East Asian Math. J.
Vol. 38 (2022), No. 1, pp. 043–063
http://dx.doi.org/10.7858/eamj.2022.005

STRUCTURE JACOBI OPERATORS OF SEMI-INVARINAT SUBMANIFOLDS IN A COMPLEX SPACE FORM II

U-Hang Ki and Soo Jin Kim*

Abstract. Let M be a semi-invariant submanifold of codimension 3 with almost contact metric structure (ϕ, ξ, η, g) in a complex space form $M_{n+1}(c)$. We denote by R_ξ the structure Jacobi operator with respect to the structure vector field ξ and by $\bar{r}$ the scalar curvature of M. Suppose that R_ξ is $\phi\nabla_\xi\xi$-parallel and at the same time the third fundamental form t satisfies $dt(X, Y) = 2\theta g(\phi X, Y)$ for a scalar $\theta(\neq 2c)$ and any vector fields X and Y on M.

In this paper, we prove that if it satisfies $R_\xi\phi = \phi R_\xi$, then M is a Hopf hypersurface of type (A) in $M_{n+1}(c)$ provided that $\bar{r} - 2(n-1)c \leq 0$.

1. Introduction

A submanifold M is called a *CR submanifold* of a Kaehlerian manifold $\tilde{M}$ with complex structure J if there exists a differentiable distribution $\triangle : p \to \triangle_p \subset T_pM$ on M such that $\triangle$ is J-invariant and the complementary orthogonal distribution $\triangle^\perp$ is totally real, where T_pM denotes the tangent space at each point p in M ([1], [35]). In particular, M is said to be a *semi-invariant submanifold* provided that $\dim\triangle^\perp = 1$. The unit normal in $J\triangle^\perp$ is called the *distinguished normal* to the semi-invariant submanifold ([4], [33]). In this case, M admits an almost contact metric structure (ϕ, ξ, η, g). A typical example of a semi-invariant submanfold is real hypersurfaces in a Kaehlerian manifold. And new examples of nontrivial semi-invariant submanifolds in a complex projective space $P_n\mathbb{C}$ are constructed in [22] and [30]. Accordingly, we may expect to generalize some results which are valid in a real hypersurface to a semi-invariant submanifold.

An n-dimensional complex space form $M_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature $4c$. As is well known, complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbb{C}$, or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0$ or $c < 0$.

Received November 29, 2021; Accepted December 31, 2021.
2010 *Mathematics Subject Classification.* 53B25, 53C40, 53C42.
Key words and phrases. Semi-invariant submanifold, almost contact metric structure, structure Jacobi operator, Hopf hypersurface, scalar curvature.
* Corresponding author.

(pISSN 1226-6973, eISSN 2287-2833)

For the real hypersurface of $M_n(c), c \neq 0$, many results are known ([6]∼[8], [24]∼[26], [31], [32], etc.). One of them, Takagi ([31], [32]) classified all the homogeneous real hypersurfaces of $P_n\mathbb{C}$ as six model spaces which are said to be A_1, A_2, B, C, D and E, and Cecil-Ryan ([5]) and Kimura ([23]) proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds when the structure vector field ξ is principal.

On the other hand, real hypersurfaces in $H_n\mathbb{C}$ have been investigated by Berndt [2], Berndt and Tamura [3], Montiel and Romero [21] and so on. Berndt [2] classified all real hypersurfaces with constant principal curvatures in $H_n\mathbb{C}$ and showed that they are realized as the tubes of constant radius over certain submanifolds. Also such kinds of tubes are said to be real hypersurfaces of type A_0, A_1, A_2 or type B.

Let M be a real hypersurface of type A_1 or type A_2 in a complex projective space $P_n\mathbb{C}$ or that of type A_0, A_1 or A_2 in a complex hyperbolic space $H_n\mathbb{C}$. Now, hereafter unless otherwise stated, such hypersurfaces are said to be of *type* (A) for our convenience sake.

Characterization problems for a real hypersurface of type (A) in a complex space form $M_n(c)$ were started by Okumura ([26]) for $c > 0$ and Montiel and Romero ([24]) for $c < 0$, respectively. They proved the following :

Theorem 1.1. *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$. If it satisfies $A\phi = \phi A$, then M is locally congruent to one of the following hypersurface :*

(I) in case that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,
- (A_1) a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,
- (A_2) a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, ..., n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$;

(II) in case that $M_n(c) = H_n\mathbb{C}$,
- (A_0) a horosphere,
- (A_1) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,
- (A_2) a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, ..., n-2\}$.

Denoting by R the curvature tensor of the submanifold, we define the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ with respect to the structure vector ξ. Then R_ξ is a self adjoint endomorphism on the tangent space of a CR submanifold.

Using several conditions on the structure Jacobi operator R_ξ, characterization problems for real hypersurfaces of type (A) have recently studied (cf. [7], [11], [18]). In the provious paper [7], Cho and one of the present authors gave another characterization of real hypersurface of type (A) in a complex projective space $P_n\mathbb{C}$. Namely they prove the following :

Theorem 1.2. *Let M be a connected real hypersurface of $P_n\mathbb{C}$. If it satisfies (1) $R_\xi A\phi = \phi A R_\xi$ or (2) $R_\xi\phi = \phi R_\xi, R_\xi A = A R_\xi$, then M is of type (A), where A denotes the shape operator of M.*

On the other hand, semi-invariant submanifolds of codimension 3 in a complex space form $M_{n+1}(c)$ have been studied in [9], [12]∼[15], [17], [19]∼[22] and so on by using properties of induced almost contact metric structure and those of the third fundamental form of the submanifold.

