g(LX,Z)LY, \end{aligned} \tag{2.15}$$

$$\begin{aligned} &(\nabla_X A)Y - (\nabla_Y A)X - l(X)KY + l(Y)KX \\ &- m(X)LY + m(Y)LX = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\}, \end{aligned} \tag{2.16}$$

$$(\nabla_X K)Y - (\nabla_Y K)X + l(X)AY - l(Y)AX - t(X)LY + t(Y)LX = 0, \tag{2.17}$$

$$\begin{aligned} (\nabla_X L)Y - (\nabla_Y L)X + m(X)AY - m(Y)AX& \\ + t(X)KY - t(Y)KX = 0&, \end{aligned} \tag{2.18}$$

where R is the Riemann-Christoffel curvature tensor on M, and those of the Ricci by

$$\begin{aligned} (\nabla_X l)(Y) - (\nabla_Y l)(X) + g(KAX,Y) - g(AKX,Y)& \\ + m(X)t(Y) - m(Y)t(X) = 0&, \end{aligned} \tag{2.19}$$

$$\begin{aligned} (\nabla_X m)(Y) - (\nabla_Y m)(X) + g(LAX,Y) - g(ALX,Y)& \\ + t(X)l(Y) - t(Y)l(X) = 0&, \end{aligned} \tag{2.20}$$

$$\begin{aligned} (\nabla_X t)(Y) - (\nabla_Y t)(X) + g(LKX,Y) - g(KLX,Y)& \\ + l(X)m(Y) - l(Y)m(X) = 2cg(\phi X,Y)&. \end{aligned} \tag{2.21}$$

In what follows, to write our formulas in a convention form, we denote by $\alpha = \eta(A\xi), \beta = \eta(A^2\xi), TrA = h, TrA^{(2)} = k, Tr({}^tAA) = h_{(2)}$ and for a function f we denote by ∇f the gradient vector field of f.

Now, we put $\nabla_\xi \xi = U$ in the sequel. Then U is orthogonal to ξ because of (2.5). From now on we put

$$A\xi = \alpha\xi + \mu W, \tag{2.22}$$

where W is a unit vector field orthogonal to ξ. Then we have

$$U = \mu\phi W \tag{2.23}$$

because of (2.5). So, W is orthogonal to U. Further, we have

$$\mu^2 = \beta - \alpha^2. \tag{2.24}$$

From (2.22) and (2.23) we have

$$\phi U = -A\xi + \alpha\xi, \tag{2.25}$$

which together with (2.5) and (2.22) yields

$$g(\nabla_X \xi, U) = \mu g(AW, X), \quad \mu g(\nabla_X W, \xi) = g(AU, X) \tag{2.26}$$

because W is orthogonal to ξ.

Differentiating (2.25) covariantly along M and using (2.5) and (2.6), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX, \tag{2.27}$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha - 2kL\xi. \tag{2.28}$$

Because of (2.5), (2.26) and (2.27), we verify that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha - 2k(K\xi - k\xi). \tag{2.29}$$

In the next place, the Jacobi operators R_ξ is given by

$$R_\xi X = R(X,\xi)\xi = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX - m(X)K\xi - l(X)L\xi, \tag{2.30}$$

where we have used (2.9), (2.10) and (2.15).

Suppose that $R_\xi \phi = \phi R_\xi$ holds on M. Then from (2.30) we have

$$\alpha(\phi AX - A\phi X) = g(A\xi, X)U + g(U,X)A\xi + 2kLX - 2k\{l(X)\xi + \eta(X)L\xi\}, \tag{2.31}$$

where we have used (2.5), (2.8) and (2.12).

3. The Third Fundamental Forms of Semi-Invariant Submanifolds

In this section we shall suppose that M is a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ and that the third fundamental form t satisfies

$$dt = 2\theta\omega, \quad \omega(X,Y) = g(\phi X, Y) \tag{3.1}$$

for a certain scalar θ and any vector fields X and Y on M, where d denotes the exterior differential operator. Then (2.21) reformed as

$$g(LKX,Y) - g(KLX,Y) + l(X)m(Y) - l(Y)m(X) = -2(\theta - c)g(\phi X, Y),$$

or, using (2.14)

$$g(LKX,Y) + l(X)m(Y) = -(\theta - c)g(\phi X, Y), \tag{3.2}$$

which together with (2.9)∼(2.11) implies that

$$KL\xi = kL\xi, \quad LK\xi = 0. \tag{3.3}$$

Differentiating (3.1) covariantly along M and using (2.6) and the first Bianchi identity, we find

$$(X\theta)\omega(Y,Z) + (Y\theta)\omega(Z,X) + (Z\theta)\omega(X,Y) = 0,$$

which implies $(n-2)X\theta = 0$. Thus $\theta(\geq c)$ is constant if $n > 2$.

For the case where $\theta = c$ in (3.1) we have $dt = 2c\omega$. In this case, the normal connection of M is said to be *L-flat*([18]).

Lemma 3.1. *Let M be a semi-invariant submanifold with L-flat normal connection in $M_{n+1}(c)$, $c \neq 0$. If $A\xi = \alpha\xi$, then we have $\nabla^\perp C = 0$ and $A^{(2)} = A^{(3)} = 0$.*

Proof. From (3.2) we have

$$T_r({}^tA^{(2)}A^{(2)}) - \|K\xi\|^2 + \|L\xi\|^2 = 2(n-1)(\theta - c)$$

because of (2.7), (2.9) and (2.12), which implies

$$\|A^{(2)} - k\eta \otimes \xi\|^2 + \|L\xi\|^2 = 2(n-1)(\theta - c),$$

where $\|F\|^2 = g(F, F)$ for any vector field F on M. Thus, by our hypothesis $\theta = c$, we have $A^{(2)} = k\eta \otimes \xi$.

In the same way, we see from (2.8), (2.10), (2.13) and (3.2) that $A^{(3)} = 0$. And hence $m(X) = -k\eta(X)$ and $l = 0$ because of (2.9). Therefore, it suffices to show that $k = 0$. Using these facts, (2.19) reformed as

$$k\{\eta(X)A\xi - g(A\xi, X)\xi\} = k(\eta(X)t - t(X)\xi),$$

which together with $A\xi = \alpha\xi$ gives

$$k(t - t(\xi)\xi) = 0. \tag{3.4}$$

We also have from (2.18)

$$k\{\eta(X)(AY + t(Y)\xi) - \eta(Y)(AX + t(X)\xi)\} = 0,$$

which implies $k(h - \alpha) = 0$. Form this and (3.4) we verify that $k = 0$. This completes the proof. □

Applying (3.2) by ϕ and taking account of (2.7) and (2.13), we find

$$K^2X + \eta(X)K^2\xi + l(X)L\xi = (\theta - c)(X - \eta(X)\xi), \tag{3.5}$$

which implies $\eta(X)K^2\xi - g(K^2\xi, X)\xi = 0$. Thus, it follows that

$$K^2\xi = (\|K\xi\|^2)\xi \tag{3.6}$$

by virtue of (2.9). Thus, (3.5) becomes

$$K^2X + l(X)L\xi + \|K\xi\|^2\eta(X)\xi = (\theta - c)(X - \eta(X)\xi).$$

Putting $X = L\xi$ in (2.8) and taking account of (2.12) and (3.3), we obtain

$$L^2\xi = kK\xi + (\|K\xi\|^2 + k^2)\xi. \tag{3.7}$$

If we put $X = L\xi$ in (3.2) and make use of (2.13) and (3.2), we find

$$(\theta - c - \|K\xi\|^2)L\xi = 0.$$

Similarly, we verify, using (3.2) and (3.7), that

$$(\theta - c - \|L\xi\|^2 - k^2)(\|K\xi\|^2 - k^2) = 0.$$

Let $\|L\xi\| \neq 0$ at every point of M and suppose that this subset does not void. Then we have $\|K\xi\|^2 = \theta - c$ and $\|L\xi\|^2 + k^2 = \theta - c$ on the subset. Using these facts, we can verify that (for detail, see (2.22) and (2.24) of [13])

$$\nabla k = 2AL\xi, \tag{3.8}$$

$$\nabla_X L\xi = t(X)K\xi - AKX - kAX \tag{3.9}$$

on the set. Differentiating (3.8) covariantly and taking the skew-symmetric part obtained, we find

$$(\theta - 2c)(\eta(X)K\xi - m(X)\xi) = 0,$$

where we have used (2.12), (2.16), (3.3) and (3.9), which shows that $(\theta - 2c)(m(X) + k\eta(X)) = 0$ and hence $\theta = 2c$ on this subset. Thus, from the first equation of (2.3) we have

Lemma 3.2. *Let M be a semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$, $c \neq 0$ satisfying (3.1). If $\theta - 2c \neq 0$, then $\nabla^{\perp} C = -k\xi E$ on M.*

In the following we assume that M satisfies (3.1) with $\theta - 2c \neq 0$. Then we have

$$L\xi = 0, \quad K\xi = k\xi \tag{3.10}$$

because of (2.9). It is, using (3.10), clear that (2.7), (2.8) and (3.2) are reduced respectively to

$$\phi LX = KX - k\eta(X)\xi, \tag{3.11}$$

$$L = K\phi, \tag{3.12}$$

$$g(LKX, Y) + (\theta - c)g(\phi X, Y) = 0. \tag{3.13}$$

From the last two equations, we obtain

$$L^2 X = (\theta - c)(X - \eta(X)\xi). \tag{3.14}$$

Further, if we take account of (3.10), then the other structure equations (2.16)∼(2.21) reformed as

$$(3.15)\qquad \begin{aligned}&(\nabla_X A)Y-(\nabla_Y A)X\\&=k\{\eta(Y)LX-\eta(X)LY\}+c\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\},\end{aligned}$$

$$(3.16)\qquad (\nabla_X K)Y-(\nabla_Y K)X=t(X)LY-t(Y)LX,$$

$$(3.17)\qquad (\nabla_X L)Y-(\nabla_Y L)X=k\{\eta(X)AY-\eta(Y)AX\}-t(X)KY+t(Y)KX,$$

$$(3.18)\qquad KAX-AKX=k\{\eta(X)t-t(X)\xi\},$$

$$(3.19)\qquad LAX-ALX=(Xk)\xi-\eta(X)\nabla k+k(\phi AX+A\phi X),$$

where we have used (2.5).

