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ANALYSIS OF NON-INTEGER ORDER THERMOELASTIC TEMPERATURE DISTRIBUTION AND THERMAL DEFLECTION OF THIN HOLLOW CIRCULAR DISK UNDER THE AXI-SYMMETRIC HEAT SUPPLY

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ABSTRACT. Analysis of non-integer order thermoelastic temperature distribution and it's thermal deflection of thin hollow circular disk under the axi-symmetric heat supply is investigated. Initially, the disk is kept at zero temperature. For $t > 0$ the parametric surfaces are thermally insulated and axi-symmetric heat supply on the thickness of the disk. The governing heat conduction equation has been solved by integral transform technique, including Mittag-Leffler function. The results have been computed numerically and illustrated graphically with the help of PTC-Mathcad.

1. INTRODUCTION

Biot [1] introduced the generalization of the classical coupled thermoelasticity theory. Lord and Shulman [2] introduced the generalized dynamical theory of thermoelasticity with one relaxation time, for the isotropic body. Ishihara et al. [3] discussed the transient thermo elastoplastic bending problems of circular plate making use of the strain increment theorem. Kar et al. [4] considered the generalized thermoelastic problem of a hollow sphere under thermal shock. Gaikwad and Ghadle [5] discussed the steady-state thermoelastic problem for a finite length hollow circular cylinder. Some contribution of this theory are given [6, 7, 8, 9, 12, 13]. The nonhomogeneous heat conduction problem of a thin hollow circular disk and its thermal deflection under heat generation was solved in [10].

The thermoelastic analysis and its deformation of a thin hollow circular disk subject to a partially distributed and axisymmetric heat supply on the upper surface studied in [11]. Studied the time-fractional heat conduction problem in a thin hollow circular disk and its thermal deflection in [14]. Analyzed the transient thermoelastic temperture distribution of a thin circular plate and its thermal deflection under uniform heat generation in [15]. discussed the

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transient thermoelastic stress analysis for thin circular plate due to uniform internal heat generation in [16]. Introduced the generalized theory of magneto-thermo-viscoelastic spherical cavity problem under fractional order derivative using the state space approach in [17]. Recently, many thermoelastic problems have been discussed [18, 19, 20, 21, 22].

2. Problem Formulation

We consider a thin, hollow circular disk of thickness h occupying space $D : a \leq r \leq b$, $-h/2 \leq z \leq h/2$ initially the disk is kept at zero temperature. For $t > 0$ the parametric surfaces are thermally insulated and axi-symmetric heat supply on the thickness of the disk.

A mathematical model is prepared considering analysis of non-integer order thermoelastic temperature distribution and it's deflection of thin hollow circular disk under the axi-symmetric heat supply using Caputo type time fractional heat conduction equation of order α.
The definition of Caputo type fractional derivative given by [23]

$$D^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int_0^t \dfrac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & \text{n-1} < \alpha < n; \\ \dfrac{df(t)}{dt}, & \text{n=1}. \end{cases}$$

For finding the Laplace transform, the Caputo derivative requires information of the initial values of the function $f(t)$ and its integer derivative of the order $k = 1, 2, ..., n-1$.

$$L\{D^{\alpha} f(t); s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \qquad n-1 < \alpha < n$$

. Also, the definition of Riemann-Liouville fractional derivative given by [23]

$$_aD_t^{\alpha} = \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha} f(\tau) d\tau, \quad n-1 < \alpha < n.$$

