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On n-skew Lie Products on Prime Rings with Involution

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Abstract. Let R be a $*$-ring and $n \geq 1$ be an integer. The objective of this paper is to introduce the notion of n-skew centralizing maps on $*$-rings, and investigate the impact of these maps. In particular, we describe the structure of prime rings with involution $'*'$ such that ${}_*[x, d(x)]_n \in Z(R)$ for all $x \in R$ (for $n = 1, 2$), where $d : R \to R$ is a nonzero derivation of R. Among other related results, we also provide two examples to prove that the assumed restrictions on our main results are not superfluous.

1. Introduction

This research is motivated by the recent work's of Ali-Dar [1], Qi-Zhang [5] and Hou-Wang [3]. However, our approach is different from that of the authors of [5] and [3]. A ring R with an involution $'*'$ is called a $*$-ring or ring with involution $'*'$. Throughout, we let R be a ring with involution $'*'$ and $Z(R)$, the center of the ring R. Moreover, the sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution is called the first kind if

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$Z(R) \subseteq H(R)$, otherwise $S(R) \cap Z(R) \neq (0)$(see [2] for details). A ring R is said to be 2-torsion free if $2x = 0$ (where $x \in R$) implies $x = 0$. A ring R is called prime if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$. A derivation on R is an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

For any $x, y \in R$, the symbol $[x, y]$ will denote the Lie product $xy - yx$ and the symbol $_*[x, y]$ will denote the skew Lie product $xy - yx^*$, where $'*'$ is an involution on R. In a recent paper, Hou and Wang [3] extended the concept of skew Lie product as follows: for an integer $n \geq 1$, the n-skew Lie product of any two elements x and y is defined by $_*[x, y]_n =_* [x,_* [x, y]_{n-1}]$, where $_*[x, y]_0 = y$, $_*[x, y] = xy - yx^*$ and $_*[x, y]_2 = x^2y - 2xyx^* + y(x^*)^2$. Obviously, for $n = 1$, the skew Lie product and n-skew Lie product coincides. Note that, for n=2, we call it 2-skew Lie product. In [3], Hou and Wang studied the strong 2-skew commutativity preserving maps in prime rings with involution. In fact, they described the form of strong 2-skew commutativity preserving maps on a unital prime ring with involution that contains a non-trivial symmetric idempotent. In [5], Qi and Zhang studied the properties of n-skew Lie product on prime rings with involution and as an application, they characterized n-skew commuting additive maps, i.e.; an additive mapping f on R into itself such that $_*[x, f(x)]_n = 0$ for all $x \in R$. In definition of n-skew commuting mapping (defined in [5]), if we consider that f is any map (not necessarily additive) then it is more reasonable to call f a n-skew commuting. To give its precise definition, we make a slight modification in Qi and Zhang's definition for n-skew commuting mapping. For an integer $n \geq 1$, a map f of a $*$-ring R into itself is called n-skew commuting mapping on R if $_*[x, f(x)]_n = 0$ for every $x \in R$. For an integer $n \geq 1$, a map f of a $*$-ring R into itself is called n-skew centralizing mapping on R if $_*[x, f(x)]_n \in Z(R)$ for every $x \in R$. In particular, for $n = 1$, 2, we call them 1-skew commuting (resp. 1-skew centralizing) and 2-skew commuting (resp. 2-skew centralizing) mapping.

The objective of this paper is to introduce the notion of n-skew centralizing mappings on $*$-rings. Further, we investigate the impact of these mappings and describe the nature of prime $*$-rings which satisfy certain $*$-differential identities. In particular, for an integer $n \geq 1$ we prove that if a 2-torsion free prime ring R with involution $'*'$ of the second kind which admits a nonzero derivation d such that $_*[x, d(x)]_n \in Z(R)$ for all $x \in R$ $(n = 1, 2)$, then R is commutative. Moreover, some more related results are obtained. Further more, examples prove that, the assumed curtailment can not be relaxed as given.

2. Main Results

In order to study the effect of n-skew centralizing mappings, we need the following two lemmas for developing the proofs of our main results. We begin our discussion with the following lemmas:

Lemma 2.1. *Let R be a 2-torsion free prime ring with involution $'*'$ of the second kind. If $x^2x^* \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. Linearization of $x^2x^* \in Z(R)$ gives that $x^2y^* + xyx^* + xyy^* + yxx^* + yxy^* + y^2x^* \in Z(R)$ for all $x, y \in R$. Taking $x = -x$ in the last expression and combine it with the above relation, we get

$$x^2y^* + xyx^* + yxx^* \in Z(R) \text{ for all } x, y \in R. \tag{2.1}$$

Substituting ky for y, where $k \in S(R) \cap Z(R)$, we obtain $(-x^2y^* + xyx^* + yxx^*)k \in Z(R)$ for all $x, y \in R$. Invoking the primeness of R and using the fact that $S(R) \cap Z(R) \neq (0)$, we get

$$-x^2y^* + xyx^* + yxx^* \in Z(R) \text{ for all } x, y \in R. \tag{2.2}$$

Combining (2.1) and (2.2), we conclude that $x^2y^* \in Z(R)$. Replacing y by y^*, we get $x^2y \in Z(R)$ for all $x, y \in R$. This can be further written as $[x^2y, w] = 0$ for all $x, y, w \in R$. Replacing y by ry, we get $x^2r[y, w] = 0$ for all $x, y, w, r \in R$. Hence by the primeness of the ring R, we are force to conclude that R is commutative. □