Putting $X=\xi$ in (3.18) and using (3.10), we find

$$(3.20)\qquad KA\xi=kA\xi+k(t-t(\xi)\xi).$$

Replacing X by ξ in (3.19) and using (2.5), (3.10) and (3.12), we get

$$(3.21)\qquad KU=(\xi k)\xi-\nabla k+kU.$$

If we apply (3.20) by ϕ and make use of (2.22) (3.11) and (3.12), then we find

$$(3.22)\qquad KU=k(t\phi-U),$$

which together with (3.21) yields

$$(3.23)\qquad \nabla k=(\xi k)\xi+k(-t\phi+2U).$$

If we transform (3.19) by ϕ and take account of (2.22), (3.11) and the last equation, then we obtain

$$\phi ALX-KAX=-k\{t-t(\xi)\xi\}\eta(X)+2\mu\eta(X)W+2g(A\xi,X)\xi-AX-\phi A\phi X\},$$

which connected to (3.18) gives

$$(3.24)\qquad \phi AL=-LA\phi.$$

Since θ is constant if $n>2$, differentiating (3.14) covariantly, we find

$$(\nabla_X L^2)Y=(c-\theta)\{\eta(Y)\phi AX+g(\phi AX,Y)\xi\},$$

or, using (3.13) and (3.17), it is verified that (see, [13])

$$\begin{aligned}2(\nabla_X L)LY =&(\theta - c)\{2t(X)\phi Y - \eta(Y)(\phi A + A\phi)X + g((A\phi - \phi A)X, Y)\xi\\ &- \eta(X)(\phi A - A\phi)Y\} - k\{\eta(Y)(AL + LA)X\\ &- g((AL + LA)X, Y)\xi - \eta(X)(LA - AL)Y\},\end{aligned}$$

which together with (3.10) and (3.22) yields

$$\begin{aligned}(\theta - c)(A\phi - \phi A)X + (k^2 + \theta - c)(u(X)\xi + \eta(X)U)\\ + k\{(AL + LA)X + k\{-t(\phi X)\xi + \eta(X)\phi \circ t\} = 0,\end{aligned} \tag{3.25}$$

where $u(X) = g(U, X)$ for any vector X.

In the following we consider the case where (2.22) with $\mu = 0$, that is $A\xi = \alpha\xi$. Differentiating this covariantly and using (2.5), we find

$$(\nabla_X A)\xi = -A\phi AX + \alpha\phi AX + (X\alpha)\xi,$$

which together with (3.10) and (3.15) gives

$$-2A\phi AX + \alpha(\phi A + A\phi)X + 2c\phi X = \eta(X)\nabla\alpha - (X\alpha)\xi. \tag{3.26}$$

If we put $X = \xi$ in this and using (2.22) with $\mu = 0$, then we find

$$\nabla\alpha = (\xi\alpha)\xi. \tag{3.27}$$

Differentiating the second equation of (3.10) covariantly along M, and using (2.5), we find $\nabla_X m = -(Xk)\xi + k\phi AX$, from which taking the skew-symmetric part and making use of (2.20) with $l = 0$,

$$LAX - ALX - k(\phi AX + A\phi X) = (Xk)\xi - \eta(X)\nabla k.$$

Since $A\xi = \alpha\xi$ was assumed, we then have

$$\nabla k = (\xi k)\xi \tag{3.28}$$

because of (3.10). From the last two equations, it follows that

$$LA - AL = k(\phi A + A\phi). \tag{3.29}$$

If we put $X = \xi$ in (3.18) and remember (2.22) with $\mu = 0$ and (3.10), then we get

$$k(t(X) - t(\xi)\eta(X)) = 0. \tag{3.30}$$

Since we have $A\xi = \alpha\xi$, differentiating (3.28) covariantly, and taking the skew-symmetric part obtained, we get

$$(\xi k)(A\phi + \phi A) = 0. \tag{3.31}$$

From this and (3.27) we can write (3.26) as $\alpha(A^2\phi + c\phi) = 0$. By the properties of the almost contact metric structure, it follows that

$$\xi k\{h_{(2)} - \alpha^2 + 2c(n-1)\} = 0,$$

which implies $\xi k = 0$ if $c > 0$.

4. Commuting Structure Jacobi Operators

We will continue our arguments under the same hypotheses $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ as those stated in section 3. Further suppose, throughout this paper, that $R_\xi\phi = \phi R_\xi$, which means that the eigenspace of the structure Jacobi operator R_ξ is invariant by the structure operator ϕ. Then (2.31) reformed as

$$\alpha(\phi AX - A\phi X) = g(A\xi, X)U + g(U, X)A\xi + 2kLX \tag{4.1}$$

by virtue of (3.10).

Transforming this by A, and taking the trace obtained, we have $g(A^2\xi, U) = 0$ because of (3.25), which together with (2.22) yields

$$\mu g(AW, U) = 0. \tag{4.2}$$

Applying (4.1) by L and using (2.25), (3.11) and (3.19), we find

$$\begin{aligned}\alpha\{AKX - k\eta(X)A\xi - \phi ALX\} + g(LU, X)A\xi + g(KU, X)U \\ = -2kL^2X,\end{aligned} \tag{4.3}$$

which together with (3.18) and (3.22) yields

$$\begin{aligned}&k\alpha\{t(X)\xi - \eta(X)t + g(A\xi, X)\xi - \eta(X)A\xi\} \\ &\quad + g(LU, X)A\xi - g(A\xi, X)LU - u(X)KU + g(KU, X)U = 0,\end{aligned}$$

where $u(X) = g(U, X)$ for any vector X. If we take the inner product with ξ to this and use (3.10), then we get

$$k\alpha\{t(X) - t(\xi)\eta(X) + g(A\xi, X) - \alpha\eta(X)\} + \alpha g(LU, X) = 0. \tag{4.4}$$

Combining the last two equations and taking account of (2.24), we obtain

$$\mu(w(X)LU - g(LU, X)W) + u(X)KU - g(KU, X)U = 0, \tag{4.5}$$

where $w(X) = g(W, X)$ for any vector X.

In the previous paper [13] we prove the following proposition.

Proposition 4.1. *Let M be a real $(2n-1)$-dimensional$(n > 2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$. If it satisfies $dt = 2\theta\omega$ for a scalar $\theta \neq 2c$ and $\mu = g(A\xi, W) = 0$, then we have $k = 0$.*

Sketch of Proof. This fact was proved for $c > 0$ (see, Proposition 3.5 of [13]). But, regardless of the sign of c this one is established. However, only $\xi k = 0$ and $\xi\alpha = 0$ should be newly certified. We are now going to prove, using (4.1), that $\xi k = 0$.

Now, let Ω_1 be a set of points such that $\xi k \neq 0$ on M and suppose that Ω_1 be nonvoid. Then we have

$$A\phi + \phi A = 0, \quad LA = AL$$

on Ω_1 because of (3.29) and (3.31). We discuss our arguments on Ω_1.

From (4.1) we have $\alpha\phi A + kL = 0$ because of $\mu = 0$, which together with (3.11) gives $\alpha AY + kKY = (\alpha^2 + k^2)\eta(Y)\xi$. Differentiating this covariantly along Ω_1 and using (3.27) and (3.28), we find

$$\begin{aligned}&(X\alpha)AY + \alpha(\nabla_X A)Y + (\xi k)\eta(X)KY + k(\nabla_X K)Y\\ &\qquad = 2(\alpha(\xi\alpha) + k(\xi k))\eta(X)\eta(Y) + (\alpha^2 + k^2)\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\},\end{aligned}$$

from which, taking the skew-symmetric part and making use of (3.16), we obtain

$$\begin{aligned}&(X\alpha)AY - (Y\alpha)AX + \alpha((\nabla_X A)Y - (\nabla_Y A)X) + k(t(X)LY - t(Y)LX)\\ &\qquad = (\alpha^2 + k^2)(\eta(Y)\phi AX - \eta(X)\phi AY).\end{aligned}$$

If we take the inner product ξ to this and remember (3.10), (3.15) and the fact that $\mu = 0$, then we have $c\alpha = 0$, which together with (4.1) yields $kL = 0$, a contradiction because of (3.14). In the same way we see from (3.27) that $\xi\alpha = 0$. This completes the proof. □

We set $\Omega = \{p \in M : k(p) \neq 0\}$, and suppose that Ω is nonempty. In the rest of this paper, we discuss our arguments on the open subset Ω of M. So, by Proposition 4.1 we see that $\mu \neq 0$ on Ω.