The temperature of the hollow circular disk $T(r, z, t)$ at time t satisfying the time fractional differential equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g(r,z,t)}{k_t} = \frac{1}{k}\frac{\partial^{\alpha} T}{\partial t^{\alpha}}, \tag{2.1}$$

with the boundary conditions,

$$kD_{RL}^{I-\alpha} \frac{\partial T}{\partial r} = 0, \qquad \text{at } r = a, \text{ for } t > 0, \tag{2.2}$$

$$kD_{RL}^{I-\alpha} \frac{\partial T}{\partial r} = 0, \qquad \text{at } r = b, \text{ for } t > 0, \tag{2.3}$$

$$kD_{RL}^{I-\alpha} \frac{\partial T}{\partial z} = Q_0 f(r,t), \qquad \text{at } z = h/2, \text{ for } t > 0, \tag{2.4}$$

$$kD_{RL}^{I-\alpha} \frac{\partial T}{\partial z} = 0, \qquad \text{at } z = -h/2, \text{ for } t > 0, \tag{2.5}$$

where, k, k_t are the thermal diffusivity and thermal conductivity, $D_{RL}^{-\alpha}T(r,\varphi,z,t)$ for $\alpha > 0$ is the Riemann-Liouville fractional integral $I^{\alpha}T(r,\varphi,z,t)$ and initial conditions,

$$T = 0, \qquad \text{at } t = 0, 0 < \alpha \leq 2\ , \tag{2.6}$$

$$\frac{\partial T}{\partial t} = 0, \qquad \text{at } t = 0, 1 < \alpha \leq 2\ . \tag{2.7}$$

The differential equation satisfied the deflection function $w(r,t)$ defined in [10] as

$$\nabla^2\nabla^2 w = -\frac{1}{(1-\nu)D}\nabla^2 M_T, \tag{2.8}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \tag{2.9}$$

and M_T is the thermal moment of the disk, ν is the Poisson's ratio of the disk material, D is the flexural rigidity of the disk denoted by

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{2.10}$$

The term M_T is defined as

$$M_T = a_t E \int_{-h/2}^{h/2} z\ T(r,z,t)dz, \tag{2.11}$$

a_t and E are the coefficients of the linear thermal expansion and the Young modulus respectively.

For the plane deformation, the boundary conditions are given as:

$$\frac{\partial w}{\partial r} = 0 \qquad \text{at } r = a \text{ and } r = b. \tag{2.12}$$

Initially, $T = w = 0$, at $t = 0$.
Equations (2.1) to (2.12) constitute the mathematical formulation of the problem under consideration.

3. Solution of the heat conduction problem

To obtain the expression for temperature function $T(r,z,t)$; Firstly we apply the finite Fourier transform and its inverse transform over the variable z in the range $h/2 \leq z \leq -h/2$. Secondly we apply finite Hankel transform and its inverse transform over the variable r in the range $a \leq r \leq b$ defined in [24] to Eq. (2.1) and also using the boundary and initial conation. Finally, we apply Laplace and their inverse, one obtains the expressions of the temperature $T(r,z,t)$ as:

$$T(r,z,t) = Q_0 \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} K(\eta_p,z)K_0(\beta_m,r)\frac{1}{k(\beta_m^2+\eta_p^2)}[t^{\alpha-1}E_{\alpha,\alpha}(-k(\beta_m^2+\eta_p^2)t^{\alpha})].b_{mp}. \tag{3.1}$$

where

$$b_{mp} = \left\{ \int_{-h/2}^{h/2} \int_{r'=a}^{b} K(\eta_p, z')r'.K_0(\beta_m, r').f(r', t').dr'dz' + \int_{t'=0}^{t} [1 - E_\alpha(-k(\beta_m^2 + \eta_p^2)t^\alpha)] \right.$$
$$\left. \times \left[\frac{k}{k_t} \int_{-h/2}^{h/2} \int_{r'=a}^{b} K(\eta_p, z')r'.K_0(\beta_m, r').g(r', z', t').dr'dz' \right] dt' \right\}.$$

4. DETERMINATION OF THERMAL DEFLECTION

Assume the solution of (2.8) satisfying conditions (2.12) as

$$w(r,t) = \sum_{m=1}^{\infty} C_m(t) \left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)} \right], \tag{4.1}$$

where β_1, β_2, $\beta_3, \ldots$ are the positive root of transcendental equation

$$\frac{J_0'(\beta a)}{J_0'(\beta b)} - \frac{Y_0'(\beta a)}{Y_0'(\beta b)} = 0.$$