Lemma 2.2. *Let R be a 2-torsion free prime ring with involution $'*'$ of the second kind. If ${}_*[x, x^*]_2 \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. By the hypothesis, we have

$$x^2x^* - 2x(x^*)^2 + (x^*)^3 \in Z(R) \text{ for all } x \in R. \tag{2.3}$$

Replacing x by kx in (2.3) where $k \in S(R) \cap Z(R)$, we get

$$-x^2x^*k^3 - 2x(x^*)^2k^3 - (x^*)^3k^3 \in Z(R) \text{ for all } x \in R. \tag{2.4}$$

By (2.3) and (2.4), we conclude that $-4x(x^*)^2k^3 \in Z(R)$ for all $x \in R$. Since R is 2-torsion free prime ring and $S(R) \cap Z(R) \neq (0)$, we obtain $x(x^*)^2 \in Z(R)$ for all $x \in R$. On linearizing we get

$$xx^*y^* + xy^*x^* + x(y^*)^2 + y(x^*)^2 + yx^*y^* + yy^*x^* \in Z(R) \text{ for all } x, y \in R. \tag{2.5}$$

Taking $x = -x$ in (2.5) and using (2.5), we obtain

$$xx^*y^* + xy^*x^* + y(x^*)^2 \in Z(R) \text{ for all } x, y \in R. \tag{2.6}$$

Substitute ky for y, where $k \in S(R) \cap Z(R)$ in (2.6), we get $2y(x^*)^2k \in Z(R)$ for all $x, y \in R$. This implies that $yx^2 \in Z(R)$ for all $x, y \in R$. Henceforth, using the same arguments as we have used in Lemma 2.1, we conclude that R is commutative. This proves the lemma. □

Theorem 2.3. *Let R be a 2-torsion free prime ring with involution $'*'$ of the second kind. If R admits a nonzero derivation d such that ${}_*[x, d(x)]_n \in Z(R)$ for all $x \in R$ $(n = 1, 2)$, then R is commutative.*

proof. **Case** (i) First we discuss the case, “when $n = 1$” and “i.e.”,

$${}_*[x, d(x)] \in Z(R) \text{ for all } x \in R.$$

Linearizing the above expression, we get

$${}_*[x, d(y)] + *[y, d(x)] \in Z(R) \text{ for all } x, y \in R.$$

That is,

$$xd(y) - d(y)x^* + yd(x) - d(x)y^* \in Z(R) \text{ for all } x, y \in R.$$

This further implies that

$$[xd(y), r] - [d(y)x^*, r] + [yd(x), r] - [d(x)y^*, r] = 0 \text{ for all } x, y, r \in R.$$

Hence

$$\begin{aligned} x[d(y), r] + [x, r]d(y) - d(y)[x^*, r] - [d(y), r]x^* \\ +y[d(x), r] + [y, r]d(x) - d(x)[y^*, r] - [d(x), r]y^* = 0 \text{ for all } x, y, r \in R. \end{aligned} \tag{2.7}$$

Replacing y by hy in (2.7), where $h \in H(R) \cap Z(R)$ and using it, we have

$$(x[y, r] + [x, r]y - y[x^*, r] - [y, r]x^*)d(h) = 0 \text{ for all } x, y, r \in R.$$

Using the primeness of R, we obtain either $d(h) = 0$ or

$$x[y, r] + [x, r]y - y[x^*, r] - [y, r]x^* = 0 \text{ for all } x, y, r \in R.$$

First we consider the situation

$$x[y, r] + [x, r]y - y[x^*, r] - [y, r]x^* = 0 \text{ for all } x, y, r \in R. \tag{2.8}$$

Substituting kx for x in (2.8), where $k \in S(R) \cap Z(R)$ and combining it with (2.8), we get

$$2(x[y, r] + [x, r]y)k = 0 \text{ for all } x, y, r \in R.$$

Since R is 2-torsion free prime ring, we deduce that

$$x[y, r] + [x, r]y = 0 \text{ for all } x, y, r \in R.$$

Replacing x by z, where $z \in Z(R)$, we get $[y, r]z = 0$ for all $y, r \in R$. Henceforth, we conclude that R is commutative. Now consider the case $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. This implies that $d(k) = 0$ for all $k \in S(R) \cap Z(R)$. Replacing y by ky in (2.7), where $k \in S(R) \cap Z(R)$ with $d(k) = 0$ and adding with (2.7), we get

$$2(x[d(y), r] + [x, r]d(y) - d(y)[x^*, r] - [d(y), r]x^* + y[d(x), r] + [y, r]d(x))k = 0$$

for all $x, y, r \in R$. Since R is 2-torsion free ring and $S(R) \cap Z(R) \neq (0)$, the above relation implies that

$$x[d(y), r] + [x, r]d(y) - d(y)[x^*, r] - [d(y), r]x^* + y[d(x), r] + [y, r]d(x) = 0 \text{ for all}$$

$x, y \in R$. Taking $y = h$, where $h \in H(R) \cap Z(R)$ and using the fact that $d(h) = 0$, we get $[d(x), r]h = 0$ for all $x, r \in R$ and $h \in H(R) \cap Z(R)$. This yields that $[d(x), r] = 0$ for all $x, r \in R$. Hence in view of Posner's [4] first theorem, R is commutative.