We notice here that the following fact :
Remark 4.2. $\alpha \neq 0$ on Ω.

In fact, if not, then we have $\alpha = 0$ on this subset. We discuss our arguments on such a place. So (4.1) reformed as

$$\mu(w(X)U + u(X)W) + 2kLX = 0 \tag{4.6}$$

because of (2.22) with $\alpha = 0$. Putting $X = U$ or W in this we have respectively

$$LU = -\frac{\mu\beta}{2k}W, \quad LW = -\frac{\mu}{2k}U \tag{4.7}$$

by virtue of (2.24) with $\alpha = 0$. Using this and (3.14), we can write (4.3) as

$$-\frac{\beta^2}{2k}w(X)W + g(KU, X)U = -2k(\theta - c)(X - \eta(X)\xi).$$

Taking the inner product with W to this, we obtain $\beta^2 = 4k^2(\theta - c)$.

On the other hand, combining (4.6) and (4.7) to (3.14) we also have $\beta^2 = 4(n-1)k^2(\theta - c)$, which implies $(n-2)(\theta - c)k = 0$, a contradiction because of our assumption and Lemma 3.1. Thus, $\alpha = 0$ is not impossible on Ω.

Now, putting $X = U$ in (4.4) and remembering Remark 4.2, we find $kt(U) + g(LU, U) = 0$.

By the way, replacing X by U in (4.1) and using (2.22) and (2.25), we find

$$\alpha(\phi AU + \mu AW) = \mu^2 A\xi + 2kLU.$$

If we take the inner product with U and make use of (4.2) and Proposition 4.1, then we obtain $g(LU, U) = 0$ and hence $t(U) = 0$.

By putting $X = U$ in (4.5), we then have

$$KU = \tau U, \tag{4.8}$$

where τ is given by $\tau\mu^2 = g(KU, U)$ by virtue of Proposition 4.1. Applying this by ϕ and using (3.12), we find

$$LU = \tau\mu W. \tag{4.9}$$

It is, using (4.8) and (4.9), seen that

$$\tau^2 = \theta - c. \tag{4.10}$$

because of (3.13).

Remark 4.3. $\Omega = \emptyset$ if $\theta = c$.

Since we have $\theta = c$, then (3.14) gives $L = 0$ and thus $KX = k\eta(X)\xi$ by virtue of (3.11). Hence, (3.17) reformed as

$$k\{\eta(X)AY - \eta(Y)AX + \eta(X)t(Y)\xi - t(X)\eta(Y)\xi\} = 0,$$

which shows $k(t(X) + g(A\xi, X) - \sigma\eta(X)) = 0$, where we have put $\sigma = \alpha + t(\xi)$. Thus, the last two equations imply

$$AX = \eta(X)A\xi + g(A\xi, X)\xi - \alpha\eta(X)\xi.$$

Since U is orthogonal to ξ and W, it is clear that $AU = 0$ and $AW = \mu\xi$.

If we put $X = \mu W$ in (4.1) and remember (2.23) and the fact that $L = 0$, then we obtain $\mu^2 U = 0$ and hence $A\xi = \alpha\xi$. Owing to Lemma 3.1, we conclude that $k = 0$ and thus $\Omega = \emptyset$.

By Remark 4.3, we may only consider the case where $\tau \neq 0$ on Ω. Because of (3.22) and (4.8) we have

$$t(\phi X) = (1 + \frac{\tau}{k})g(U, X). \tag{4.11}$$

Therefore, by properties of the almost contact metric structure, it is clear that

$$t = t(\xi)\xi - \mu(1 + \frac{\tau}{k})W. \tag{4.12}$$

Using (2.22), we can write (3.20) as

$$\mu KW = k\mu W + k(t - t(\xi)\xi),$$

which together with (4.12) implies that

$$KW = -\tau W \tag{4.13}$$

because of Proposition 4.1.

If we take account of (3.25) and (4.11), then we find

$$\tau^2(A\phi X - \phi AX) + \tau(\tau - k)(u(X)\xi + \eta(X)U) + k(ALX + LAX) = 0. \tag{4.14}$$

From (2.15) the Ricci tensor S of type (1,1) of M is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X + kKX - K^2X - L^2X$$

by virtue of (3.10).

By the way, we see, using (3.12)∼(3.14), that

$$K^2X = (\theta - c)(X - \eta(X)\xi) + k^2\eta(X)\xi. \tag{4.15}$$

Substituting this and (3.14) into the last equation and using (4.10), we obtain

$$SX = \{c(2n+1) - 2(\theta - c)\}X + (2(\theta - c) - k^2 - 3c)\eta(X)\xi + hAX - A^2X + kKX, \tag{4.16}$$

which connected to (3.10) yields

$$S\xi = 2c(n-1)\xi + hA\xi - A^2\xi. \tag{4.17}$$

Differentiating (4.8) covariantly along Ω, we find

$$(\nabla_X K)U + K\nabla_X U = \tau\nabla_X U,$$

which together with (3.16) and (4.9) yields

$$\begin{aligned}\mu\tau(t(X)w(Y) - t(Y)w(X)) &+ g(K\nabla_X U, Y) - g(K\nabla_Y U, X)\\ &= \tau\{g(\nabla_X U, Y) - g(\nabla_Y U, X)\}.\end{aligned} \tag{4.18}$$

By the way, because of (2.22) and (2.24), we can write (2.29) as

$$\nabla_\xi U = 3\phi AU + \alpha\mu W - \mu^2\xi + \phi\nabla\alpha. \tag{4.19}$$

Replacing X by ξ in (4.18) and taking account of the last two relationships, we find

$$\mu^2(\tau - k)\xi + \mu\tau(t(\xi) - 2\alpha)W + \mu(k - \tau)AW + 3(LAU - \tau\phi AU) = \tau\phi\nabla\alpha - L\nabla\alpha, \tag{4.20}$$

where we have used the first equation of (2.26).

In a direct consequence of (3.12) and (4.8), we obtain

$$\mu LW = \tau U \tag{4.21}$$

because of $\mu \neq 0$ on Ω.

In the same way as above, we see from (4.13)

$$\frac{\tau}{\mu}\{t(X)u(Y) - t(Y)u(X)\} + g(K\nabla_X W, Y) - g(K\nabla_Y W, X) = \tau\{g(\nabla_Y W, X) - g(\nabla_X W, Y)\}. \tag{4.22}$$

In the next place, from (2.22) and (2.25) we have $\phi U = -\mu W$. Differentiating this covariantly and using (2.6), we find

$$g(AU, X)\xi - \phi\nabla_X U = (X\mu)W + \mu\nabla_X W.$$

Putting $X = \xi$ in this and making use of (2.29), we get

$$\mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (\xi\mu)W, \tag{4.23}$$

which enables us to obtain

$$W\alpha = \xi\mu. \tag{4.24}$$

5. Ricci Tensors of Semi-invariant Submanifolds

We will continue our arguments under the same hypotheses $R_\xi\phi = \phi R_\xi$ and $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ as those in section 3. Further, we assume that $S\xi = g(S\xi, \xi)\xi$ is satisfied on a semi-invariant submanifold of codimension 3 in $M_{n+1}(c), c \neq 0$. Then we have from (4.17)

$$A^2\xi = hA\xi + (\beta - h\alpha)\xi. \tag{5.1}$$

From this, and (2.22) and (2.24) we see that

$$AW = \mu\xi + (h - \alpha)W. \tag{5.2}$$

In the next place, differentiating (5.2) covariantly along Ω, we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi + X(h - \alpha)W + (h - \alpha)\nabla_X W. \tag{5.3}$$

By taking the inner product with W to this and using (2.26) and (5.2), we obtain

$$g((\nabla_X A)W, W) = -2g(AU, X) + Xh - X\alpha \tag{5.4}$$

because W is a unit orthogonal vector to ξ.