It can be easily shown that

$$\frac{\partial w}{\partial r} = \sum_{m=1}^{\infty} C_m(t) \left[\frac{J_0'(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0'(\beta_m r)}{Y_0'(\beta_m b)} \right],$$
$$\frac{\partial w}{\partial r} = 0 \qquad \text{at } r = a \text{ and } r = b,$$

hence the solution (4.1) satisfies the Eq. (2.8).
Now,

$$\nabla^2 \nabla^2 w = \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right)^2 \sum_{m=1}^{\infty} C_m(t) \left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)} \right], \tag{4.2}$$

Using the well-known result

$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) J_0(\beta_m r) = -\beta_m^2 J_0(\beta_m r),$$
$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) Y_0(\beta_m r) = -\beta_m^2 Y_0(\beta_m r),$$

in Eq. (4.2), one obtains

$$\nabla^2 \nabla^2 w = \sum_{m=1}^{\infty} C_m(t) \beta_m^4 \left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)} \right].$$

Substituting Eq. (3.1) in Eq. (2.11), one obtains

$$M_T = -\sqrt{\frac{2}{\pi}} 2a_t E h \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{[\cos(\eta_p h/2) + 1]}{\eta_p^2} \frac{K_0(\beta_m, r)}{k(\beta_m^2 + \eta_p^2)} [t^{\alpha-1} E_{\alpha,\alpha}(-k(\beta_m^2 + \eta_p^2)t^\alpha)].b_{mp},$$

Now,

$$\nabla^2 M_T = -\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)\sqrt{\frac{2}{h}}2a_t Eh \sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{[\cos(\eta_p h/2)+1]}{\eta_p^2} \\ K_0(\beta_m, r)\frac{1}{k(\beta_m^2+\eta_p^2)}[t^{\alpha-1}E_{\alpha,\alpha}(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{mp}, \tag{4.3}$$

solving Eq. (4.3), one obtains

$$\nabla^2 M_T = -\sqrt{\frac{2}{\pi}}2a_t Eh \sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{[\cos(\eta_p h/2)+1]}{\eta_p^2}\frac{\beta_m^2 K_0(\beta_m, r)}{k(\beta_m^2+\eta_p^2)}[t^{\alpha-1}E_{\alpha,\alpha}(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{mp}, \tag{4.4}$$

Substituting Eq. (4.2) and (4.4) into Eq. (2.8), one obtains

$$\sum_{m=1}^{\infty} C_m(t)\beta_m^4\left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right] = -\sqrt{\frac{2}{\pi}}2a_t Eh\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{[\cos(\eta_p h/2)+1]}{\eta_p^2}\beta_m^2 \\ \frac{\pi}{\sqrt{2}}\frac{\beta_m J_0'(\beta_m b).Y_0'(\beta_m b)}{\left[1-\dfrac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)}\right]^{1/2}}\left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right]\frac{1}{k(\beta_m^2+\eta_p^2)}[t^{\alpha-1}E_{\alpha,\alpha}(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{mp}, \tag{4.5}$$

Solving Eq. (4.5), one obtains

$$C_m(t) = -\sqrt{\frac{2}{\pi}}2a_t Eh\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{[\cos(\eta_p h/2)+1]}{\eta_p^2}\frac{1}{\beta_m^2} \\ \times\frac{\pi}{\sqrt{2}}\frac{\beta_m J_0'(\beta_m b).Y_0'(\beta_m b)}{\left[1-\dfrac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)}\right]^{1/2}}\frac{1}{k(\beta_m^2+\eta_p^2)}[t^{\alpha-1}E_{\alpha,\alpha}(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{mp}. \tag{4.6}$$