Case *(ii)* Now, we prove the result for $n = 2$ i.e.,

$$_*[x, d(x)]_2 \in Z(R) \text{ for all } x \in R.$$

On expansion we acquire

$$x^2 d(x) - 2xd(x)x^* + d(x)(x^*)^2 \in Z(R) \text{ for all } x \in R. \tag{2.9}$$

Replacing x by xh in (2.9), where $h \in H(R) \cap Z(R)$, we obtain

$$(x^3 - 2x^2x^* + x(x^*)^2)d(h)h^2 \in Z(R) \text{ for all } x \in R.$$

Then by the primeness of R we are force to conclude that either $d(h)h^2 = 0$ or $x^3 - 2x^2x^* + x(x^*)^2 \in Z(R)$ for all $x \in R$. First we consider the case

$$x^3 - 2x^2x^* + x(x^*)^2 \in Z(R) \text{ for all } x \in R. \tag{2.10}$$

Substituting kx for x in (2.10), where $k \in S(R) \cap Z(R)$, we get $(x^3 + 2x^2x^* + x(x^*)^2)k^3 \in Z(R)$ for all $x \in R$. This further implies that

$$x^3 + 2x^2x^* + x(x^*)^2 \in Z(R) \text{ for all } x \in R. \tag{2.11}$$

Subtracting (2.10) from (2.11) and using 2-torsion freeness of R, we obtain $x^2x^* \in Z(R)$ for all $x \in R$. Therefore, by Lemma 2.1, R is commutative. Now consider the second case $d(h)h^2 = 0$ for all $h \in H(R) \cap Z(R)$. This implies that $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. Therefore, $d(k) = 0$ for all $k \in S(R) \cap Z(R)$. Replacing x by kx in (2.9), where $k \in S(R) \cap Z(R)$ and using the fact that $d(k) = 0$, we obtain

$$x^2 d(x)k^3 + 2xd(x)x^*k^3 + d(x)(x^*)^2k^3 \in Z(R) \text{ for all } x \in R. \tag{2.12}$$

Application of (2.9) yields $4xd(x)x^*k^3 \in Z(R)$ for all $x \in R$. Since R is 2-torsion free ring and $S(R) \cap Z(R) \neq (0)$, we get

$$xd(x)x^* \in Z(R) \text{ for all } x \in R. \tag{2.13}$$

Putting $x + h$ in place of x, where $h \in H(R) \cap Z(R)$, we arrive at

$$xd(x)h + d(x)x^*h + d(x)h^2 \in Z(R) \text{ for all } x \in R. \tag{2.14}$$

Taking $x = -x$ in (2.14) and then combining it with the obtained relation, we get $2d(x)h^2 \in Z(R)$. This implies that $d(x) \in Z(R)$ for all $x \in R$, since the involution $'*'$ is of the second kind. Hence, by Posner's [4] first theorem, R is commutative. $\square$

Theorem 2.4. *Let R be a 2-torsion free prime ring with involution $'*'$ of the second kind. If R admits a nonzero derivation d such that $d({}_*[x,x^*]_n) \in Z(R)$ for all $x \in R$ $(n = 1, 2)$, then R is commutative.*

proof. **Case** (i) Firstly we are focus to discuss the case when $n = 1$ i.e.,

$$d({}_*[x,x^*]) \in Z(R) \quad \text{for all} \quad x \in R.$$

Linearizing this, we get

$$d({}_*[x,y^*]) + d(*[y,x^*]) \in Z(R) \quad \text{for all} \quad x, y \in R.$$

This implies that

(2.15)

$$d(x)[y^*,r]+[d(x),r]y^*+x[d(y^*),r]+[x,r]d(y^*)-d(y^*)[x^*,r]-[d(y^*),r]x^*-y^*[d(x^*),r]$$

$$-[y^*,r]d(x^*) + d(y)[x^*,r] + [d(y),r]x^* + y[d(x^*),r] + [y,r]d(x^*) - d(x^*)[y^*,r]-$$

$$[d(x^*),r]y^* - x^*[d(y^*),r] - [x^*,r]d(y^*) = 0 \quad \text{for all} \quad x, y, r \in R.$$

Replacing y by hy, where $h \in H(R) \cap Z(R)$ in (2.15), we obtain

$$(x[y^*,r] + [x,r]y^* - y^*[x^*,r] - [y^*,r]x^* + y[x^*,r]+$$

$$[y,r]x^* - x^*[y^*,r] - [x^*,r]y^*)d(h) = 0 \quad \text{for all} \quad x, y, r \in R.$$

Using the primeness of R, we have either $d(h) = 0$ or

(2.16)

$$x[y^*,r] + [x,r]y^* - y^*[x^*,r] - [y^*,r]x^* + y[x^*,r] + [y,r]x^* - x^*[y^*,r] - [x^*,r]y^* = 0$$

for all $x, y, r \in R$. We first consider the relation (2.16). Replacing y by ky, where $k \in S(R) \cap Z(R)$ in (2.16), we get $2(y[x^*,r] + [y,r]x^*)k = 0$ for all $x, y, r \in R$. Since R is 2-torsion free ring and $S(R) \cap Z(R) \neq (0)$, we obtain $y[x^*,r] + [y,r]x^* = 0$ for all $x, y, r \in R$. Taking $x = k$, where $k \in S(R) \cap Z(R)$, we get $-[y,r]k = 0$ for all $y, r \in R$. Thus $-[y,r] = 0$ for all $y, r \in R$. That is, R is commutative. Now consider $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. This implies that $d(k) = 0$ for all $k \in S(R) \cap Z(R)$. Replacing y by ky, where $k \in S(R) \cap Z(R)$ in (2.15) and making use of (2.15), we get

$$2(d(y)[x^*,r] + [d(y),r]x^* + y[d(x^*),r] + [y,r]d(x^*))k = 0 \quad \text{for all} \quad x, y, r \in R.$$