Applying (5.3) by ξ and using (2.26), we also obtain

$$\mu g((\nabla_X A)W, \xi) = (h - 2\alpha)g(AU, X) + \mu(X\mu), \tag{5.5}$$

which connected to (3.15) gives

$$\mu(\nabla_\xi A)W = (h - 2\alpha)AU + \mu\nabla\mu - k\mu LW - cU, \tag{5.6}$$

or, using (3.10), (3.15) and (5.5),

$$\mu(\nabla_W A)\xi = (h - 2\alpha)AU - 2cU + \mu\nabla\mu. \tag{5.7}$$

Putting $X = \xi$ in (5.4) and taking account of (5.5), we have

$$W\mu = \xi h - \xi\alpha. \tag{5.8}$$

Replacing X by ξ in (5.3) and using (5.6), we find

$$\begin{aligned}(h - 2\alpha)AU - k\mu LW - cU + \mu\nabla\mu + \mu(A\nabla_\xi W - (h - \alpha)\nabla_\xi W)\\ = \mu(\xi\mu)\xi + \mu^2 U + \mu(\xi h - \xi\alpha)W.\end{aligned}$$

Substituting (4.23) and (4.24) into this and making use of (4.21), we find

$$\begin{aligned}3A^2U - 2hAU + (\alpha h - \beta - c - k\tau)U + A\nabla\alpha + \frac{1}{2}\nabla\beta - h\nabla\alpha\\ = 2\mu(W\alpha)\xi + (2\alpha - h)(\xi\alpha)\xi + \mu(\xi h)W.\end{aligned} \tag{5.9}$$

On the other hand, if we put $X = \mu W$ in (4.1) and take account of (2.23), (2.24) and (5.2), then we find $\alpha AU + (\beta - h\alpha + 2k\tau)U = 0$, which shows

$$AU = \lambda U, \tag{5.10}$$

where the function λ is defined, using Remark 4.2, by

$$\alpha\lambda = h\alpha - \beta - 2k\tau. \tag{5.11}$$

Differentiating (5.10) covariantly along Ω, we find

$$(\nabla_X A)U + A\nabla_X U = (X\lambda)U + \lambda\nabla_X U.$$

If we take the skew-symmetric part of this, then we get

$$\begin{aligned}\mu(k\tau - c)(\eta(Y)w(X) - \eta(X)w(Y)) + g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\\ = (X\lambda)u(Y) - (Y\lambda)u(X) + \lambda(g(\nabla_X U, Y) - g(\nabla_Y U, X)),\end{aligned}$$

where we have used (2.22), (2.25), (3.15) and (4.9). Replacing X by U in this and using (5.10), we get

$$A\nabla_U U - \lambda\nabla_U U = (U\lambda)U - \mu^2\nabla\lambda. \tag{5.12}$$

Taking the inner product with W to this and remembering (5.2), we obtain

$$\mu g(\xi, \nabla_U U) + \mu^2(W\lambda) + (h - \alpha - \lambda)g(W, \nabla_U U) = 0. \tag{5.13}$$

By the way, from $KU = \tau U$, we have

$$(\nabla_X K)U + K\nabla_X U = \tau\nabla_X U, \tag{5.14}$$

which implies that $g((\nabla_X K)U, U) = 0$. Because of (3.16), (4.9) and the last relationship give $(\nabla_U K)U = 0$, which connected to (4.13) and (5.14) yields $g(W, \nabla_U U) = 0$. Thus, (5.13) reformed as

$$\mu g(\xi, \nabla_U U) + \mu^2(W\lambda) = 0.$$

However, the first term of this vanishes identically because of (2.26) and (5.2), which shows $\mu(W\lambda) = 0$ and hence

$$W\lambda = 0. \tag{5.15}$$

In the same way, we verify, using (2.26) and (5.2), that

$$\xi\lambda = 0. \tag{5.16}$$

Now, differentiating (2.25) covariantly and using (2.5), we find

$$(\nabla_X A)\xi + A\phi AX = (X\alpha)\xi + \alpha\phi AX + (X\mu)W + \mu\nabla_X W.$$

If we put $X = \mu W$ in this and use (5.2), (5.7) and (5.10), then we find

$$\mu^2\nabla_W W - \mu\nabla\mu = (2h\lambda - 3\alpha\lambda + \alpha^2 - \alpha h - 2c)U - \mu(W\alpha)\xi - \mu(W\mu)W. \tag{5.17}$$

Lemma 5.1. *If M satisfies* (4.1), (5.2) *and $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$, then we have on Ω*

$$\nabla k = (k - \tau)U. \tag{5.18}$$

Proof. Using (3.21) and (4.8) we have

$$Xk = (\xi k)\eta(X) + (k - \tau)u(X)$$

for any vector field X. Differentiating this covariantly along Ω and taking the skew-symmetric part obtained, we find

$$\begin{aligned}(5.19)\qquad &\eta(Y)X(\xi k)-\eta(X)Y(\xi k)+(\xi k)\{\eta(X)u(Y)-\eta(Y)u(X)\\ &+g(\phi AX,Y)-g(\phi AY,X)\}+(k-\tau)du(X,Y)=0,\end{aligned}$$

where we have used (2.5).

Now, we take an orthonormal frame filed $\{e_0=\xi, e_1=W, e_2,\cdots,e_{n-1}, e_n=\phi e_1=\frac{1}{\mu}U, e_{n+1}=\phi e_2,\cdots,e_{2n-2}=\phi e_{n-1}\}$ of M. Taking the trace of (2.27), we obtain

$$\sum_{i=0}^{2n-2} g(\phi\nabla_{e_i}U, e_i)=\xi\alpha-\xi h.$$

Putting $X=\phi e_i$ and $Y=e_i$ in (5.19) and summing up for $i=1,2,\cdots,n-1$, we have

$$(k-\tau)\sum_{i=0}^{2n-2} du(\phi e_i, e_i)=\xi k(\alpha-h),$$

where we have used (2.22), (2.25), (5.2) and (5.10). Combining the last two relationships, we get

$$(5.20)\qquad (h-\alpha)\xi k=(k-\tau)(\xi h-\xi\alpha).$$

By the way, if we put $X=\mu W$ in (3.25) and take account of (2.22), (3.10) and (5.2), we obtain

$$(\theta-c)\{AU-(h-\alpha)U\}+k\tau\{AU+(h-\alpha)U\}=0,$$

which connected to (4.9) and (5.10) yields

$$(5.21)\qquad \lambda(k+\tau)+(h-\alpha)(k-\tau)=0.$$

From this we have

$$(h-\alpha+\lambda)\nabla k+(k-\tau)(\nabla h-\nabla\alpha)+(k+\tau)\nabla\lambda=0.$$

So we have $(h-\alpha+\lambda)\xi k+(k-\tau)(\xi h-\xi\alpha)=0$ with the aid of (5.16). From this and (5.20) we see that $(2h-2\alpha+\lambda)\xi k=0$.

If $\xi k\neq 0$ on Ω, then we have $\lambda=2(\alpha-h)$, which together with (5.21) implies that $(h-\alpha)(k+3\tau)=0$ on this subset. We discuss our arguments on such a place. So we have $h-\alpha=0$ from the last equation and hence $\lambda=0$. Consequently we have $\mu^2+2k\tau=0$ by virtue of (2.24) and (5.11). Differentiation with respect to ξ gives $\mu(\xi\mu)+\tau(\xi k)=0$.

However, if we take the inner product with U to (5.7) and remember (2.24), (5.10) and the fact that $h-\alpha=0$ and $\lambda=0$, then we have $\mu\nabla\mu=(\mu^2+k\tau+c)U$

and consequently $\xi\mu = 0$. Hence we have $\tau(\xi k) = 0$, a contradiction. Thus, we have (5.18). This completes the proof. $\square$

Lemma 5.2. *Under the same hypotheses as those stated in Lemma 5.1, we have* $k - \tau \neq 0$ *on* Ω.

Proof. If not, then we have $k - \tau = 0$ on an open subset of Ω. We discuss our argument on such a place. Then we have $\lambda = 0$ because of (5.21) and Remark 4.3. So (5.10) and (5.11) turn out respectively to

$$AU = 0, \tag{5.22}$$

$$\beta - h\alpha + 2\tau^2 = 0. \tag{5.23}$$

We also have from (4.11) $t = t(\xi)\xi - 2\phi U$, which shows $t(Y) = t(\xi)\eta(Y) - 2g(\phi U, Y)$ for any vector Y. Differentiating this covariantly and using (2.5), (2.6) and (5.22), we find

$$(\nabla_X t)Y = X(t(\xi))\eta(Y) + t(\xi)g(\phi AX, Y) - 2g(\phi\nabla_X U, Y),$$

from which, taking the skew-symmetric part with respect to X and Y and using (3.1),

$$\begin{aligned} 2\theta g(\phi X, Y) = {} & X(t(\xi))\eta(Y) - Y(t(\xi))\eta(X) + t(\xi)\{g(\phi AX, Y) - g(\phi AY, X)\} \\ & + 2\{g(\phi\nabla_Y U, X) - g(\phi\nabla_X U, Y)\}. \end{aligned}$$

On the other hand, we verify from (2.27) that

$$\begin{aligned} & g(\phi\nabla_X U, Y) - g(\phi\nabla_Y U, X) + (X\alpha)\eta(Y) - (Y\alpha)\eta(X) \\ & \quad = -2cg(\phi X, Y) - 2g(A\phi AX, Y) + \alpha(g(\phi AX, Y) - g(\phi AY, X)). \end{aligned}$$

Combining the last two equations, it follows that

$$\begin{aligned} & 2(\theta - 2c)g(\phi X, Y) + t(\xi)\{g(\phi AX, Y) - g(\phi AY, X)\} \\ & \quad = X(t(\xi))\eta(Y) - Y(t(\xi))\eta(X) + 2\{2g(A\phi AX, Y) + \alpha(g(\phi AX, Y) \\ & \quad - g(\phi AY, X)) + (X\alpha)\eta(Y) - (Y\alpha)\eta(X)\}. \end{aligned}$$

Putting $Y = \xi$ in this and remembering (5.22), we find

$$X(t(\xi)) + 2(X\alpha) = \{\xi(t(\xi)) + 2\xi\alpha\}\eta(X) + (t(\xi) + 2\alpha)u(X). \tag{5.24}$$