Finally, substituting Eq. (4.6) in Eq. (4.1), one obtains the expression for the thermal deflection $w(r,t)$ as

$$w(r,t) = -\sqrt{\frac{2}{\pi}}2a_t Eh\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{[\cos(\eta_p h/2)+1]}{\eta_p^2}\frac{1}{\beta_m^2}\frac{1}{k(\beta_m^2+\eta_p^2)} \\ \times\frac{\pi}{\sqrt{2}}\frac{\beta_m J_0'(\beta_m b).Y_0'(\beta_m b)}{\left[1-\dfrac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)}\right]^{1/2}}\left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right][t^{\alpha-1}E_{\alpha,\alpha}(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{mp}.$$

5. Special Case and Numerical Results

Set $f(r,t) = (r^2-a^2)^2(r^2-b^2)^2(1-e^{-At})$, where r is the radius measured in meter and $A > 0$.

The first five positive root of the transcendental equation $\dfrac{J_0'(\beta a)}{J_0'(\beta b)} - \dfrac{Y_0'(\beta a)}{Y_0'(\beta b)} = 0$ as defined in [24] are $\beta_1 = 3.1965$, $\beta_2 = 6.3123$, $\beta_3 = 9.4445$, $\beta_4 = 12.5812$, $\beta_5 = 15.7199$.
The cooper material was chosen for purpose of numerical calculation for a thin circular hollow disk. The numerical calculation and graphs has been carried out with help of computational mathematical software PTC Mathcad Prime-3.1.

Table 1: Material constants

ρ = 2707 kg/m^3	k = 84.18 m^2/s	k_t = 386 W/(m. K)
μ = 26.67	a_t = 16.5 $\times$ 10^{-6} 1/K	E=70 GPa
ν = 0.35 h = 0.1 m	a = 1 m b = 2 m	c_p = 896 J/kg.K

In this paper, we have obtained solutions to time fractional heat conduction equation of thin hollow circular disk and its thermal deflection with the help of Caputo time fractional derivative. It is assumed that, for $t > 0$ the heat is generated within the solid at rate $t =$ 200 W/m^3. The numerical calculation is carried out according to the values of parameters α reflecting the characteristic features of the solution for various orders of the time-fractional derivative. Their distinguishing values of the parameter α are considered, $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha \leq 2$ depicting weak, normal and strong conductivity. Figure 1 and Figure 2 shows the variation of temperature and thermal deflection in radial distance r at instants $\alpha = 0.50$ for time parameter $t = 0.25, 0.50, 0.75, 1$. Figure 1, indicate the variation of temperature in radial direction for the different time parameter. Due to the internal heat generation at a constant rate, initially the temperature is maximum at the inner circular edge ($r = 1$)and it decreases with increase the time, becomes zero towards the outer circular edge ($r = 2$). Figure 2, shows the variation of thermal deflection in radial direction for the different time parameter $t = 0.25, 0.50, 0.75, 1$. It is observed that the thermal deflection is maximum at the inner surface of the disk and it will be zero on the boundary surface. Figure 3 and Figure 4 indicates the variation of temperature and thermal deflection in radial direction at instants $t = 0.50$ for the different values of fractional order parameter $\alpha = 0.25, 0.50, 0.75, 1$.

Figure 3, shows the variation of temperature in radial direction for the parameter $\alpha = 0.25, 0.50, 0.75, 1$. It is clear that the temperature is maximum at upper surface of solid and decreases to zero at $r = 1.8$ then follows the uniform pattern. Figure 4, shows the variation of thermal deflection in the radial direction for the different values of fractional order parameters $\alpha = 0.25, 0.50, 0.75, 1$. It is observe that the due to the internal heat generation the thermal deflection is maximum at $r = 1.3$ and it will be zero on the boundary surface.