This implies that

$$d(y)[x^*,r] + [d(y),r]x^* + y[d(x^*),r] + [y,r]d(x^*) = 0 \quad \text{for all} \quad x, y, r \in R.$$

Taking $x = h$ where $h \in H(R) \cap Z(R)$ and using $d(h) = 0$, we arrive at $h[d(y),r] = 0$ for all $y, r \in R$. Then by the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain $[d(y),r] = 0$ for all $y, r \in R$. Hence by Posner's [4] first theorem, R is commutative.

Case (*ii*) Next, for $n=2$ we have

$$d({}_*[x,x^*]_2)\in Z(R) \text{ for all } x\in R.$$

On expansion we get

$$d(x^2x^*)-2d(x(x^*)^2)+d((x^*)^3)\in Z(R) \text{ for all } x\in R. \tag{2.17}$$

Linearization of (2.17) yields

$$\begin{aligned} &d(x^2y^*)+d(xyx^*)+d(xyy^*)+d(yxx^*)+d(yxy^*)+d(y^2x^*)-2d(xx^*y^*)-2d(xy^*x^*)\\ &-2d(x(y^*)^2)-2d(y(x^*)^2)-2d(yx^*y^*)-2d(yy^*x^*)+d((x^*)^2y^*)+d(x^*y^*x^*)\\ &+d(x^*(y^*)^2)+d(y^*(x^*)^2)+d(y^*x^*y^*)+d((y^*)^2x^*)\in Z(R) \text{ for all } x,y\in R. \end{aligned} \tag{2.18}$$

Substituting $-x$ for x in (2.18) and combining the obtained relation with (2.18), we obtain

$$\begin{aligned} &2(d(x^2y^*)+d(xyx^*)+d(yxx^*)-2d(xx^*y^*)-2d(xy^*x^*)-2d(y(x^*)^2)\\ &+d((x^*)^2y^*)+d(x^*y^*x^*)+d(y^*(x^*)^2))\in Z(R) \text{ for all } x,y\in R. \end{aligned}$$

Since R is 2-torsion free, the last relation gives

$$\begin{aligned} &d(x^2y^*)+d(xyx^*)+d(yxx^*)-2d(xx^*y^*)-2d(xy^*x^*)-2d(y(x^*)^2)\\ &+d((x^*)^2y^*)+d(x^*y^*x^*)+d(y^*(x^*)^2)\in Z(R) \text{ for all } x,y\in R. \end{aligned} \tag{2.19}$$

Replacing y by hy, where $h\in H(R)\cap Z(R)$ in (2.19) and intermix it with (2.19), we come to

$$\begin{aligned} &(x^2y^*+xyx^*+yxx^*-2xx^*y^*-2xy^*x^*-2y(x^*)^2\\ &+(x^*)^2y^*+x^*y^*x^*+y^*(x^*)^2)d(h)\in Z(R) \text{ for all } x,y\in R. \end{aligned}$$

By the primeness of the ring R, we get either $d(h)=0$ or

$$x^2y^*+xyx^*+yxx^*-2xx^*y^*-2xy^*x^*-2y(x^*)^2+(x^*)^2y^*+x^*y^*x^*+y^*(x^*)^2\in Z(R) \tag{2.20}$$

for all $x,y\in R$. Replacing y by ky, where $k\in S(R)\cap Z(R)$ in (2.20), we arrive at

$$2(xyx^*+yxx^*-2y(x^*)^2)\in Z(R) \text{ for all } x,y\in R.$$

This implies that

$$xyx^*+yxx^*-2y(x^*)^2\in Z(R) \text{ for all } x,y\in R.$$

Substituting kx for x in the last relation, we conclude that $y(x^*)^2\in Z(R)$ for all $x,y\in R$. Now proceed as we have already done in Lemma 2.1, we conclude that R is commutative. “Considering the second case in which we have $d(h)=0$ for all $h \in H(R) \cap Z(R)$." This implies that $d(k) = 0$ for all $k \in S(R) \cap Z(R)$. Now replacing x by kx in (2.19) and using "$d(k) = 0$, we get"

$$(-d(x^2y^*) + d(xyx^*) + d(yxx^*) + 2d(xx^*y^*) + 2d(xy^*x^*) - 2d(y(x^*)^2)$$

$$-d((x^*)^2y^*) - d(x^*y^*x^*) - d(y^*(x^*)^2))k \in Z(R) \text{ for all } x, y \in R.$$

Since $S(R) \cap Z(R) \neq (0)$, the last expression implies that

$$-d(x^2y^*) + d(xyx^*) + d(yxx^*) + 2d(xx^*y^*) + 2d(xy^*x^*) - 2d(y(x^*)^2)$$

$$-d((x^*)^2y^*) - d(x^*y^*x^*) - d(y^*(x^*)^2) \in Z(R) \text{ for all } x, y \in R.$$