Substituting this into the last equation, we obtain

$$\begin{aligned} 2(\theta - 2c)g(\phi X, Y) = {} & (t(\xi) + 2\alpha)(u(X)\eta(Y) - u(Y)\eta(X) \\ & + g(\phi AX, Y) - g(\phi AY, X)) + 4g(A\phi AX, Y). \end{aligned}$$

If we put $X = \mu W$ in this and take account of (2.23), (5.2) and (5.22), then we get

$$2(\theta - 2c) = (t(\xi) + 2\alpha)(h - \alpha). \tag{5.25}$$

In the next step, differentiating (4.13) covariantly, we find

$$(\nabla_X K)W + K\nabla_X W + \tau \nabla_X W = 0,$$

from which, taking the skew-symmetric part and using (3.16) and (4.9),

$$\begin{aligned} \frac{\tau}{\mu}(t(Y)u(X) - t(X)u(Y)) + g(K\nabla_X W, Y) - g(K\nabla_Y W, X) \\ = \tau\{(\nabla_Y W)X - (\nabla_X W)Y\}. \end{aligned} \tag{5.26}$$

If we put $X = \xi$ in this and make use of (2.26), (4.23) and (5.22), then, we find

$$K\nabla\alpha + \tau\nabla\alpha = 2\tau(\xi\alpha)\xi + \tau(2\alpha + t(\xi))U. \tag{5.27}$$

Replacing X by W in (5.26) and making use of (5.17), we have

$$\mu(K\nabla\mu + \tau\nabla\mu) = 2\tau(\mu^2 - \alpha^2 + h\alpha + 2c)U + 2\mu\tau(W\alpha)\xi.$$

If we take the inner product with U to this and take account of (4.8), then we obtain $\mu(U\mu) = (\mu^2 - \alpha^2 + h\alpha + 2c)\mu^2$, which together with (2.24) and (5.23) gives

$$\mu(U\mu) = 2(\mu^2 + \tau^2 + c)\mu^2. \tag{5.28}$$

On the other hand, differentiating (5.22) covariantly with respect to ξ, we find $(\nabla_\xi A)U + A\nabla_\xi U = 0$, which together with (4.19) (5.1) and (5.22) implies that

$$(\nabla_\xi A)U + (\alpha h - \beta)A\xi - \alpha(\beta - h\alpha)\xi + A\phi\nabla\alpha = 0.$$

Applying by ϕ, we have

$$\phi(\nabla_\xi A)U + (\alpha h - \beta)U + \phi A\phi\nabla\alpha = 0. \tag{5.29}$$

Since we see from (3.15)

$$(\nabla_U A)\xi - (\nabla_\xi A)U = \mu(\tau^2 + c)W$$

by virtue of (2.25), (3.10) and (4.9), it follows that

$$\phi(\nabla_U A)\xi = \phi(\nabla_\xi A)U + (\tau^2 + c)U. \tag{5.30}$$

We also have from (2.27)

$$\nabla_X U + g(A^2\xi, X)\xi = \phi(\nabla_X A)\xi + \phi A\phi AX + \alpha AX,$$

which connected to (5.22) gives $\nabla_U U = \phi(\nabla_U A)\xi$. Thus, (5.30) reformed as

$$\nabla_U U = \phi(\nabla_\xi A)U + (\tau^2 + c)U.$$

Combining this to (5.29) and using (5.23), it follows that

$$\nabla_U U = (c - \tau^2)U - \phi A\phi\nabla\alpha. \tag{5.31}$$

If we apply by A and take account of (5.12) with $\lambda = 0$ and (5.22), then we have $A\phi A\phi\nabla\alpha = 0$.

Now, taking the inner product with U to (5.30) and making use of (2.22) $\sim$ (2.25) and (5.2), we obtain

$$\mu(U\mu) = (c - \tau^2)\mu^2 + (h - \alpha)U\alpha. \tag{5.32}$$

However, applying (5.27) by U and using (4.8), we find $2U\alpha = (t(\xi) + 2\alpha)\mu^2$, which connected to (5.25) gives $(h - \alpha)U\alpha = (\theta - 2c)\mu^2$. Substituting (5.28) and this into (5.32), we find $2\mu^2 + 3c + 3\tau^2 = \theta$, which together with (4.10) gives $\mu^2 + \tau^2 + c = 0$ and consequently μ is constant. Thus, we see, using (2.24) and (5.23), that

$$\alpha(h - \alpha) = \tau^2 - c. \tag{5.33}$$

Therefore, $\alpha(h - \alpha) = const$. Differentiation gives

$$(h - \alpha)\nabla\alpha + \alpha(\nabla h - \nabla\alpha) = 0,$$

which connected to (5.8) implies that $(h-\alpha)\xi\alpha = 0$, where we have used $\mu = const$. Accordingly we have $\xi\alpha = 0$ by virtue of (5.33) and the fact that $\theta - 2c \neq 0$.

Using (4.10) and (5.33), we can write (5.25) as

$$2(\theta - 2c)\alpha = (\theta - 2c)(t(\xi) + 2\alpha).$$

Thus, it follows that $t(\xi) = 0$ provided that $\theta - 2c \neq 0$. Hence, (5.24) turns out to be $\nabla\alpha = \alpha U$, which implies $du = 0$. Therefore, it is clear that $\nabla_U U = 0$ because of $\mu = const$, which connected to (5.31) yields $(c - \tau^2)U = \alpha\phi A\phi U$. So we have $c - \tau^2 = \alpha(h - \alpha)$, where we have used (2.23), (2.25) and (5.2). From this and (5.33) it follows that $\theta - 2c = 0$, a contradiction. Hence, Lemma 5.2 is proved. □

Lemma 5.3. *Under the same hypotheses as those in Lemma 5.1, we have*

$$\nabla\alpha = (h - 3\lambda)U. \tag{5.34}$$

Proof. Because of Lemma 5.1 and Lemma 5.2, we can write (5.19) as $du(X, Y) = 0$, that is, $g(\nabla_X U, Y) - g(\nabla_Y U, X) = 0$. Putting $X = \xi$ in this, and using (2.26) and (4.19), we find

$$3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha + \mu AW = 0,$$

which together with (2.22), (2.25), (5.2) and (5.10) implies that

$$\phi\nabla\alpha + (h - 3\lambda)\mu W = 0.$$

Thus, it follows that

$$\nabla\alpha = (\xi\alpha)\xi + (h - 3\lambda)U. \tag{5.35}$$

We are now going to prove that $\xi\alpha = 0$.

Differentiation (5.21) with respect to ξ gives $\xi h - \xi\alpha = 0$ with the aid of (5.16), Lemma 5.1 and Lemma 5.2.

Using (5.10), (5.35) and this fact, we can write (5.9) as

$$\frac{1}{2}\nabla\beta + (2h\lambda + \alpha h - \beta - c - k\tau - h^2)U = \{2\mu(W\alpha) + \alpha(\xi\alpha)\}\xi. \tag{5.36}$$

Since we have $W\mu = 0$ because of (5.8), if we take the inner product ξ to the last equation and take account of (2.24), then we obtain $\alpha(W\alpha) = 0$ and hence $W\alpha = 0$ by virtue of Remark 4.2.

Differentiating (5.11) with respect to ξ and making use of (5.16), Lemma 5.1 and the fact that $\xi h - \xi\alpha = 0$, we find

$$\xi\beta = (h + \alpha - \lambda)\xi\alpha.$$

On the other hand, if we differentiate (2.24) with respect to ξ and remember $W\alpha = 0$ and (4.24), then we have $\xi\beta = 2\alpha(\xi\alpha)$. From this and the last relationship we get $(\lambda + \alpha - h)\xi\alpha = 0$.

Now, if $\xi\alpha \neq 0$ on Ω, the we have $\lambda = h - \alpha$ on this subset. We discuss our arguments on this subset. Then (5.21) yields $\lambda k = 0$ and hence $\lambda = 0$ and $h - \alpha = 0$. So (5.35) and (5.36) are reduced respectively to

$$\nabla\alpha = (\xi\alpha)\xi + \alpha U, \qquad \frac{1}{2}\nabla\beta = \alpha(\xi\alpha)\xi + (\beta + k\tau + c)U.$$

We also have from (5.11) $\beta = \alpha^2 - 2k\tau$, which together with (5.18) implies that

$$\frac{1}{2}\nabla\beta = \alpha\nabla\alpha - \tau(k - \tau)U.$$

Combining above equations, it follows that $\tau^2 = c$, that is, $\theta - 2c = 0$, a contradiction. This completes the proof of Lemma 5.3. □

6. Proof of Main Theorem

First of all, we will prove the following lemma.

Lemma 6.1. *Let M be a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ satisfying $dt = 2\theta\omega$*

for a scalar $\theta \neq 2c$. Suppose that M satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $S\xi = g(S\xi,\xi)\xi$. Then the distinguished normal is parallel in the normal bundle, where S denotes the Ricci tensor of M.