Now, let M be a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ such that the third fundamental form t satisfies $dt(X, Y) = 2\theta\omega$ for a scalar $\theta(\neq 2c)$, where $\omega(X, Y) = g(\phi X, Y)$ for any vector fields X and Y on M. We denote by A and S the shape operator in the direction of the distinguished normal and the Ricci tensor of M, respectively.

In the preceding work [22], it is proved that the submanifold M above is a Hopf hypersurface in $P_n\mathbb{C}$ provided that $A\xi = \alpha\xi$ and $\theta - 2c < 0$ for $c > 0$.

Further, Ki and Song ([21]) proved that if it satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $S\xi = g(S\xi, \xi)\xi$, then M is a Hopf hypersurface of type (A) in $M_n(c)$ provided that the scalar curvature $\bar{r}$ of M holds $\bar{r} - 2(n-1)c \leq 0$. This is a semi-invariant version of the main theorem stated in [18].

Moreover, one of the present authors and Kurihara [17] proved also that if it satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi A = AR_\xi$, then M is the same time type as above.

In this paper, we consider a semi-invariant submanifold of codimension 3 in $M_n(c)$, satisfying $R_\xi\phi = \phi R_\xi$ and that R_ξ is $\phi\nabla_\xi\xi$-parallel. In this case, we prove that M is a Hopf hypersurface of type (A) provided that the scalar curvature $\bar{r}$ of M holds $\bar{r} - 2(n-1)c \leq 0$.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the semi-invariant are supposed to be orientable.

2. Preliminaries

Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold equipped with parallel almost complex structure J and a Riemannian metric tensor G which is J-Hermitian. Let M be a real $(2n-1)$ -dimensional Riemannian manifold isometrically immersed in $\tilde{M}$ by the immersion $i : M \to \bar{M}$. In the sequel we identify $i(M)$ with M itself. We denote by g the Riemannian metric tensor on M from that of $\tilde{M}$.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation with respect to the metric tensor G on $\tilde{M}$ and by ∇ the one on M. Then the Gauss formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)C + g(KX, Y)D + g(LX, Y)E \tag{2.1}$$

for any vector fields X and Y tangent to M and any mutually orthogonal vectors C, D and E normal to M, where A, K and L are the *second fundamental forms* with respect to C, D and E respectively.

In the following we consider that M is a real $(2n-1)$-dimensional *semi-invariant submanifold* of codimension 3 in $\tilde{M}$ of real dimension $2(n+1)$. Then we can choose a local orthonormal frame field

$$\{e_1, \cdots, e_{n-1}, Je_1, \cdots, Je_{n-1}, e_0 = \xi, J\xi = C, D = JE, E\}$$

on the tangent bundle $T\tilde{M}$ such that $e_1, \cdots, e_{n-1}, Je_1, \cdots, Je_{n-1}, \xi \in TM$ and $C, D, E \in T^\perp M$, where $T^\perp M$ is the normal bundle (cf. [14], [17]). Then equations of Weingarten are also given by

$$\begin{aligned} \tilde{\nabla}_X C &= -AX + l(X)D + m(X)E, \\ \tilde{\nabla}_X D &= -KX - l(X)C + t(X)E, \\ \tilde{\nabla}_X E &= -LX - m(X)C - t(X)D \end{aligned} \tag{2.2}$$

because C, D and E are mutually orthogonal, where l, m and t being the *third fundamental forms.*

Now, let ϕ be the restriction of J on M, then we have

$$JX = \phi X + \eta(X)C, \quad \eta(X) = g(\xi, X), \quad JC = -\xi \tag{2.3}$$

for any vector field X on M ([34]). From this it is, using Hermitian property of J, verified that the aggregate (ϕ, ξ, η, g) is an *almost contact metric structure* on M, that is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\xi, X) = \eta(X),$$
$$\phi\xi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X and Y.

In the sequel, we denote the normal components of $\tilde{\nabla}_X C$ by $\nabla^\perp C$. The distinguished normal C is said to be *parallel* in the normal bundle if we have $\nabla^\perp C = 0$, that is, l and m vanish identically.

Using the Kaehler condition $\tilde{\nabla} J = 0$ and the Gauss and Weingarten formulas,we obtain from (2.3)

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \phi AX, \tag{2.5}$$

$$KX = \phi LX - m(X)\xi, \tag{2.6}$$

$$LX = -\phi KX + l(X)\xi \tag{2.7}$$

for any vectors X and Y on M. From the last two equations, we have

$$g(K\xi, X) = -m(X), \tag{2.8}$$

$$g(L\xi, X) = l(X). \tag{2.9}$$

Using the frame field $\{e_0 = \xi, e_1, \cdots, e_{n-1}, \phi e_1, \cdots, \phi e_{n-1}\}$ on M it follows from (2.6) $\sim$ (2.9) that

$$T_r K = \eta(K\xi) = -m(\xi), \quad T_r L = \eta(L\xi) = l(\xi), \tag{2.10}$$

where T_r means that the notation of trace.

Now, we retake D and E, there is no loss of generality such that we may assume $T_rL = 0$(cf. [17], [22]). So we have

$$l(\xi) = 0. \tag{2.11}$$

In what follows, to write our formulas in a convention form, we denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, $\gamma = \eta(A^3\xi)$, $T_rA = h$, $T_rK = k$, $T_r({}^tAA) = h_{(2)}$ and for a function f we denote by ∇f the gradient vector field of f.