Combining this with (2.19) and using the fact that R is 2-torsion free ring, we arrive at

$$d(xyx^*) + d(yxx^*) - 2d(y(x^*)^2) \in Z(R) \text{ for all } x, y \in R.$$

Replacing x by kx, where $k \in S(R) \cap Z(R)$ and combining it with previous expression, we obtain $2d(y(x^*)^2) \in Z(R)$ for all $x, y \in R$. Replacing x by h, where $h \in H(R) \cap Z(R)$ we come to $d(y)h^2 \in Z(R)$ for all $y \in R$. This implies that $d(y) \in Z(R)$ for all $y \in R$. Hence, R is commutative by Posner's [4] first theorem. □

Theorem 2.5. *Let R be a 2-torsion free prime ring with involution $'*'$ of the second kind. If R admits a derivation d such that ${}_*[x, d(x)]_2 +_* [x, x^*]_2 \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. By the hypothesis we assume that

$$(2.21) \qquad {}_*[x, d(x)]_2 +_* [x, x^*]_2 \in Z(R) \text{ for all } x \in R.$$

If we take $d = 0$. Then, application of Lemma 2.2, yields the required result. Now consider the case $d \neq 0$ and on expansion of (2.21), we get

$$(2.22) \quad x^2d(x) - 2xd(x)x^* + d(x)(x^*)^2 + x^2x^* - 2x(x^*)^2 + (x^*)^3 \in Z(R) \text{ for all } x \in R.$$

Replacing x by xh in (2.22), where $h \in H(R) \cap Z(R)$, we obtain $(x^3 - x^2x^* + x(x^*)^2)d(h)h^2 \in Z(R)$ for all $x \in R$. Now by the primeness of R we get either $x^3 - 2x^2x^* + x(x^*)^2 \in Z(R)$ for all $x \in R$ or $d(h)h^2 = 0$. Now, we suppose that

$$(2.23) \qquad x^3 - 2x^2x^* + x(x^*)^2 \in Z(R) \text{ for all } x \in R.$$

This is same as the relation (2.10) in Theorem 2.3 and hence we conclude that R is commutative. Now we consider the case $d(h)h^2 = 0$ for all $h \in H(R) \cap Z(R)$. Since R is prime ring, so we get $d(h) = 0$. This also implies that $d(k) = 0$ for all $k \in S(R) \cap Z(R)$. Replacing x by xk in (2.22) and combining with (2.22) we arrive at $(2xd(x)x^* - x^2x^* - (x^*)^3)k^3 \in Z(R)$ for all $x \in R$. Since $S(R) \cap Z(R) \neq (0)$, so by the primeness of R, we get

$$(2.24) \qquad 2xd(x)x^* - x^2x^* - (x^*)^3 \in Z(R) \text{ for all } x \in R.$$

Linearization of (2.24) gives
(2.25)
$2xd(x)y^* + 2xd(y)x^* + 2xd(y)y^* + 2yd(x)x^* + 2yd(x)y^* + 2yd(y)x^* - x^2y^* - xyx^*$

$-xyy^* - yxx^* - yxy^* - y^2x^* - (x^*)^2y^* - x^*y^*x^* - x^*(y^*)^2 - y^*(x^*)^2 - y^*x^*y^* - (y^*)^2x^*$

$\in Z(R)$ for all $x, y \in R$. Replacing x by $-x$ in (2.25) and combining the obtained relation with (2.25), we obtain

$2xd(x)y^* + 2xd(y)x^* + 2yd(x)x^* - x^2y^* - xyx^* - yxx^* - (x^*)^2y^* - x^*y^*x^* - y^*(x^*)^2$

$\in Z(R)$ for all $x, y \in R$. Taking $x = h$, where $h \in H(R) \cap Z(R)$, we get

$$(2d(y) - 4y^* - 2y)h^2 \in Z(R) \quad \text{for all} \quad y \in R.$$

The primeness of R yields that $d(y) - 2y^* - y \in Z(R)$ for all $y \in R$. Replacing y by ky and on solving, we get $y \in Z(R)$ for all $y \in R$. Hence, this conclude that R is commutative. □

Theorem 2.6. *Let R be a 2-torsion free prime ring with involution $'*'$ of the second kind. If R admit two distinct derivations d_1 and d_2 such that ${}_*[x, d_1(x)]_2 -_* [x, d_2(x)]_2 \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We assume that

$$(2.26) \qquad {}_*[x, d_1(x)]_2 -_* [x, d_2(x)]_2 \in Z(R) \quad \text{for all} \quad x \in R.$$

If either d_1 or d_2 is zero, then we get the required result by Theorem 2.3 above. Now consider both d_1, d_2 are non-zero. Expansion of (2.26) yields that

$$(2.27) \quad x^2d_1(x) - 2xd_1(x)x^* + d_1(x)(x^*)^2 - x^2d_2(x) + 2xd_2(x)x^* - d_2(x)(x^*)^2 \in Z(R)$$

for all $x \in R$. Replacing x by xh in (2.27), where $h \in H(R) \cap Z(R)$ and on simplifying with the help of (2.27), we get

$$(x^3 - 2x^2x^* + x(x^*)^2)(d_1(h) - d_2(h))h^2 \in Z(R) \quad \text{for all} \quad x \in R.$$