Proof. Because of (5.19), Lemma 5.1 and Lemma 5.2, we have $du = 0$. So we have from (5.14)

$$g(K\nabla_X U, Y) - g(K\nabla_Y U, X) + \mu\tau\{t(X)w(Y) - t(Y)w(X)\} = 0,$$

where we have used (3.16) and (4.9). Putting $X = \xi$ in this and using (2.25), (2.26), (4.19) and (5.10), we find

$$K(3\lambda\mu W + \alpha A\xi - \beta\xi + \phi\nabla\alpha) + k\mu AW + \mu\tau t(\xi)W = 0,$$

which connected to (2.22), (3.10), (3.12), (4.13), (5.2) and (5.34) gives

$$\tau t(\xi) + (h-\alpha)(k+\tau) = 0, \tag{6.1}$$

or, using (5.21)

$$\tau(k-\tau)t(\xi) = \lambda(k+\tau)^2. \tag{6.2}$$

On the other hand, differentiating (4.12) covariantly along Ω, and taking account of (2.5), (2.6), (5.10) and (5.18), we get

$$\begin{aligned}(\nabla_X t)Y = X(t(\xi))\eta(Y) + t(\xi)g(\phi AX, Y) + \frac{\tau}{k^2}(k-\tau)\mu u(X)w(Y)\\ - (1+\frac{\tau}{k})\{\lambda u(X)\eta(Y) - g(\phi\nabla_X U, Y) + t(\nabla_X Y),\end{aligned}$$

from which taking the skew-symmetric part and using (2.25) and (3.1),

$$\begin{aligned}&2\theta g(\phi X, Y) + \frac{\tau}{k^2}(k-\tau)\mu(u(Y)w(X) - u(X)w(Y))\\ &\quad = X(t(\xi))\eta(Y) - Y(t(\xi))\eta(X) + t(\xi)\{g(\phi AX, Y) - g(\phi AY, X)\}\\ &\quad - (1+\frac{\tau}{k})\{\lambda(u(X)\eta(Y) - u(Y)\eta(X)) - g(\phi\nabla_X U, Y) + g(\phi\nabla_Y U, X)\}.\end{aligned} \tag{6.3}$$

By the way, we have from (2.27) and (3.15)

$$\begin{aligned}g(\phi\nabla_X U, Y) - g(\phi\nabla_Y U, X) + (h+\lambda-3\alpha)(u(X)\eta(Y) - u(Y)\eta(X))\\ = 2cg(\phi X, Y) - 2g(A\phi AX, Y) + \alpha(g(\phi AX, Y) - g(\phi AY, X)),\end{aligned}$$

where we have used (3.10), (5.10) and (5.34).

Combining the last two equations, we obtain

$$\begin{aligned}2\theta g(\phi X, Y) + \frac{\tau}{k^2}(k-\tau)\mu(u(Y)w(X) - u(X)w(Y)) - t(\xi)(g(\phi AX, Y) - g(\phi AY, X))\\ = X(t(\xi))\eta(Y) - Y(t(\xi))\eta(X) + (1+\frac{\tau}{k})\{2cg(\phi X, Y) + (h-3\lambda)(u(X)\eta(Y)\\ -u(Y)\eta(X)) - 2g(A\phi AX, Y) + \alpha(g(\phi AX, Y) - g(\phi AY, X))\}.\end{aligned}$$

Putting $Y = \xi$ in this and making use of (2.5) and (5.10), we find

$$X(t(\xi)) = \xi(t(\xi))\eta(X) + \{t(\xi) + (1 + \frac{\tau}{k})(\lambda + \alpha - h)\}u(X), \tag{6.4}$$

which together with (6.1) yields

$$X(t(\xi)) = \xi(t(\xi))\eta(X) + (1 + \frac{\tau}{k})(\lambda + t(\xi))u(X).$$

Substituting this into the last equation and using (5.21), we find

$$2\theta g(\phi X, Y) + \frac{\tau}{k^2}\mu(k - \tau)(w(X)u(Y) - w(Y)u(X)) \tag{6.5}$$

$$= (1 + \frac{\tau}{k})\{(h - 2\lambda + t(\xi))(u(X)\eta(Y) - u(Y)\eta(X))$$
$$+2cg(\phi X, Y) + 2g(A\phi AX, Y) + (h + t(\xi))(g(\phi AX, Y) - g(\phi AY, X))\}.$$

Differentiating (6.1) covariantly and remembering (5.18), we find

$$\tau X(t(\xi)) = (\alpha - h)(k - \tau)u(X) + (k + \tau)(X\alpha - Xh),$$

which connected to (5.21) yields

$$\tau X(t(\xi)) = (k + \tau)(X\alpha - Xh + \lambda u(X)). \tag{6.6}$$

By the way, we see, using (5.20), Lemma 5.1 and Lemma 5.2, that $\xi h - \xi\alpha = 0$. Thus, from the last equation, it follows that $\xi(t(\xi)) = 0$ and hence (6.4) can be written as

$$X(t(\xi)) = \{t(\xi) + (1 + \frac{\tau}{k})(\lambda - h + \alpha)\}u(X),$$

which together with (6.1) gives

$$\tau X(t(\xi)) = \{(k + 2\tau + \frac{\tau^2}{k})(\alpha - h) + \tau\lambda(1 + \frac{\tau}{k})\}u(X).$$

Combining this to (6.6), we get

$$(k + \tau)(\nabla\alpha - \nabla h + \lambda U) = (1 + \frac{\tau}{k})\{(k + \tau)(\alpha - h) + \tau\lambda\}U,$$

which together with (5.21) gives

$$k(\nabla\alpha - \nabla h) = 2\tau(\lambda + \alpha - h)U, \tag{6.7}$$

where we have used $k + \tau \neq 0$.

If we differentiate (6.2) and take account of Lemma 5.1 and itself, we find

$$\lambda(k + \tau)^2 U + \tau(k - \tau)\nabla t(\xi) = (k + \tau)^2\nabla\lambda + 2\lambda(k^2 - \tau^2)U,$$

which together with (6.6), and Lemma 5.1 and Lemma 5.2 implies that $(k+\tau)\nabla\lambda = (k-\tau)(\nabla\alpha - \nabla h) + 2\tau\lambda U$, or using (5.21) and (6.7),

$$(k+\tau)\nabla\lambda = 6\tau\lambda U. \tag{6.8}$$

Now, if we put $X = U$ and $Y = W$ in (6.5) and using (2.23), (5.2) and (5.10), then we find

$$2\theta + \frac{\tau}{k^2}(k-\tau)\mu^2 = (1+\frac{\tau}{k})\{2c - 2\lambda(h-\alpha) + (t(\xi)+h)(\lambda+h-\alpha)\}.$$

By the way, it is seen, using (5.11) and (5.21), that $(k-\tau)^2\mu^2 + 2k(\alpha\lambda + \tau k - \tau^2) = 0$. Thus, the last equation can be written as

$$\begin{aligned}\theta k(k-\tau) - \tau\alpha\lambda(k-\tau) - \tau^2(k-\tau)^2 \\ = c(k^2-\tau^2) + \lambda^2(k+\tau)^2 - \tau\lambda(k+\tau)(t(\xi)+h).\end{aligned}$$

If we multiply $k-\tau$ to this and take account of (4.10), (5.21) and (6.2), then we obtain

$$\lambda^2(k+\tau)^2 + 2\tau\alpha\lambda(k-\tau) + (k-\tau)^2(\tau^2 - c) = 0. \tag{6.9}$$

Differentiating this covariantly and using (5.18) and (6.8), we find

$$\tau(k-\tau)\nabla(\alpha\lambda) + 6\tau\lambda^2(k+\tau)U = \tau\lambda\{2\lambda(k+\tau) + \alpha(k-\tau)\}U,$$

which implies

$$(k-\tau)\nabla(\alpha\lambda) = \lambda\{\alpha(k-\tau) - 4\lambda(k+\tau)\}U.$$

From this and (5.21) and (5.34), we have

$$\alpha(k-\tau)\nabla\lambda + 6\tau\lambda^2 U = 0,$$

which together with (6.8) yields $\lambda\{\alpha(k-\tau) + \lambda(k+\tau)\} = 0$. Thus, it follows that $\alpha(k-\tau) + \lambda(k+\tau) = 0$ by virtue of (6.9), which connected to (5.21) gives $h = 2\alpha$. Further, we have from the last relationship $(k+\tau)\nabla\lambda + (k-\tau)\nabla\alpha = 0$, which together with (5.34) and (6.8) gives $6\tau\lambda + (k-\tau)(2\alpha - 3\lambda) = 0$. Thus, it follows that $(8\tau - 5k)\lambda = 0$, and hence $5k = 8\tau$ because of (6.9).
So, we see, using (5.18), that k is a constant on Ω and hence $U = 0$, a contradiction. This completes the proof. □

According to Lemma 6.1 we can prove the following :

Lemma 6.2. *Under the same hypotheses as those in Lemma 6.1, we have* $A^{(2)} = A^{(3)} = 0$ *provided that* $\bar{r} - 2(n-1)c \leq 0$.

Remark 6.3. This lemma proved in [13] for the case where $\theta - 2c < 0$ and $c > 0$. But, we need the condition $\bar{r} - 2c(n-1) \leq 0$ for the case where $c < 0$, where $\bar{r}$ is the scalar curvature of M. So we introduce the outline of the proof.

