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HANKEL DETERMINANT PROBLEMS FOR CERTAIN SUBCLASSES OF SAKAGUCHI TYPE FUNCTIONS DEFINED WITH SUBORDINATION

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Abstract. The present investigation is concerned with the estimation of initial coefficients, Fekete-Szegö inequality, second Hankel determinants, Zalcman functionals and third Hankel determinants for certain subclasses of Sakaguchi type functions defined with subordination in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. The results derived in this paper will pave the way for the further study in this direction.

1. Introduction

Let $\mathcal{A}$, be the class of all analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ defined in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0)-1 = 0$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$, consisting of functions which are univalent in E. For two analytic functions f and g in E, f is said to be subordinate to g (symbolically $f \prec g$) if there exists a function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. Further, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

A very famous result in the theory of univalent functions was Bieberbach's conjecture established by Bieberbach [5]. According to this conjecture, for $f \in \mathcal{S}$, $|a_n| \leq n$, $n = 2, 3, ...$ This conjecture remained as a challenge for the mathematicians for a long time. Finally, it was L. De-Branges [7], who proved this conjecture in 1985. During the course of proving this conjecture, various results related to the coefficients were established and some new subclasses of $\mathcal{S}$ were developed. The well-known classes of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$, respectively.

Sakaguchi [21] introduced the class $\mathcal{S}_s^*$ consisting of analytic functions $f \in \mathcal{A}$ and satisfying the condition

$$Re\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > 0 \text{ or } \frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1+z}{1-z}.$$

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The functions in the class $\mathcal{S}_s^*$ are known as the starlike functions with respect to symmetric points.

Later on, Das and Singh [6] introduced the class $\mathcal{K}_s$ consisting of analytic functions $f \in \mathcal{A}$, known as convex functions with respect to symmetric points, which satisfy the condition

$$Re\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right) > 0 \text{ or } \frac{2(zf'(z))'}{(f(z)-f(-z))'} \prec \frac{1+z}{1-z}.$$

Clearly, $f \in \mathcal{K}_s$ if and only if $zf' \in \mathcal{S}_s^*$.

Sokol and Stankiewicz [25] introduced the class $\mathcal{SL}^*$ consisting of analytic functions $f \in \mathcal{A}$ and satisfying the condition

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1 \text{ or } \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}.$$

The superordinate function $\sqrt{1+z}$ maps the unit disc E onto the right-half of the lemniscate of Bernoulli which has the equation

$$(x^2+y^2)^2 - 2(x^2-y^2) = 0.$$

From time to time, various subclasses of $\mathcal{S}$ were studied by subordinating to the function $\sqrt{1+z}$ by various researchers including, Najafzadeh et al. [17], Singh et al. [24], Ali et al. [1], Sokol and Thomas [26] and Ullah et al. [27]. Getting inspired from these works, now we define the following subclasses of Sakaguchi type functions by subordinating to $\sqrt{1+z}$.

Let $\mathcal{S}_s^*(\mathcal{L})$ denote the class which consists of analytic functions $f \in \mathcal{A}$ satisfying the condition

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) \prec \sqrt{1+z}.$$

By $\mathcal{K}_s(\mathcal{L})$, we denote the class which consists of analytic functions $f \in \mathcal{A}$ satisfying the condition

$$\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right) \prec \sqrt{1+z}.$$

For the complex sequence $a_n, a_{n+1}, a_{n+2}, ...,$, the $q \times q$ Hankel matrix, named after Hermann Hankel (1839-1873), is defined as

$$\begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{pmatrix} \text{ where } q \in \mathbb{N} - \{1\}.$$

We observe that the Hankel matrix has constant positive slopping diagonals whose entries also satisfy:

$$a_{ij} = a_{i-1,j+1} (i \in \mathbb{N} - \{1\}; j \in \mathbb{N}).$$

For basic properties of the hankel matrix, we refer to [9, 10]. In 1976, Noonan and Thomas [18] stated the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Particularly, for $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$, the Hankel determinant simplifies respectively to

$$H_2(1) = a_3 - a_2^2 \text{ and } H_2(2) = a_2a_4 - a_3^2.$$

Easily, one can observe that the Fekete-Szegö functional is $H_2(1)$. Fekete and Szegö [8] then further generalised the estimate $|a_3 - \mu a_2^2|$ where μ is real and $f \in \mathcal{S}$. Also $H_2(2)$ is called the second Hankel determinant.