This implies either $x^3 - 2x^2x^* + x(x^*)^2 \in Z(R)$ for all $x \in R$ or $(d_1(h) - d_2(h))h^2 = 0$. If $x^3 - 2x^2x^* + x(x^*)^2 \in Z(R)$ for all $x \in R$, then by using the same steps as we have used after (2.10), we arrive at $x^2x^* \in Z(R)$ for all $x \in R$. Thus R is commutative, by Lemma 2.1. On the other hand, if $(d_1(h) - d_2(h))h^2 = 0$ for all $h \in H(R) \cap Z(R)$. Then we are force to conclude that $d_1(h) = d_2(h)$ and hence $d_1(k) = d_2(k)$ for all $k \in S(R) \cap Z(R)$. Replacing x by kx in (2.27), and combining with (2.27) by using the fact that $d_1(k) = d_2(k)$, we get

$$4(xd_1(x)x^* - xd_2(x)x^*)k^3 \in Z(R) \quad \text{for all} \quad x \in R.$$

Since R is 2-torsion free and $S(R) \cap Z(R) \neq (0)$, the last relation gives

$$(2.28) \qquad xd_1(x)x^* - xd_2(x)x^* \in Z(R) \quad \text{for all} \quad x \in R.$$

Linearizing (2.28), we obtain

(2.29)
$$xd_1(x)y^*+yd_1(x)x^*+yd_1(x)y^*+xd_1(y)x^*+xd_1(y)y^*+yd_1(y)x^*-xd_2(x)y^*-xd_2(y)x^*$$
$$-xd_2(y)y^*-yd_2(x)x^*-yd_2(x)y^*-yd_2(y)x^*\in Z(R)\ \text{ for all }\ x,y\in R.$$

Replacing x by $-x$ in (2.29) and combining the obtained result with (2.29), we get

$$2(xd_1(x)y^*+yd_1(x)x^*+xd_1(y)x^*-xd_2(x)y^*-xd_2(y)x^*-yd_2(x)x^*)\in Z(R)$$

for all $x,y\in R$. Since R is 2-torsion free ring, the above expression yields

$$xd_1(x)y^*+yd_1(x)x^*+xd_1(y)x^*-xd_2(x)y^*-xd_2(y)x^*-yd_2(x)x^*\in Z(R) \tag{2.30}$$

for all $x,y\in R$. Replacing x by kx in (2.30) and on solving with the help (2.30) and using the fact that $d_1(k)=d_2(k)$, we get $(xd_1(x)-xd_2(x))y^*\in Z(R)$ for all $x,y\in R$. Replacing y by h, where $h\in H(R)\cap Z(R)$. Then by the primeness of R and $S(R)\cap Z(R)\neq(0)$ condition force that $xd_1(x)-xd_2(x)\in Z(R)$ for all $x\in R$. Linearizing this we get $xd_1(y)+yd_1(x)-xd_2(y)-yd_2(x)\in Z(R)$ for all $x,y\in R$. Taking y by h where $h\in H(R)\cap Z(R)$ and using $d_1(h)=d_2(h)$, we obtain $d_1(x)-d_2(x)\in Z(R)$ for all $x\in R$. This can be further written as

$$[d_1(x),r]-[d_2(x),r]=0\ \text{ for all }\ x,r\in R. \tag{2.31}$$

Replacing x by xr in (2.31), we get $[x,r](d_1(r)-d_2(r))=0$ for all $x,r\in R$. Substitute xu for x in the last relation, we obtain $[x,r]u(d_1(r)-d_2(r))=0$ for all $x,r,u\in R$. Then by the primeness of R, for each fixed $r\in R$, we get either $[x,r]=0$ for all $x\in R$ or $d_1(r)-d_2(r)=0$. Define $A=\{r\in R\mid [x,r]=0$ for all $x\in R\}$ and $B=\{r\in R\mid d_1(r)-d_2(r)=0\}$. Clearly, A and B are additive subgroups of R whose union is R. Hence by Brauer's trick, either $A=R$ or $B=R$. If $A=R$, then $[x,r]=0$ for all $x,r\in R$. This implies that R is commutative. If $B=R$, then $d_1(r)=d_2(r)$ for all $r\in R$, which is a contradiction to our assumption. Hence, we conclude that R is commutative. □

Theorem 2.7. *Let R be a 2-torsion free prime ring with involution $'*'$ of the second kind. If R admit derivations d_1, d_2 such that at least one of them is nonzero and satisfies $d_1({}_*[x,x^*]_2)+_*[x,d_2(x^*)]_2\in Z(R)$ for all $x\in R$, then R is commutative.*

Proof. We are given that d_1 and d_2 are derivations of R such that

$$d_1({}_*[x,x^*]_2)+_*[x,d_2(x^*)]_2\in Z(R)\ \text{ for all }\ x\in R. \tag{2.32}$$

If d_2 is zero then by Theorem 2.4, we get R is commutative. If d_1 is zero then we have ${}_*[x,d_2(x^*)]_2\in Z(R)$ for all $x\in R$. Expansion of last relation gives

$$x^2d_2(x^*)-2xd_2(x^*)x^*+d_2(x^*)(x^*)^2\in Z(R)\ \text{ for all }\ x\in R. \tag{2.33}$$