6. CONCLUSIONS

We investigate the temperature and thermal deflection in a thin, hollow circular disk in a theory of thermoelasticity based on fractional heat conduction with the Caputo time-fractional derivative of order $0 < \alpha < 2$. The present method is based on the direct method, using the

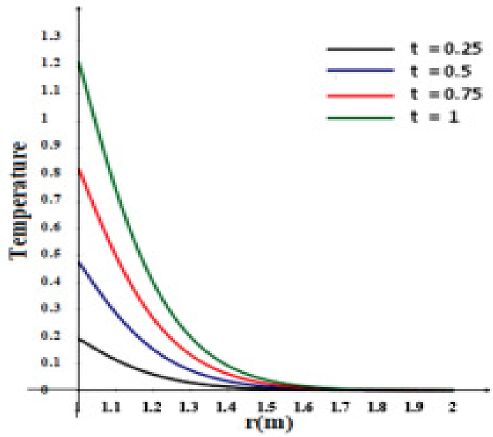

Figure 1: Temperature distribution at $\alpha = 0.5$ and different values of t.

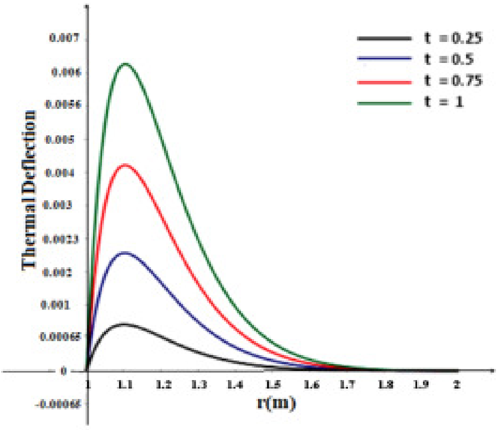

Figure 2: Thermal deflection at $\alpha = 0.5$ and different values of t.

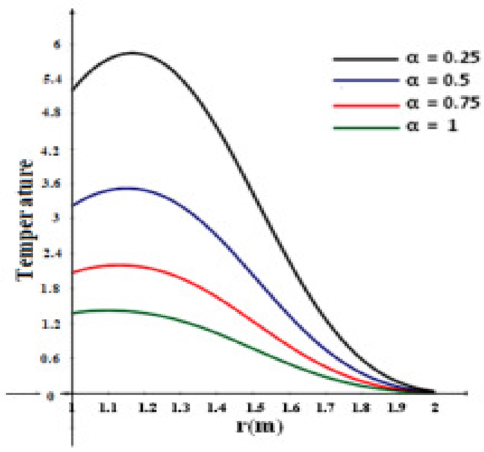

Figure 3: Temperature distribution at $t = 0.5$ and different values of α.

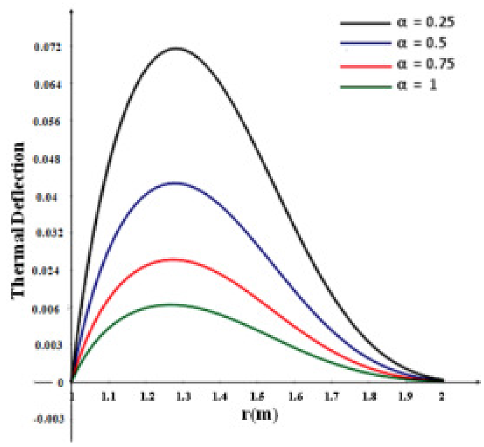

Figure 4: Thermal deflection at $t = 0.5$ and different values of α.

finite Hankel transform, the generalized finite Fourier transform and Laplace transform.The numerical results show the significant influence of the order of time derivative on the temperature and thermal deflection with radial coordinate. The fractional order parameter α within the range $0 < \alpha < 1$ and $1 < \alpha < 2$ represent the weak and strong conductivity, while $\alpha = 1$ represents the normal conductivity. The results presented here will be useful for researchers working in material sciences also useful in studying the thermal characteristics of various bodies in real-life engineering problems, mathematical biology by considering the time fractional derivative in the field equations.

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