The functional $J_{n,m}(f) = a_n a_m - a_{m+n-1}$, $n, m \in \mathbb{N} - \{1\}$, was investigated by Ma [15] and it is known as generalized Zalcman functional. The functional $J_{2,3}(f) = a_2a_3 - a_4$ is a specific case of the generalized Zalcman functional. Various authors computed the upper bound for the functional $J_{2,3}(f)$ over different subclasses of analytic functions.

The Hankel determinant in the case $q = 3, n = 1$,

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

is called the Third Hankel determinant.
For $f \in \mathcal{S}$ and $a_1 = 1$, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

By using the triangle inequality, the above equation yields

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|... \tag{1}$$

For various subclasses of $\mathcal{S}$, the second Hankel determinants has been extensively studied by various authors including Mehrok and Singh [16], Janteng et al. [11] and many others, while the third Hankel determinants were studied by the authors including Babalola [4], Shanmugam et al. [22], Altinkaya and Yalcin [2] and Singh and Singh [23].

In the present paper, we study Fekete-Szegö inequalities, Second Hankel determinants, Zalcman functionals and third Hankel determinants for the classes $\mathcal{S}_s^*(\mathcal{L})$ and $\mathcal{K}_s(\mathcal{L})$.
To prove our result, we shall make use of the following lemmas:

By $\mathcal{P}$, we denote the class of analytic functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

whose real parts are positive in E.

LEMMA 1.1. [12, 19] *If $p \in \mathcal{P}$, then*

$$|p_k| \leq 2, k \in \mathbb{N}..., \tag{2}$$

$$\left|p_2 - \frac{p_1^2}{2}\right| \leq 2 - \frac{|p_1|^2}{2}..., \tag{3}$$

$$|p_{i+j} - \mu p_i p_j| \leq 2, 0 \leq \mu \leq 1..., \tag{4}$$

and for complex number ρ, we have

$$|p_2 - \rho p_1^2| \leq 2max\{1, |2\rho - 1|\}... \tag{5}$$

LEMMA 1.2. [13, 14] *If $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for $|x| \leq 1$ and $|z| \leq 1$.

LEMMA 1.3. [3] *Let $p \in \mathcal{P}$, then*

$$|Jp_1^3 - Kp_1p_2 + Lp_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|.$$

In particular, it is proved in [19] that

$$|p_1^3 - 2p_1p_2 + p_3| \leq 2.$$

LEMMA 1.4. [20] *Let m, n, l and r satisfy the inequalities $0 < m < 1$, $0 < r < 1$ and*

$$8r(1-r)\left[(mn - 2l)^2 + (m(r + m) - n)^2\right] + m(1-m)(n-2rm)^2 \leq 4m^2(1-m)^2r(1-r).$$

If $p \in \mathcal{P}$, then

$$\left|lp_1^4 + rp_2^2 + 2mp_1p_3 - \frac{3}{2}np_1^2p_2 - p_4\right| \leq 2.$$

2. The class $\mathcal{S}_s^*(\mathcal{L})$

THEOREM 2.1. *If $f \in \mathcal{S}_s^*(\mathcal{L})$, then*

$$|a_2| \leq \frac{1}{4}..., \tag{6}$$

$$|a_3| \leq \frac{1}{4}..., \tag{7}$$

$$|a_4| \leq \frac{1}{8}..., \tag{8}$$

and

$$|a_5| \leq \frac{1}{8}... \tag{9}$$

Proof. As $f \in \mathcal{S}_s^*(\mathcal{L})$, by the principle of subordination, we have

$$\frac{2zf'(z)}{f(z)-f(-z)} = \sqrt{1+w(z)}... \tag{10}$$

Define $p(z) = \dfrac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$, which implies $w(z) = \dfrac{p(z)-1}{p(z)+1}$.
Also $\dfrac{2zf'(z)}{f(z)-f(-z)} = 1 + 2a_2 z + 2a_3 z^2 + (4a_4 - 2a_2a_3)z^3 + (4a_5 - 2a_3^2)z^4 + ...$
Again $\sqrt{1+w(z)} = \left(\dfrac{2p(z)}{p(z)+1}\right)^{\frac{1}{2}} = 1 + \dfrac{1}{4}p_1 z + \left(\dfrac{p_2}{4} - \dfrac{5p_1^2}{32}\right) z^2$
$+ \left(\dfrac{13p_1^3}{128} - \dfrac{5p_1p_2}{16} + \dfrac{p_3}{4}\right) z^3 + \left(\dfrac{-141p_1^4}{2048} + \dfrac{39p_1^2p_2}{128} - \dfrac{5p_3p_1}{16} - \dfrac{5p_2^2}{32} + \dfrac{p_4}{4}\right) z^4 + ...$
On comparing (10), it yields

$$1 + 2a_2 z + 2a_3 z^2 + (4a_4 - 2a_2a_3)z^3 + (4a_5 - 2a_3^2)z^4 + ... = 1 + \frac{1}{4}p_1 z + \left(\frac{p_2}{4} - \frac{5p_1^2}{32}\right) z^2$$
$$+ \left(\frac{13p_1^3}{128} - \frac{5p_1p_2}{16} + \frac{p_3}{4}\right) z^3 + \left(\frac{-141p_1^4}{2048} + \frac{39p_1^2p_2}{128} - \frac{5p_3p_1}{16} - \frac{5p_2^2}{32} + \frac{p_4}{4}\right) z^4 + ... \tag{11}$$