Replacing x by hx, where $h\in H(R)\cap Z(R)$ in (2.33) and combining the obtained expression, we get ${}_*[x,x^*]_2d_2(h)h^2\in Z(R)$ for all $x\in R$. Now applying the primeness of the ring R, we get either ${}_*[x,x^*]_2\in R$ or $d(h)h^2=0$. If ${}_*[x,x^*]_2\in Z(R)$

for all $x \in R$, then by Lemma 2.2, we get R is commutative. Now consider the second case in which we have $d_2(h)h^2 = 0$ for all $h \in H(R) \cap Z(R)$. This implies that $d_2(h) = 0$, from here we get $d_2(k) = 0$ for all $k \in S(R) \cap Z(R)$. Replacing x by kx in (2.33) and using the fact that $d_2(k) = 0$, we get $4xd_2(x^*)x^*k^3 \in Z(R)$ for all $x \in R$. This implies that $xd_2(x^*)x^* \in Z(R)$ for all $x \in R$. Arguing as above after (2.13), we conclude that R is commutative.

Now consider the second case in which both d_1 and d_2 are nonzero. On expansion of (2.32), we have

(2.34)
$$d_1(x^2x^*) - 2d_1(x(x^*)^2) + d_1((x^*)^3) + x^2d_2(x^*) - 2xd_2(x^*)x^* + d_2(x^*)(x^*)^2 \in Z(R)$$

for all $x \in R$. Replacing x by hx, where $h \in H(R) \cap Z(R)$ in (2.34) and solving with the help of (2.34), we get

$$_*[x, x^*]_2(3d_1(h) + d_2(h))h^2 \in Z(R) \quad \text{for all} \quad x \in R.$$

By the primeness of the ring R, we get either $_*[x, x^*]_2 \in Z(R)$ for all $x \in R$ or $(3d_1(h)+d_2(h))h^2 = 0$ for all $h \in H(R) \cap Z(R)$. If $_*[x, x^*]_2 \in Z(R)$ for all $x \in R$, then by Lemma 2.2, we get R is commutative. Now consider the case $(3d_1(h)+d_2(h))h^2 = 0$. This implies that $d_2(h) = -3d_1(h)$ and hence $d_2(k) = -3d_1(k)$ for all $k \in S(R) \cap Z(R)$. Now substituting kx for x, where $k \in S(R) \cap Z(R)$ in (2.34) and combining the obtained result with (2.34), we get $4(d_1(x(x^*)^2) + xd_2(x^*)x^*)k^3 \in Z(R)$ for all $x \in R$. Since R is 2-torsion free ring and $S(R) \cap Z(R) \neq (0)$, then invoking the primeness of R we obtain $d_1(x(x^*)^2) + xd_2(x^*)x^* \in Z(R)$ for all $x \in Z(R)$. Linearization to the last expression gives

(2.35)
$$d_1(xx^*y^*)+d_1(xy^*x^*)+d_1(x(y^*)^2)+d_1(y(x^*)^2)+d_1(yx^*y^*)+d_1(yy^*x^*)+xd_2(x^*)y^*$$
$$+xd_2(y^*)x^*+xd_2(y^*)y^*+yd_2(x^*)x^*+yd_2(x^*)y^*+yd_2(y^*)x^* \in Z(R) \quad \text{for all} \ x, y \in R.$$

Replacing x by $-x$ in (2.35), we get

$$2(d_1(xx^*y^*)+d_1(xy^*x^*)+d_1(y(x^*)^2)+xd_2(x^*)y^*+xd_2(y^*)x^*+yd_2(x^*)x^*) \in Z(R)$$

for all $x, y \in R$. Since R is 2-torsion free ring, we get

(2.36)
$$d_1(xx^*y^*) + d_1(xy^*x^*) + d_1(y(x^*)^2) + xd_2(x^*)y^* + xd_2(y^*)x^* + yd_2(x^*)x^* \in Z(R)$$

for all $x, y \in R$. Substituting ky for y, where $k \in S(R) \cap Z(R)$ in (2.36) and combining with (2.36) with use of $d_2(k) = -3d_1(k)$, we arrive at

(2.37)
$$2d_1(y(x^*)^2)k + 2yd_2(x^*)x^*k - xx^*y^*d_1(k) + y(x^*)^2d_1(k) + 2xy^*x^*d_1(k) \in Z(R)$$

for all $x, y \in R$ and $k \in S(R) \cap Z(R)$. Substitute ky for y in (2.37) yields

$$2d_1(y(x^*)^2)k^2 + 2yd_2(x^*)x^*k^2 + xx^*y^*d_1(k)k + y(x^*)^2d_1(k)k \tag{2.38}$$

$$-2xy^*x^*d_1(k)k + 2y(x^*)^2d_1(k)k \in Z(R) \text{ for all } x, y \in R.$$

Subtracting (2.38) form (2.37), we get $(-2xx^*y^* + 4xy^*x^* - 2y(x^*)^2)d_1(k)k \in Z(R)$ for all $x, y \in R$. Since R is 2-torsion free prime ring and $S(R) \cap Z(R) \neq (0)$, the last expression forces that either $xx^*y^* - 2xy^*x^* + y(x^*)^2 \in Z(R)$ for all $x, y \in R$ or $d_1(k)k = 0$. Suppose

$$xx^*y^* - 2xy^*x^* + y(x^*)^2 \in Z(R) \text{ for all } x, y \in R. \tag{2.39}$$