From (2.10) we also have

$$m(\xi) = -k. \tag{2.12}$$

From (2.6) and (2.7) we get

$$\eta(X)l(\phi Y) - \eta(Y)l(\phi X) = m(Y)\eta(X) - m(X)\eta(Y),$$

which together with (2.12) gives

$$l(\phi X) = m(X) + k\eta(X), \tag{2.13}$$

which tells us, using (2.11), that

$$m(\phi X) = -l(X), \tag{2.14}$$

where we have used (2.9) and (2.11).

Taking the inner product with LY to (2.6) and using (2.9), we get

$$g(KLX, Y) + g(LKX, Y) = -\{l(X)m(Y) + l(Y)m(X)\}. \tag{2.15}$$

Now, we put $\nabla_\xi \xi = U$ in the sequel. Then U is orthogonal to ξ because of (2.5).

We put

$$A\xi = \alpha\xi + \mu W, \tag{2.16}$$

where W is a unit vector orthogonal to ξ. Then we have

$$U = \mu\phi W \tag{2.17}$$

by virtue of (2.5). Thus, W is also orthogonal to U. Further, we have

$$\mu^2 = \beta - \alpha^2. \tag{2.18}$$

From (2.16) and (2.17) we have

$$\phi U = -A\xi + \alpha\xi. \tag{2.19}$$

If we take account of (2.5), (2.10) and (2.19), then we find

$$g(\nabla_X \xi, U) = \mu g(AW, X). \tag{2.20}$$

Since W is orthogonal to ξ, we can, using (2.5) and (2.17), see that

$$\mu g(\nabla_X W, \xi) = g(AU, X). \tag{2.21}$$

Differentiating (2.19) covariantly along M and using (2.4) and (2.5), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX. \tag{2.22}$$

From now on we shall suppose that M is a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ and that the third fundamental form t satisfies

$$dt = 2\theta\omega, \quad \omega(X,Y) = g(\phi X, Y) \tag{2.23}$$

for any vector fields X and Y and a certain scalar θ, where d denotes the exterior differential operator. Then we can verify that (see [15], [17])

$$l = 0 \tag{2.24}$$

provided that $\theta - 2c \neq 0$ and hence

$$m(X) = -k\eta(X) \tag{2.25}$$

because of (2.13). Using these facts, (2.8) and (2.9) turn out respectively to

$$K\xi = k\xi, \quad L\xi = 0. \tag{2.26}$$

Because of (2.24) and (2.25), we can also write respectively (2.6) and (2.7) as

$$KX = \phi LX + k\eta(X)\xi, \tag{2.27}$$

$$L = -\phi K. \tag{2.28}$$

In the rest of this paper, we shall suppose that $\tilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature 4c, which called a *complex space form* and denote by $M_{n+1}(c)$. Then equations of the Gauss is given by

$$\begin{aligned} R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X \\ - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY \\ + g(KY,Z)KX - g(KX,Z)KY + g(LY,Z)LX - g(LX,Z)LY. \end{aligned} \tag{2.29}$$

If we take account of (2.24) and (2.25), then equations of the Codazzi and Ricci are given respectively by

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = k\{\eta(Y)LX - \eta(X)LY\} \\ + c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\}, \end{aligned} \tag{2.30}$$

$$(\nabla_X K)Y - (\nabla_Y K)X = t(X)LY - t(Y)LX, \tag{2.31}$$

$$(\nabla_X L)Y - (\nabla_Y L)X = k\{\eta(X)AY - \eta(Y)AX\} - t(X)KY + t(Y)KX, \tag{2.32}$$

$$g((KA-AK)X,Y)=k\{\eta(X)t(Y)-t(X)\eta(Y)\}, \tag{2.33}$$

$$\begin{gathered} LAX-ALX=(Xk)\xi-\eta(X)\nabla k+k(\phi AX+A\phi X), \\ g((LK-KL)X,Y)=-2(\theta-c)g(\phi X,Y), \end{gathered} \tag{2.34}$$

which together with (2.15) and (2.24) yields

$$g(LKX,Y)=-(\theta-c)g(\phi X,Y). \tag{2.35}$$

From (2.28) and this, we obtain

$$L^2X=(\theta-c)(X-\eta(X)\xi). \tag{2.36}$$

By properties of the almost contact metric structure we have from (2.35)

$$T_r({}^tKK)-\|K\xi\|^2+\|L\xi\|^2=2(n-1)(\theta-c),$$

where we have used (2.6), (2.9) and (2.10), which connected to (2.8) gives

$$\|K-m\otimes\xi\|^2+\|L\xi\|^2=2(n-1)(\theta-c). \tag{2.37}$$

In the same way, using (2.7), (2.11), (2.14), (2.35) we see that

$$\|m+k\xi\|^2-\|L\xi\|^2-T_r({}^tLL)=2(n-1)(\theta-c). \tag{2.38}$$

Differentiating (2.23) covariantly along M and making use of (2.4) and the first Bianchi identity, we find

$$(X\theta)\omega(Y,Z)+(Y\theta)\omega(Z,X)+(Z\theta)\omega(X,Y)=0,$$

which implies $(n-2)X\theta=0$. Therefore, θ is a constant if $n>2$.

For the case where $\theta=c$ in (2.23) we have $dt=2c\omega$. In this case, the normal connection M is said to be *L-flat* (see [27]).

Using (2.37) and (2.38) we can verify that the following lemma (see [15],[22]) :

Lemma 2.1. *Let M be a semi-invariant submanifold with L-flat normal connection in $M_{n+1}(c)$, $c\neq 0$. If $A\xi=\alpha\xi$, then we have $\nabla^{\perp}C=0$ and $K=L=0$ on M.*

Putting $X=\xi$ in (2.33) and using (2.26), we find

$$KA\xi=kA\xi+k\{t'-t(\xi)\xi\}, \tag{2.39}$$

where t' is the associated vector of the 1-form t.