Equating the coefficients of z, z^2, z^3 and z^4 in (11) and on simplifying, we obtain

$$a_2 = \frac{p_1}{8}..., \tag{12}$$

$$a_3 = \frac{p_2}{8} - \frac{5p_1^2}{64}..., \tag{13}$$

$$a_4 = \frac{21p_1^3}{1024} - \frac{9p_1p_2}{128} + \frac{p_3}{16}..., \tag{14}$$

and

$$a_5 = -\frac{1}{16}\left[\frac{29}{128}p_1^4 + \frac{p_2^2}{2} + \frac{5p_3p_1}{4} - \frac{34}{32}p_1^2p_2 - p_4\right]... \tag{15}$$

Using (2) in (12), it gives (6).
From (13), we have

$$|a_3| = \frac{1}{8}\left|p_2 - \frac{5}{8}p_1^2\right|.$$

By using (5), it yields

$$|a_3| \le \frac{1}{8}.2max\left\{1, \frac{1}{4}\right\},$$

which gives (7).
On applying Lemma 3 in (14), (8) can be easily obtained.
Using Lemma 4 in (15), the result (9) is obvious.

□

Theorem 2.2. *If $f \in \mathcal{S}_s^*(\mathcal{L})$, then*

$$|a_3 - \rho a_2^2| \le \frac{1}{4}max\left\{1, \left|\frac{1+\rho}{4}\right|\right\}... \tag{16}$$

Proof. From (12) and (13), we have

$$|a_3 - \rho a_2^2| = \frac{1}{8}\left|p_2 - \left(\frac{5}{8} + \frac{1}{8}\rho\right)p_1^2\right|... \tag{17}$$

An application of (5) in (17) leads us to (16).
For $\rho = 1$, (16) gives

$$|a_3 - a_2^2| \leq \frac{1}{4}... \tag{18}$$

□

THEOREM 2.3. *If $f \in \mathcal{S}_s^*(\mathcal{L})$, then*

$$|a_2 a_3 - a_4| \leq \frac{1}{8}... \tag{19}$$

Proof. From (12),(13) and (14), we have

$$|a_2 a_3 - a_4| = \left|\frac{31}{1024}p_1^3 - \frac{11}{128}p_1 p_2 + \frac{1}{16}p_3\right|... \tag{20}$$

By implementing the triangle inequality and Lemma 3 in (20), it leads us to (19).

□

THEOREM 2.4. *If $f \in \mathcal{S}_s^*(\mathcal{L})$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{1}{16}... \tag{21}$$

The bound is sharp.

Proof. Using (12), (13) and (14), we have

$$|a_2 a_4 - a_3^2| = \frac{1}{8192}\left|-29p_1^4 + 88p_1^2 p_2 + 64p_1 p_3 - 128p_2^2\right|.$$

Substituting for p_2 and p_3 from Lemma 2 and letting $p_1 = p$, we get

$$|a_2 a_4 - a_3^2| = \frac{1}{8192}\left|-p^4 + 12p^2(4-p^2)x + 32p(4-p^2)(1-|x|^2)z - 16(4-p^2)(8-p^2)x^2\right|.$$

Since $|p| = |p_1| \leq 2$, by using (2), we may assume that $p \in [0, 2]$. Then by using triangle inequality and $|z| \leq 1$ with $|x| = t \in [0, 1]$, we obtain

$$|a_2 a_4 - a_3^2| \leq \frac{1}{8192}\left[p^4 + 32p(4-p^2) + 12p^2(4-p^2)t + 16(4-p^2)(8-2p-p^2)t^2\right] = F(p,t).$$

Then

$$\frac{\partial F}{\partial t} = \frac{1}{8192}\left[12(4-p^2)p^2 + 32t(4-p^2)(8-2p-p^2)\right] \geq 0.$$

Therefore $F(p,t)$ is an increasing function of t.
So $max.F(p,t) = F(p,1) = \frac{1}{8192}[5p^4 - 144p^2 + 512] = G(p)$.
Now $G'(p) = 0$ gives $p = 0$. Also $G''(p) = \frac{1}{8192}[60p^2 - 288]$ which is negative for each $p \in [0, 2]$.
This implies $max.G(p) = G(0) = \frac{512}{8192}$.