The sketch of Proof. By Lemma 2.2 and Lemma 6.1, we have $k=0$ and hence $m=0$ on M because of (3.10). Thus, (3.15)∼(3.20) turn out to be

$$(\nabla_X A)Y-(\nabla_Y A)X=c\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\}, \tag{6.10}$$

$$(\nabla_X K)Y-(\nabla_Y K)X=t(X)LY-t(Y)LX, \tag{6.11}$$

$$(\nabla_X L)Y-(\nabla_Y L)X=0, \tag{6.12}$$

$$KA-AK=0, \quad LA-AL=0, \tag{6.13}$$

Since we have $K\xi=0$ because of (3.10), differentiating $K\xi=0$ covariantly along M and using (2.5) and (3.12), we find

$$(\nabla_X K)\xi=-LAX. \tag{6.14}$$

If we take account of Lemma 5.2 and (4.10), then (4.15) reformed as

$$K^2X=\tau'(X-\eta(X)\xi), \tag{6.15}$$

where $\tau'=\theta-c$.

Differentiating (6.15) covariantly along M and using (2.5), we find

$$(\nabla_X K)KY+K(\nabla_X K)Y=-\tau'\{\eta(Y)\phi AX+g(\phi AX,Y)\xi\}.$$

Using the quite same method as those used to (3.26) from (3.14), we can derive from the last equation the following :

$$\begin{aligned}2(\nabla_X K)KY=\tau'\{&-2t(X)\phi Y+\eta(X)(\phi A-A\phi)Y\\ &+g((\phi A-A\phi)X,Y)\xi+\eta(Y)(\phi A+A\phi)X\},\end{aligned} \tag{6.16}$$

where we have used (3.13) and (6.11).

By the way, if we take the trace of K in (6.11), we have $\sum_i \nabla_{e_i}Ke_i=Lt$ because of (3.10). If we use this fact to (6.16), we obtain

$$KLt=\tau'(\phi t+U),$$

where we have used (2.5), which together with (3.11) gives $\tau'U=0$ and consequently $U=0$ on M, that is $A\xi=\alpha\xi$ because of (2.25). Therefore, if we take account of Lemma 5.3 and (3.26), then we obtain

$$\tau'(A\phi-\phi A)=0. \tag{6.17}$$

In the following, we assume that $\tau'\neq 0$ on M. Then, from this and (6.10) we can verify the following (cf. [6], [16]) :

$$A^2=\alpha A+c(I-\eta\otimes\xi), \tag{6.18}$$

$$(6.19)\qquad (\nabla_X A)Y = -c(\eta(Y)\phi X + g(\phi X, Y)\xi).$$

Using (6.17), we can write (6.16) as

$$K(\nabla_X K)Y = \tau'\{-t(X)\phi Y + \eta(X)\phi AY + g(\phi AX, Y)\xi\}.$$

If we transform this by K and make use of (3.12), (6.11), (6.14) and (6.15), then we have

$$(6.20)\qquad (\nabla_X K)Y = t(X)LY - \eta(X)ALX - \eta(Y)LAX - g(ALX, Y)\xi.$$

Differentiating (3.12) covariantly along M and using (2.6) and the last equation, we find

$$(6.21)\qquad (\nabla_X L)Y = -t(X)KY + \eta(X)AKY + \eta(Y)AKX + g(AKX, Y)\xi.$$

If we take the trace of L in this and remember (3.20) and the fact that $TrA^{(2)} = TrA^{(3)} = 0$ and $A\xi = \alpha\xi$, we verify that

$$(6.22)\qquad Tr(AA^{(2)}) = 0,$$

which connected to (6.18) gives

$$(6.23)\qquad Tr(A^2A^{(2)}) = 0.$$

For the orthonormal frame field $\{e_0, e_1, \cdots, e_{2n-2}\}$ already selected, we write $g(e_j, e_i) = g_{ji}, g(\phi e_i, e_j) = \phi_{ij}, (g_{ji})^{-1} = g^{ji}, g(Ae_i, e_j) = A_{ij}$ and $\nabla_{e_i}X = (\nabla_i X^h)e_h$ for any vector $X = X^i e_i$. And the Einstein summation convention will be used. Then (6.20) can be written as

$$\nabla_k K_{ji} = t_k L_{ji} - \xi_k A_{jr} L_i{}^r - \xi_i A_{kr} L_j{}^r - \xi_j A_{ir} L_k{}^r.$$

Differentiating this covariantly along M and taking account of (2.5), (3.20), (6.18), (6.19) and itself, we find

$$\begin{aligned}\nabla_h\nabla_k K_{ji} = {} & (\nabla_h t_k)L_{ji} - c(K_{jh}\xi_k\xi_i + K_{ki}\xi_j\xi_h + 2K_{ih}\xi_j\xi_k) + B_{hkji}\\ & - \alpha(\xi_j\xi_h A_{kr}K_i{}^r + \xi_k\xi_i A_{jr}K_h{}^r + 2\xi_j\xi_k A_{ir}K_h{}^r)\\ & + (A_{hs}\phi_j{}^s)(A_{kr}L_i{}^r) + (A_{hs}\phi_k{}^s)(A_{ir}L_j{}^r) + (A_{hs}\phi_i{}^s)(A_{jr}L_k{}^r),\end{aligned}$$

where B_{hkji} is a certain tensor with $B_{hkji} = B_{khji}$, from which, taking the skew-symmetric part with respect to h and k, and making use of (3.1), (6.17) and the Ricci identity for K_{ji} (for detail, see (4.17) of [13]),

$$\begin{aligned}(6.24)\qquad & R_{khjr}K_i{}^r + R_{khir}K_j{}^r\\ & = 2\theta\phi_{hk}L_{ji} - c\{\xi_j(\xi_k K_{ih} - \xi_h K_{ik}) + \xi_i(\xi_k K_{jh} - \xi_h K_{jk})\}\\ & - \alpha\{\xi_j(\xi_k A_{ir}K_h{}^r - \xi_h A_{ir}K_k{}^r) + \xi_i(\xi_k A_{jr}K_h{}^r - \xi_h A_{jr}K_k{}^r)\}\\ & + (A_{hs}\phi_j{}^s)(A_{kr}L_i{}^r) - (A_{ks}\phi_j{}^s)(A_{hr}L_i{}^r) + (A_{hs}\phi_i{}^s)(A_{kr}L_j{}^r)\\ & - (A_{ks}\phi_i{}^s)(A_{hr}L_j{}^r) + 2(A_{hs}\phi_k{}^s)(A_{jr}L_i{}^r).\end{aligned}$$

Multiplying (6.24) with ϕ^{kh} and summing for k and h, and using (3.1), (3.11), (3.12), (6.17) and (6.18), we find

$$\phi^{kh}(R_{khjr}K_i{}^r + R_{khir}K_j{}^r) = 4\{c - (n-1)\theta\}L_{ji} + 2(h+\alpha)A_{jr}L_i{}^r. \tag{6.25}$$

On the other hand, from (2.15) we see, using (3.12), (6.15), (6.17) and (6.18), that

$$\phi^{kh}(R_{khir}K_j{}^r + R_{khjr}K_i{}^r) = 4\{2\theta - (2n+3)c\}L_{ji} - 4\alpha A_{jr}L_i{}^r,$$

which connected to (6.25) implies that (for detail, see (4.19) of [13])

$$(h+3\alpha)AL = 2\{(n+1)\theta - 2(n+2)c\}L,$$

which connected to (3.14) yields

$$(h+3\alpha)(AX - \alpha\eta(X)\xi) = 2\{(n+1)\theta - 2(n+2)c\}(X - \eta(X)\xi).$$

Taking the trace of (6.26), we have

$$(h+3\alpha)(h-\alpha) = 4(n-1)\{(n+1)\theta - 2c(n+2)\},$$

which implies

$$(h-\alpha)^2 + 4\alpha(h-\alpha) = \delta, \tag{6.26}$$

where we put

$$\delta = 4(n-1)\{(n+1)\theta - 2c(n+2)\}. \tag{6.27}$$

In the same way as above, by using properties of A and (2.15), (6.22), (6.23) and (6.25), we obtain (for detail, see (4.21) of [13])

$$(4\theta - 12c - h_{(2)} - 3\alpha^2)AK = \{4c\alpha - (\theta - 2c)(h-\alpha)\}K,$$

which connected to (6.15) yields

$$(4\theta - 12c - h_{(2)} - 3\alpha^2)(h-\alpha) = 2(n-1)\{4c\alpha - (\theta-2c)(h-\alpha)\}. \tag{6.28}$$

Since we have $h_{(2)} = \alpha h + 2c(n-1)$ from (6.18), combining (6.27) to (6.28), we obtain

$$(\theta - 3c)(h-\alpha) = 2(n-1)\alpha(\theta - 2c). \tag{6.29}$$

On the other hand, from (4.16) we verify that the scalar curvature $\bar{r}$ of M is given by

$$\bar{r} = 4c(n^2-1) - 4(n-1)\tau' + h^2 - h_{(2)},$$

which connected to (6.18) gives

$$\bar{r} = 2c(n-1)(2n+1) - 4(n-1)\tau' + h(h-\alpha). \tag{6.30}$$

By the way, it is seen, using (4.10), that $\theta - 3c \neq 0$ for $c < 0$. We also have $\theta - 3c \neq 0$ for $c > 0$,