If we apply this by ϕ and use (2.19), (2.26) and (2.28), then we get

$$g(KU,X)=k\{t(\phi X)-u(X)\}, \tag{2.40}$$

where $u(X)=g(U,X)$ for any vector field X.

Replacing X by ξ in (2.34) and using (2.5), (2.26) and (2.28), we get

$$KU = (\xi k)\xi - \nabla k + kU. \tag{2.41}$$

which together with (2.40) gives

$$Xk = (\xi k)\eta(X) + k\{2u(X) - t(\phi X)\}. \tag{2.42}$$

If we apply (2.34) by ϕ and take account of (2.27) and the last equation, then we find

$$\begin{aligned}\phi ALX - KAX = -k\{(t' - t(\xi)\xi)\eta(X) + 2\eta(X)(A\xi - \alpha\xi)\\ + 2g(A\xi, X)\xi - AX + \phi A\phi X\},\end{aligned}$$

or, using (2.26), (2.34) and (2.35) we have $\phi AL + LA\phi = 0$.

Since θ is constant if $n > 2$, differentiating (2.36) covariantly, we get

$$2L\nabla_X L = (c - \theta)\{\eta(X)\phi A + g(\phi A, X)\xi\},$$

or, using (2.26), (2.32), (2.34) and (2.35), it is verified that (see [19])

$$\begin{aligned}(\theta - c)(A\phi - \phi A)X + (k^2 + \theta - c)(u(X)\xi + \eta(X)U)\\ + k\{(AL + LA)X + k\{-t(\phi X)\xi + \eta(X)\phi\circ t\} = 0.\end{aligned} \tag{2.43}$$

Taking the trace of this, we obtain

$$kT_r(AL) = 0. \tag{2.44}$$

In the previous paper [17], [22] the following Lemma was proved.

Lemma 2.2. *If M satisfies $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ and $\mu = 0$ in $M_{n+1}(c)$, $c \neq 0$, then we have $k = 0$ on M.*

We set $\Omega = \{p \in M : k(p) \neq 0\}$, and suppose that Ω is not empty. In the rest of this paper, we discuss our arguments on the open subset Ω of M. So, by Lemma 2.2, we see that $\mu \neq 0$ on Ω.

3. Semi-invariant submanifolds satisfying $R_\xi\phi = \phi R_\xi$

We introduce the structure Jacobi operator R_ξ with respect to the structure vector field ξ which is defined by $R_\xi X = R(X, \xi)\xi$ for any vector field X. Then we have from (2.29)

$$\begin{aligned}R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + \eta(K\xi)KX - \eta(KX)K\xi\\ + \eta(L\xi)LX - \eta(LX)L\xi.\end{aligned}$$

Since l and m are dual 1-forms of $L\xi$ and $K\xi$ respectively because of (2.8) and (2.9), the last relationship is reformed as

$$R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX + m(X)K\xi - l(X)L\xi, \quad (3.1)$$

where we have used (2.8)∼(2.12).

We will continue now, our arguments under the same hypotheses $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ as in section 3. Then, by virtue of (2.25) and (2.26) we can write (3.1) as

$$R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX - k^2\eta(X)\xi. \quad (3.2)$$

In the next step suppose, throughout this paper, that $R_\xi\phi = \phi R_\xi$. Then from (3.2) we have

$$\alpha(\phi A - A\phi)X = g(A\xi, X)U + g(U, X)A\xi + 2kLX, \quad (3.3)$$

where we have used (2.25), (2.26) and (2.28).

Transforming this by A, and taking the trace obtained, we have $g(A^2\xi, U) = 0$ because of (2.44), which together with (2.16) yields

$$\mu g(AW, U) = 0. \quad (3.4)$$

Applying (3.3) by L and using (2.19), (2.27) and (2.34), we find

$$\begin{aligned} \alpha\{AKX - k\eta(X)A\xi - \phi ALX\} + g(LU, X)A\xi + g(KU, X)U& \\ = -2kL^2X,& \end{aligned} \quad (3.5)$$

which together with (2.33) and (2.40) yields

$$\begin{aligned} &k\alpha\{t(X)\xi - \eta(X)t' + g(A\xi, X)\xi - \eta(X)A\xi\} \\ &\quad + g(LU, X)A\xi - g(A\xi, X)LU - u(X)KU + g(KU, X)U = 0. \end{aligned}$$

If we take the inner product with ξ to this and use (2.26), then we get

$$k\alpha\{t(X) - t(\xi)\eta(X) + g(A\xi, X) - \alpha\eta(X)\} + \alpha g(LU, X) = 0. \quad (3.6)$$

Combining the last two equations and taking account of (2.18), we obtain

$$\mu\{w(X)LU - g(LU, X)W\} + u(X)KU - g(KU, X)U = 0, \quad (3.7)$$

where $w(X) = g(W, X)$ for any vector X.

Remark 1. $\alpha \neq 0$ on Ω.

In fact, if not, then we have $\alpha = 0$ on this subset. We discuss our arguments on such a place. So (3.3) reformed as

$$\mu\{w(X)U + u(X)W\} + 2kLX = 0 \quad (3.8)$$

because of (2.16) with $\alpha = 0$. Putting $X = U$ or W in this we have respectively

$$LU = -\frac{\mu\beta}{2k}W, \quad LW = -\frac{\mu}{2k}U \tag{3.9}$$

by virtue of (2.18) with $\alpha = 0$. Using this and (2.36), we can write (3.5) as

$$-\frac{\beta^2}{2k}w(X)W + g(KU, X)U = -2k(\theta - c)(X - \eta(X)\xi).$$

Taking the inner product with W to this, we obtain $\beta^2 = 4k^2(\theta - c)$.