Substituting ky for y, where $k \in S(R) \cap Z(R)$ in (2.39) and combining with (2.39), we get $2y(x^*)^2k \in Z(R)$ for all $x, y \in R$. Taking $x = k$, we obtain $2yk^3 \in Z(R)$ for all $y \in R$. Since R is 2-torsion free prime ring and $S(R) \cap Z(R) \neq (0)$, we conclude that R is commutative. Now consider the case in which we have $d_1(k)k = 0$ for all $k \in S(R) \cap Z(R)$. This implies that $d_1(k) = 0$ for all $k \in S(R) \cap Z(R)$. This further implies that $d_2(k) = 0$. Substitute k for x in (2.36), to get

$$-2d_1(y^*)k^2 + d_1(y)k^2 - d_2(y^*)k^2 \in Z(R) \text{ for all } y \in R. \tag{2.40}$$

Replacing y by ky, where $k \in S(R) \cap Z(R)$ in (2.40) and combining the obtained relation with (2.40), finally we get $2d_1(y)k^3 \in Z(R)$ for all $y \in R$. Since R is 2-torsion free ring and $S(R) \cap Z(R) \neq (0)$, we obtain $d_1(y) \in Z(R)$ for all $y \in R$. Hence, by Posner's [4] first theorem, R is commutative. □

As an immediate consequence of the above theorem, we get the following corollary:

Corollary 2.8. *Let R be a 2-torsion free prime ring with involution $'*'$ of the second kind. If R admits a nonzero derivation d such that $d({}_*[x, x^*]_2) +_* [x, d(x^*)]_2 \in Z(R)$ for all $x \in R$, then R is commutative.*

The following example shows that the second kind involution assumption is essential in Theorem 2.3 and Theorem 2.4.

Example 2.9. Let $R = \left\{ \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \,\middle|\, \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z} \right\}$. Of course, R with matrix addition and matrix multiplication is a noncommutative prime ring. Define mappings $*, d_1, d_2 : R \longrightarrow R$ such that $\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}^* = \begin{pmatrix} \beta_4 & -\beta_2 \\ -\beta_3 & \beta_1 \end{pmatrix}$, $d_1 \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} 0 & -\beta_2 \\ \beta_3 & 0 \end{pmatrix}$ and $d_2 \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} 0 & \beta_2 \\ -\beta_3 & 0 \end{pmatrix}$.
Obviously, $Z(R) = \left\{ \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1 \end{pmatrix} \,\middle|\, \beta_1 \in \mathbb{Z} \right\}$. Then $x^* = x$ for all $x \in Z(R)$, and hence $Z(R) \subseteq H(R)$, which shows that the involution $*$ is of the first kind. Moreover, d_1 and d_2 are nonzero derivations of R such that ${}_*[x, d_1(x)]_2 \in Z(R)$ and ${}_*[x, d_1(x)]_2 -_* [x, d_2(x)]_2 \in Z(R)$ for all $x \in R$. However, R is not commutative. Hence, the hypothesis of second kind involution is crucial in Theorems 2.3 & 2.4
Our next example shows that Theorems 2.3 and 2.4 are not true for semiprime rings.

Example 2.10. Let $S = R \times \mathbb{C}$, where R is same as in Example 2.9 with involution $'*'$ and derivations d_1 and d_2 same as in above example, $\mathbb{C}$ is the ring of complex numbers with conjugate involution τ. Hence, S is a 2-torsion free noncommutative semiprime ring. Now define an involution α on S, as $(x, y)^\alpha = (x^*, y^\tau)$. Clearly, α is an involution of the second kind. Further, we define the mappings D_1 and D_2 from S to S such that $D_1(x, y) = (d_1(x), 0)$ and $D_2(x, y) = (d_2(x), 0)$ for all $(x, y) \in S$. It can be easily checked that D_1, D_2 are derivations on S and satisfying $\alpha[X, D_1(X)]_2 \in Z(S)$ and $\alpha[X, D_1(X)]_2 - \alpha[X, D_2(X)]_2 \in Z(S)$ for all $X \in S$, but S is not commutative. Hence, in Theorems 2.3 & 2.4, the hypothesis of primeness is essential.

We conclude the paper with the following Conjectures.

Conjecture 2.11. Let $n > 2$ be an integer, R be a prime ring with involution $'*'$ of the second kind and with suitable torsion restrictions on R. Next, let d be a nonzero derivation on R such that ${}_*[x, d(x)]_n \in Z(R)$ for all $x \in R$. Then what we can say about the structure of R or the form of d?

Conjecture 2.12. Let $n > 2$ be an integer, R be a prime ring with involution $'*'$ of the second kind and with suitable torsion restrictions on R. Next, let d be a nonzero derivation on R such that $d({}_*[x, x^*]_n) \in Z(R)$ for all $x \in R$. Then what we can say about the structure of R or the form of d?

Conjecture 2.13. Let $n > 2$ be an integer, R be a prime ring with involution $'*'$ of the second kind and with suitable torsion restrictions on R. Next, let d be a nonzero derivation on R such that $d({}_*[x, x^*]_n) +_* [x, d(x^*)]_n \in Z(R)$ for all $x \in R$. Then what we can say about the structure of R or the form of d?

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