In fact, if not, then we have $\theta - 3c = 0$ on this open subset. Thus, it follows, using (6.29), that

$$\alpha = 0, \quad \tau' = 2c. \tag{6.31}$$

Hence $h^2 = 4(n-1)^2 c$ on the set by virtue of (6.26) and (6.27). Using this fact and (6.31), we can write (6.30) as $\bar{r} = 2c(n-1)(4n-5)$, a contradiction because of $\bar{r} - 2c(n-1) \leq 0$ and $c > 0$. Therefore $\theta - 3c \neq 0$ is proved. Thus, we can write (6.29) as

$$h - \alpha = \frac{2(n-1)}{\theta - 3c}(\theta - 2c)\alpha.$$

Substituting this into (6.26), we obtain

$$4(n-1)(\theta - 2c)\{(n+1)\theta - 2(n+2)c\}\alpha^2 = \delta(\theta - 3c)^2,$$

which together (6.27) gives

$$\delta\{(\theta - 3c)^2 - (\theta - 2c)\alpha^2\} = 0. \tag{6.32}$$

We notice here that $\delta \neq 0$ if $c < 0$. We also see that $\delta \neq 0$ for $c > 0$. In fact, if not, then we have $\delta = 0$. Then we have by (6.27)

$$\theta - c = \frac{n+3}{n+1}c.$$

Using this fact and (6.26), we can write (6.30) as

$$\bar{r} - 2(n-1)c = \frac{4(n-1)}{n+1}(n^2 - 3)c + \varepsilon^2,$$

where $\varepsilon^2 = 0$ or $12\alpha^2$, a contradiction because $c > 0$ and $\bar{r} - 2(n-1)c \leq 0$ was assumed. Therefore (6.32) turns out to be

$$(\theta - 3c)^2 = (\theta - 2c)\alpha^2. \tag{6.33}$$

Accordingly, if we combine (6.29) to (6.33), then we obtain $\alpha(h - \alpha) = 2(n-1)(\theta - 3c)$, which together with (6.26) yields

$$h(h - \alpha) = 2(n-1)(2n-1)\tau' - 4n(n-1)c.$$

Using this, we can write (6.30) as

$$\bar{r} - 2c(n-1) = 2(n-1)(2n-3)\tau'.$$

Therefore we have $\tau' = 0$ if $\bar{r} - 2c(n-1) \leq 0$. This completes the proof of Lemma 6.2. □

Let $N_0(p) = \{v \in T_p^{\perp}(M) : A_v = 0\}$ and $H_0(p)$ be the maximal J-invariant subspace of $N_0(p)$. As a consequence of Lemma 6.2, we have $A^{(2)} = A^{(3)} = 0$, the orthogonal complement of $H_0(p)$ is invariant under parallel translation with respect to the normal connection because of $\nabla^{\perp} C = 0$. Thus, by the reduction theorem for $P_{n+1}\mathbb{C}$([19]) and $H_{n+1}\mathbb{C}$([9], [11]), there exists a totally geodesic complex space form including M in $M_{n+1}(c)$, we conclude that

Theorem 6.4. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ with constant holomorphic sectional curvature $4c$ such that the third fundamental form t satisfies $dt = 2\theta\omega$ for a nonzero scalar $\theta - 2c \neq 0$ and $\bar{r} - 2c(n-1) \leq 0$, where $\omega(X,Y) = g(\phi X, Y)$ for any vector fields X and Y on M. If M satisfies $R_\xi \phi = \phi R_\xi$ and at the same time $S\xi = g(S\xi,\xi)\xi$, then M is a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$.*

Since we have $\nabla^{\perp} C = 0$, we can write (2.16) and (4.1) as

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},$$
$$\alpha(\phi AX - A\phi X) - g(A\xi, X)U - g(U, X)A\xi = 0$$

respectively. Making use of (2.5), (2.6) and the above equations, it is prove in [16] that $g(U,U) = 0$, that is, M is a Hopf real hypersurface. Hence, we conclude that $\alpha(A\phi - \phi A) = 0$ and hence $A\xi = 0$ or $A\phi = \phi A$. Since M is a Hopf hypersurface, $A\xi = 0$ means that $\alpha = 0$. Here, we note that the case $\alpha = 0$ correspond to the case of tube of radius $\pi/4$ in $P_n\mathbb{C}$([5],[6]). But, in the case $H_n\mathbb{C}$ it is known that α never vanishes for Hopf hypersurfaces (cf.[3]) Thus, owing to Theorem 6.4, Theorem O and Theorem MR, we have

Main Theorem. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ with constant holomorphic sectional curvature $4c$ such that the Ricci tensor S satisfies $S\xi = g(S\xi,\xi)\xi$ and the third fundamental form t satisfies $dt = 2\theta\omega$ for a scalar θ $-2c(\neq 0)$ and satisfies $\bar{r} - 2c(n-1) \leq 0$, where S and $\bar{r}$ denote the Ricci tensor and the scalar curvature of M, respectively. Then $R_\xi\phi = \phi R_\xi$ holds on M if and only if M is locally congruent to one of the following hypersurfaces :*

(I) *in case that $M_n(c) = P_n\mathbb{C}$,*

- (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
- (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, ..., n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
- (T) *a tube of radius $\pi/4$ over a certain complex submanifold in $P_n\mathbb{C}$;*

(II) *in case that $M_n(c) = H_n\mathbb{C}$,*

- (A_0) *a horosphere,*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane* $H_{n-1}\mathbb{C}$*,*

(A_2) *a tube over a totally geodesic* $H_k\mathbb{C}$ *for some* $k \in \{1, ..., n-2\}$.

Remark 6.5. Because of (4.10), it is clear that $\theta \neq 0$ if $c > 0$, and $\theta - 2c \neq 0$ if $c < 0$.

Acknowledgements. The authors with to express their sincere thanks to the referee who gave us valuable suggestions and comments to improve the paper.

References

[1] A. Bejancu, *CR-submanifolds of a Kähler manifold I*, Proc. Amer. Math. Soc., **69**(1978), 135–142.

[2] J. Berndt, *Real hypersurfaces with constant principal curvatures in a complex hyperbolic space*, J. Reine Angew. Math., **395**(1989), 132–141.

[3] J. Berndt and H. Tamaru, *Cohomogeneity one actions on non compact symmetric space of rank one*, Trans. Amer. Math. Soc., **359**(2007), 3425–3438.

[4] D. E. Blair, G. D. Ludden and K. Yano, *Semi-invariant immersion*, Kodai Math. Sem. Rep., **27**(1976), 313–319.

[5] T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc., **269**(1982), 481–499.

[6] T. E. Cecil and P. J. Ryan, *Geometry of Hypersurfaces*, Springer(2015).

[7] J. T. Cho and U-H. Ki, *Real hypersurfaces of a complex projective space in terms of the Jacobi operators*, Acta Math. Hungar, **80**(1998), 155–167.

[8] J. T. Cho and U-H. Ki, *Real hypersurfaces in complex space forms with Reeb-flows symmetric Jacobi operator*, Canadian Math. Bull., **51**(2008), 359–371.

[9] J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Differential Geom., **5**(1971), 333–340.

[10] J. I. Her, U-H. Ki and S. -B. Lee, *Semi-invariant submanifolds of codimension 3 of a complex projective space in terms of the Jacobi operator*, Bull. Korean Math. Soc., **42**(2005), 93–119.

[11] S. Kawamoto, *Codimension reduction for real submanifolds of a complex hyperbolic space*, Rev. Mat. Univ. Complut. Madrid., **7**(1994), 119–128.

[12] U-H. Ki and H. Song, *Jacobi operators on a semi-invariant submanifolds of codimension 3 in a complex projective space*, Nihonkai Math J., **14**(2003), 1–16.

[13] U-H. Ki, H. Song and R. Takagi, *Submanifolds of codimension 3 admitting almost contact metric structure in a complex projective space*, Nihonkai Math J., **11**(2000), 57–86.

[14] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc., **296**(1986), 137–149.

[15] S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geometriae Dedicata., **20**(1986), 245–261.

[16] R. Niebergall and P. J. Ryan, *Real hypersurfaces in complex space form, in Tight and Taut submanifolds*, Cambridge University Press : (1998(T. E. Cecil and S. -S. Chern eds.)), 233–305.

[17] M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc., **212**(1973), 355–364.

[18] M. Okumura, *Normal curvature and real submanifold of the complex projective space*, Geom. Dedicata., **7**(1978), 509–517.

[19] M. Okumura, *Codimension reduction problem for real submanifolds of complex projective space*, Colloz. Math. Soc. Jámos Bolyai, **56**(1989), 574–585.

[20] H. Song, *Some differential-geometric properties of R-spaces*, Tsukuba J. Math., **25**(2001), 279–298.

[21] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math., **19**(1973), 495–506.

[22] R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures* I, II, J. Math. Soc. Japan, **27**(1975), 43–53, 507–516.

[23] Y. Tashiro, *Relations between the theory of almost complex spaces and that of almost contact spaces (in Japanese)*, *Sũgaku*, **16**(1964), 34–61.

[24] K. Yano, and U-H. Ki, *On* $(f, g, u, v, w, \lambda, \mu, \nu)$*-structure satisfying* $\lambda^2 + \mu^2 + \nu^2 = 1$, Kodai Math. Sem. Rep., **56**(1978), 574–585.

[25] K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, *Birkhäuser*(1983).