On the other hand, combining (3.8) and (3.9) to (2.36) we also have $\beta^2 = 4(n-1)k^2(\theta - c)$, which implies $(n-2)(\theta - c)k = 0$, a contradiction because of our assumption and Lemma 2.1. Thus, $\alpha = 0$ is not impossible on Ω.

Now, putting $X = U$ in (3.6) and remembering Remark 1, we find $kt(U) + g(LU, U) = 0$.

By the way, replacing X by U in (3.3) and using (2.16) and (2.19), we find

$$\alpha(\phi AU + \mu AW) = \mu^2 A\xi + 2kLU.$$

If we take the inner product with U and make use of (3.4) and Lemma 2.2, then we obtain $g(LU, U) = 0$ and hence $t(U) = 0$.

By putting $X = U$ in (3.7), we then have

$$KU = \tau U, \tag{3.10}$$

where τ is given by $\tau\mu^2 = g(KU, U)$ by virtue of Lemma 2.2. Applying this by ϕ and using (2.28), we find

$$LU = \tau\mu W. \tag{3.11}$$

It is, using (3.10) and (3.11), seen that

$$\tau^2 = \theta - c. \tag{3.12}$$

because of (2.35).

Remark 2. $\Omega = \emptyset$ if $\theta = c$.

Since we have $\theta = c$, then (2.36) gives $L = 0$ and thus $KX = k\eta(X)\xi$ by virtue of (2.27). Hence, (2.32) reformed as

$$k\{\eta(X)AY - \eta(Y)AX + \eta(X)t(Y)\xi - t(X)\eta(Y)\xi\} = 0,$$

which shows $k(t(X) + g(A\xi, X) - \alpha'\eta(X)) = 0$, where we have put $\alpha' = \alpha + t(\xi)$. Thus, the last two equations imply that

$$AX = \eta(X)A\xi + g(A\xi, X)\xi - \alpha\eta(X)\xi.$$

Since U is orthogonal to ξ and W, it is clear that $AU = 0$ and $AW = \mu\xi$.

If we put $X = \mu W$ in (3.3) and remember (2.17) and the fact that $L = 0$, then we obtain $\mu^2 U = 0$ and hence $A\xi = \alpha\xi$. Owing to Lemma 2.1, we conclude that $k = 0$ and thus $\Omega = \emptyset$.

By Remark 2, we may only consider the case where $\tau \neq 0$ on Ω. Because of (2.16) and (3.11) we have

$$t(\phi X) = (1 + \frac{\tau}{k})g(U, X). \tag{3.13}$$

Therefore, it is clear that

$$t(X) = t(\xi)\eta(X) - \mu(1 + \frac{\tau}{k})w(X). \tag{3.14}$$

Using (2.16), we can write (2.39) as

$$\mu KW = k\mu W + k(t - t(\xi)\xi),$$

which together with (3.14) implies that

$$KW = -\tau W \tag{3.15}$$

because of Lemma 2.2.

If we take account of (2.43) and (3.13), then we find

$$\tau^2(A\phi X - \phi AX) + \tau(\tau - k)(u(X)\xi + \eta(X)U) + k(ALX + LAX) = 0. \tag{3.16}$$

Differentiating (3.10) covariantly along Ω, we find

$$(\nabla_X K)U + K\nabla_X U = \tau\nabla_X U,$$

which together with (2.31) and (3.11) yields

$$\begin{aligned}\mu\tau\{t(X)w(Y) - t(Y)w(X)\} + g(K\nabla_X U, Y) - g(K\nabla_Y U, X)\\ = \tau\{g(\nabla_X U, Y) - g(\nabla_Y U, X)\}.\end{aligned} \tag{3.17}$$

By the way, because of (2.5) and (2.20) and (2.22) we verify that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha - 2k(K\xi - k\xi),$$

which connected to (2.16) and (2.18) gives

$$\nabla_\xi U = 3\phi AU + \alpha\mu W - \mu^2\xi + \phi\nabla\alpha. \tag{3.18}$$

Replacing X by ξ in (3.17) and taking account of the last two relationships, we find

$$\begin{aligned}\mu^2(\tau - k)\xi + \mu\tau(t(\xi) - 2\alpha)W + \mu(k - \tau)AW\\ + 3(LAU - \tau\phi AU) = \tau\phi\nabla\alpha - L\nabla\alpha,\end{aligned} \tag{3.19}$$

where we have used the first equation of (2.20).

In a direct consequence of (2.28) and (3.10), we obtain

$$\mu LW = \tau U \tag{3.20}$$

because of $\mu \neq 0$ on Ω.