Hence $|a_2a_4 - a_3^2| \leq \frac{1}{16}$.
The result is sharp for $p_1 = 0$, $p_2 = \pm 2$ and $p_3 = 0$.

□

THEOREM 2.5. *If* $f \in \mathcal{S}_s^*(\mathcal{L})$, *then*

(22) $$|H_3(1)| \leq \frac{1}{16}...$$

Proof. By using (7), (8), (9), (18), (19) and (21) in (1), the result (22) is obvious.

□

3. The class $\mathcal{K}_s(\mathcal{L})$

THEOREM 3.1. *If* $f(z) \in \mathcal{K}_s(\mathcal{L})$, *then*

(23) $$|a_2| \leq \frac{1}{8}...,$$

(24) $$|a_3| \leq \frac{1}{12}...,$$

(25) $$|a_4| \leq \frac{1}{32}...$$

and

(26) $$|a_5| \leq \frac{1}{40}...$$

Proof. As $f \in \mathcal{K}_s(\mathcal{L})$, therefore

(27) $$\frac{2(zf'(z))'}{(f(z) - f(-z))'} = \sqrt{1 + w(z)}...$$

On expanding as in Theorem 2.1, (27) yields

$$1+4a_2z+6a_3z^2+(16a_4-12a_2a_3)z^3+(20a_5-18a_3^2)z^4+... = 1+\frac{1}{4}p_1z+\left(\frac{p_2}{4}-\frac{5p_1^2}{32}\right)z^2$$

(28) $$+\left(\frac{13p_1^3}{128}-\frac{5p_1p_2}{16}+\frac{p_3}{4}\right)z^3+\left(\frac{-141p_1^4}{2048}+\frac{39p_1^2p_2}{128}-\frac{5p_3p_1}{16}-\frac{5p_2^2}{32}+\frac{p_4}{4}\right)z^4+...$$

Equating the coefficients of z, z^2, z^3 and z^4 in (28) and on simplifying, we obtain

(29) $$a_2 = \frac{p_1}{16}...,$$

(30) $$a_3 = \frac{p_2}{24} - \frac{5p_1^2}{192}...,$$

(31) $$a_4 = \frac{1}{4096}[21p_1^3 - 72p_1p_2 + 64p_3]...,$$

and

(32) $$a_5 = \frac{1}{80}\left[-\frac{29}{128}p_1^4 + \frac{17}{16}p_1^2p_2 - \frac{5p_3p_1}{4} - \frac{p_2^2}{2} + p_4\right]...$$

By using Lemma 1, Lemma 3 and Lemma 4 in (29), (30), (31) and (32) and following the procedure of Theorem 2.1, the results (23), (24), (25) and (26) can be easily obtained.

□

THEOREM 3.2. *If $f \in \mathcal{K}_s(\mathcal{L})$, then*

$$|a_3 - \rho a_2^2| \leq \frac{1}{12} max \left\{1, \left|\frac{4+3\rho}{16}\right|\right\}... \tag{33}$$

Proof. From (29) and (30), we have

$$|a_3 - \rho a_2^2| = \frac{1}{24}\left|p_2 - \left(\frac{20+3\rho}{32}\right)p_1^2\right|... \tag{34}$$

By using (5) in (34), the result (33) is obvious.
For $\rho = 1$, (33) transforms to

$$|a_3 - a_2^2| \leq \frac{1}{12}... \tag{35}$$

□

THEOREM 3.3. *If $f \in \mathcal{K}_s(\mathcal{L})$, then*

$$|a_2 a_3 - a_4| \leq \frac{1}{32}... \tag{36}$$

Proof. By using (29), (30) and (31) and on the lines of Theorem 2.3, the proof is obvious.

□

THEOREM 3.4. *If $f \in \mathcal{K}_s(\mathcal{L})$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{1}{144}... \tag{37}$$

The result is sharp.

Proof. By using (29), (30) and (31) and following the procedure of Theorem 2.4, the result (37) can be easily obtained.
The bound is sharp for $p_1 = 0$, $p_2 = \pm 2$ and $p_3 = 0$.

□

THEOREM 3.5. *If $f \in \mathcal{K}_s(\mathcal{L})$, then*

$$|H_3(1)| \leq \frac{503}{138240}... \tag{38}$$

Proof. By using (24), (25), (26), (35), (36) and (37) in (1), it yields result (38).

□

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