In the same way as above, we see from (3.15)

$$\frac{\tau}{\mu}\{t(X)u(Y) - t(Y)u(X)\} + g(K\nabla_X W, Y) - g(K\nabla_Y W, X) \\ = \tau\{g(\nabla_Y W, X) - g(\nabla_X W, Y)\}. \tag{3.21}$$

In the next place, from (2.16) and (2.19) we have $\phi U = -\mu W$. Differentiating this covariantly and using (2.4), we find

$$g(AU, X)\xi - \phi\nabla_X U = (X\mu)W + \mu\nabla_X W.$$

Putting $X = \xi$ in this and making use of (3.18), we get

$$\mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (\xi\mu)W, \tag{3.22}$$

which enables us to obtain

$$W\alpha = \xi\mu. \tag{3.23}$$

Now, if we put $X = U$ in (3.3) and take account of (2.18), (2.19) and (3.11), then we get

$$\phi AU + \mu AW = (\lambda' - \alpha)A\xi + \frac{2k\tau}{\alpha}\mu W$$

because of Remark 1, where we have put $\beta = \alpha\lambda'$. If we put

$$\lambda = \lambda' + \frac{2k\tau}{\alpha}, \tag{3.24}$$

then the last equation can be written as

$$\phi AU = \lambda A\xi - A^2\xi - 2k\tau\xi, \tag{3.25}$$

where we have used (2.16). Applying this by ϕ and using (2.5), we find

$$\phi A^2\xi = AU + \lambda U, \tag{3.26}$$

which together with (2.16) gives

$$\mu\phi AW = AU + (\lambda - \alpha)U. \tag{3.27}$$

Putting $X = AU$ in (3.3) and using (3.4), we also obtain

$$\alpha(\phi A^2 U - A\phi AU) = g(AU, U)A\xi + 2kLAU,$$

which together with (2.28) and (3.25) yields

$$\alpha\phi A^2 U = \alpha\lambda A^2\xi - \alpha A^3\xi - 2k\tau\alpha A\xi + g(AU, U)A\xi - 2k\phi KAU.$$

By the way, we have $KAU = \tau AU$ by virtue of (2.33) and (3.14). Thus, the last relationship reformed as

$$\alpha\phi A^2 U = \alpha\lambda A^2\xi - \alpha A^3\xi - 2k\tau\alpha A\xi + g(AU, U)A\xi - 2k\tau\phi AU.$$

If we take the inner product ξ to this, then we obtain

$$g(AU, U) = \gamma - \alpha\lambda^2 + 2k\tau(\lambda + \alpha). \tag{3.28}$$

Therefore, using (3.25) and this, we can write the last equation as

$$\alpha\phi A^2 U = (\alpha\lambda + 2k\tau)A^2\xi - \alpha A^3\xi + (\gamma - \alpha\lambda^2)A\xi + 4k^2\tau^2\xi. \tag{3.29}$$

4. Semi-invariant submanifolds with $\phi\nabla_\xi\xi$-parallel Jacobi operator

In the rest of this paper we will suppose that M is a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ and that the third fundamental form t satisfies $dt = 2\theta\omega$ for a scalar $\theta \neq 2c$ and at the same time $R_\xi\phi = \phi R_\xi$. Further, we assume that R_ξ is $\phi\nabla_\xi\xi$-parallel on M.

Differentiating (3.2) covariantly along M and using (2.5), we find

$$\begin{aligned} g((\nabla_X R_\xi)Y, Z) = &-(k^2 + c)\{\eta(Z)g(\nabla_X\xi, Y) + \eta(Y)g(\nabla_X\xi, Z)\} + (X\alpha)g(AY, Z) \\ &+ \alpha g((\nabla_X A)Y, Z) - g(A\xi, Z)\{g((\nabla_X A)\xi, Y) - g(A\phi AY, X)\} \\ &- g(A\xi, Y)\{g((\nabla_X A)\xi, Z) - g(A\phi AZ, X)\} + (Xk)g(KY, Z) \\ &+ kg((\nabla_X K)Y, Z) - 2k(Xk)\eta(Y)\eta(Z). \end{aligned}$$

If we put $X = W$ in this and taking account of the assumption $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$, we have

$$\begin{aligned} &(W\alpha)AY - (k^2 + c)\{g(\phi AW, Y)\xi + \eta(Y)\phi AW\} \\ &\quad + \alpha(\nabla_W A)Y - \{g((\nabla_W A)\xi, Y) + g(A\phi AW, Y)\}A\xi \\ &\quad + k(\nabla_W K)Y - \{(\nabla_W A)\xi + A\phi AW\}\eta(AY)\} = 0 \end{aligned}$$

since we have $Wk = 0$ because of (2.41) and (3.10).

Now, replacing X by ξ in (2.22) and make use of (2.5) and (3.18), we find

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha. \tag{4.1}$$

If we put $Y = \xi$ in above relationship and use (4.1), then we get

$$k(\nabla_W K)\xi = \alpha A\phi AW + (k^2 + c)\phi AW.$$

On the other hand, differentiating the first equation of (2.26) covariantly with respect to W and using (2.5) and the fact that $Wk = 0$, we find

$$(\nabla_W K)\xi + K\phi AW = k\phi AW,$$

which together with the last equation implies that

$$\alpha A\phi AW + c\phi AW + kK\phi AW = 0. \tag{4.2}$$

If we use (3.10) and (3.27) to this, then we obtain

$$\alpha A^2U + \{\alpha(\lambda-\alpha)+c\}AU + (k\tau+c)(\lambda-\alpha)U + kKAU = 0,$$

which together with (2.33) and (3.14) gives

$$\alpha A^2U + \{\alpha(\lambda-\alpha)+k\tau+c\}AU + (\lambda-\alpha)(k\tau+c)U = 0. \tag{4.3}$$

Since we have $\eta(A^2U) = 0$ because of Remark 1, using (3.1) and (3.10), we can write (4.3) as

$$R_\xi AU = (\alpha-\lambda)R_\xi U. \tag{4.4}$$

Applying (4.3) by ϕ and taking account of (2.16), (2.19) and (3.25), we find

$$\alpha\phi A^2U + (\alpha(\lambda-\alpha)+k\tau+c)(\lambda A\xi - A^2\xi - 2k\tau\xi) \\ - (\lambda-\alpha)(k\tau+c)(A\xi-\alpha\xi) = 0,$$

which connected to (3.29) yields

$$\alpha A^3\xi = (\alpha^2+k\tau-c)A^2\xi + (\gamma-\alpha^2\lambda+\alpha(k\tau+c))A\xi \\ + \{2k^2\tau^2 - k\tau(\lambda-\alpha)\alpha - 2ck\tau + c\alpha(\lambda-\alpha)\}\xi.$$

If we use (2.16), then the last relationship reformed as

$$\alpha\mu A^2W = (k\tau-c)A^2\xi + (\gamma-\lambda\alpha^2+\alpha(k\tau+c)A\xi + (k\tau-c)(2k\tau-\alpha(\lambda-\alpha))\xi,$$

which together with (3.26) gives

$$\alpha\mu\phi A^2W = (k\tau-c)AU + \{(k\tau-c)+\gamma-\lambda\alpha^2+\alpha(k\tau+c)\}U. \tag{4.5}$$

On the other hand, putting $X = AW$ in (3.3) and using (4.2), we find

$$\alpha\phi A^2W + kK\phi AW + c\phi AW = g(A^2\xi, W)U + 2kLAW,$$

which together with (2.33), (3.10), (3.14) and (3.27) yields

$$\mu\alpha\phi A^2W + (k\tau+c)(AU+(\lambda-\alpha)U) = \mu g(A^2\xi, W)U + 2k\mu LAW.$$

However, putting $X = \mu W$ in (2.34) and using (3.14), (3.20) and (3.27), we get

$$\mu LAW = (2k+\tau)AU + k(\lambda-\alpha)U.$$

Substituting this into the last equation, we obtain

$$\mu\alpha\phi A^2W = (k\tau+4k^2-c)AU + \{\mu g(A^2\xi, W) + (\lambda-\alpha)(2k^2-k\tau-c)\}U.$$

If we compare this with (4.5), then we have

$$4k^2AU = \{\gamma - \lambda\alpha^2 + 2\lambda k(\tau - k) + 2\alpha k^2 - \mu^2(\alpha + g(AW, W))\}U.$$

This, it follows that

$$AU = \sigma U, \tag{4.6}$$

where the function σ is defined by on Ω

$$4k^2\sigma = \gamma - \lambda\alpha^2 + 2\lambda k(\tau - k) + 2\alpha k^2 - \mu^2(\alpha + g(AW, W)) \tag{4.7}$$

because of Remark 1. From (4.6) we can verify that (cf. [12], [15])

$$\xi\sigma = 0, \quad W\sigma = 0. \tag{4.8}$$

Applying (4.6) by ϕ and using (2.19) and (3.25), we find

$$A^2\xi = (\lambda + \sigma)A\xi - (2k\tau + \sigma\alpha)\xi, \tag{4.9}$$

which tells us that

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi, \tag{4.10}$$

where we have put $\rho = \lambda + \sigma$. Then we have $\beta = \rho\alpha - 2k\tau - \sigma\alpha$.

Combining (2.16) and (2.18) to (4.10), we obtain

$$AW = \mu\xi + (\rho - \alpha)W \tag{4.11}$$

on Ω. Differentiating this covariantly, we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi + X(\rho - \alpha)W + (\rho - \alpha)\nabla_X W. \tag{4.12}$$

If we take the inner product with W to this and use (2.21) and (4.11), then we find

$$g((\nabla_X A)W, W) = -2g(AU, X) + X\rho - X\alpha. \tag{4.13}$$

Applying (4.12) by ξ and using (2.21), we also find

$$\mu g((\nabla_X A)W, \xi) = (\rho - 2\alpha)g(AU, X) + \mu(X\mu), \tag{4.14}$$

or using (2.30) and (3.20).

$$\mu(\nabla_\xi A)W = (\rho - 2\alpha)AU + \mu\nabla\mu - (k\tau + c)U. \tag{4.15}$$

From this we verify, using (2.26), (2.30) and (4.14), that

$$\mu(\nabla_W A)\xi = (\rho - 2\alpha)AU - 2cU + \mu\nabla\mu. \tag{4.16}$$

Putting $X = \xi$ in (4.13) and taking account of (4.14), we get

$$W\mu = \xi\rho - \xi\alpha. \tag{4.17}$$

Replacing X by ξ in (4.12) and using (3.20) and (4.15), we find

$$\begin{aligned}(\rho-2\alpha)AU+(k\tau-c)U+\mu\nabla\mu+\mu\{A\nabla_\xi W-(\rho-\alpha)\nabla_\xi W\}\\ =\mu(\xi\mu)\xi+\mu^2U+\mu(\xi\rho-\xi\alpha)W.\end{aligned}$$

Substituting (3.22) and (3.23) into this, we obtain

$$\begin{aligned}3A^2U-2\rho AU+(\alpha\rho-\beta-k\tau-c)U+A\nabla\alpha+\frac{1}{2}\nabla\beta-\rho\nabla\alpha\\ =2\mu(W\alpha)\xi+(2\alpha-\rho)(\xi\alpha)\xi+\mu(\xi\rho-\xi\alpha)W.\end{aligned}\tag{4.18}$$

Now, if we use (4.6) and (4.11), then (3.27) can be written as

$$\mu(\rho-\alpha)\phi W=(\sigma+\lambda-\alpha)U,$$

which connected to (2.17) and Lemma 2.1 gives

$$\sigma=\rho-\lambda.\tag{4.19}$$

By the way, it is seen, using (4.6), that (3.28) reformed as $\gamma=\sigma\mu^2+\alpha\lambda^2-2k\tau(\lambda+\alpha)$. Using this and (4.9) we can write (4.7) as

$$4\sigma k^2=\alpha\lambda^2-\lambda(\mu^2+\alpha^2)+2k(\alpha k-\lambda k-\tau\alpha),$$

which together with (2.18) and (3.24) yields

$$2\sigma k=(\lambda-\alpha)(\tau-k)\tag{4.20}$$

on Ω. If we combine (4.6) to (4.4), then we have $(\sigma-\alpha+\lambda)R_\xi U=0$.

Lemma 4.1. *$R_\xi U=0$ on Ω.*

Proof. Suppose that $R_\xi U\neq 0$. Then we have $\sigma=\alpha-\lambda$ on this open subset on Ω. We restrict our arguments on this subset. Then we have $\rho-\alpha=0$ because of (4.19) and hence $AW=\mu\xi$ with the aid of (4.11).

On the other hand, putting $X=\mu W$ in (2.43) and remembering (2.26), (3.14), (3.20) and the last relationship, we obtain $\tau(k+\tau)AU=0$, which connected to Remark 2 gives $AU=0$.

In fact, if $k+\tau=0$, then k is a constant, which together with (2.41) and (3.10) gives $k-\tau=0$, a contradiction. By virtue of (4.6), it follows that $\lambda-\alpha=0$. Hence, (3.24) reformed as $\mu^2+2k\tau=0$ because of (2.18), which implies that $\mu\nabla\mu+\tau\nabla k=0$.

By the way, it is clear, using (2.41) and (3.10), that

$$\nabla k=(\xi k)\xi+(k-\tau)U.\tag{4.21}$$

From the last two equations, it follows that $U\mu+\tau(k-\tau)\mu=0$.

Applying (4.18) by U and using (2.18) and the fact that $\rho=\alpha$ and $AU=0$, we find $U\mu=\mu^2+k\tau+c$. Comparing this and above relationship, we obtain $\tau^2+c=0$, that is $\theta-2c=0$, a contradiction. Thus, $R_\xi U=0$ on Ω is proved. □

Lemma 4.2. *$\xi k = 0$ on Ω.*

Proof. Replacing X by U in (3.2) and using (3.20) and Lemma 6.1, we find $\alpha AU + (k\tau + c)U = 0$, which together with (4.6) and Lemma 2.2 gives

$$\sigma\alpha + k\tau + c = 0. \tag{4.22}$$

Differentiation with respect to W and remembering (4.8) and (4.21) gives $\sigma W\alpha = 0$, which implies $W\alpha = 0$.

In fact, if not, then we have $\sigma = 0$ on this set. Hence we have $\tau^2 + c = 0$ because of (4.21) and (4.22), a contradiction because $\theta - 2c \neq 0$ was assumed. Hence $W\alpha = 0$ is proved on Ω.

Next, differentiating (4.20) with respect to W and using (4.8), (4.21) and itself, we find $W\lambda = 0$.

If we differentiate (3.24) with respect to W, and use (4.21) and the fact that $W\alpha = W\lambda = 0$, then $W\beta = 0$, which together with (2.18) yields $W\mu = 0$. Thus we see, using (4.17), that $\xi\rho - \xi\alpha = 0$, which tells, using (4.8) and (4.19), us that $\xi\lambda - \xi\alpha = 0$.

Now, differentiating (4.20) with respect to ξ and making use of (4.8) and the last equation, we find $(2\sigma + \lambda - \alpha)\xi k = 0$, which connected to (4.20) implies that $\xi k = 0$. This completes the proof. □

Putting $X = \xi$ in the first equation of Section 4, and using (2.5) and (4.1), we have

$$\begin{aligned}(\nabla_\xi R_\xi)Y = &-(k^2 + c)(u(Y)\xi + \eta(Y)U) + (\xi\alpha)AY + \alpha(\nabla_\xi A)Y \\ &+ (\xi k)KY + k(\nabla_\xi K)Y - 2k(\xi k)\eta(Y)\xi \\ &- (3AU + \nabla\alpha)g(A\xi, Y) - (3g(AU, Y) + Y\alpha)A\xi.\end{aligned} \tag{4.23}$$

By the way, from $K\xi = k\xi$, we have $(\nabla_X K)\xi + K\nabla_X\xi = (Xk)\xi + k\nabla_X\xi$, which, together with (3.10) and Lemma 4.2 gives$(\nabla_\xi K)\xi = (k - \tau)U$.

If we put $Y = \xi$ in (4.23) and take account of (4.1), Lemma 4.2 and the last equation, then we find

$$(\nabla_\xi R_\xi)\xi + \alpha AU + (k\tau + c)U = 0.$$

However, if we replace X by U in (3.3) and make use of (3.10) and Lemma 4.1, then we obtain $\alpha AU + (k\tau + c)U = 0$. Accordingly we verify that $R'_\xi = (\nabla_\xi R_\xi)\xi = 0$ on Ω. Thus, by Lemma 5.3 of [15] we conclude that $\Omega = \emptyset$, that is, $k = 0$ on M. So (2.27) becomes $K = \phi L$ which together with (2.35) yields

$$K^2X = (\theta - c)(X - \eta(X)\xi). \tag{4.24}$$

We also have $KU = 0$ because of (2.41), which connected to (4.24) gives $(\theta - c)U = 0$. Using this fact and $k = 0$, (3.43) turns out to be $(\theta - c)(A\phi - \phi A) = 0$.

In the following, we assume that $\theta - c \neq 0$ on M. Then we have

$$A\phi - \phi A = 0,$$

which implies $A\xi = \alpha\xi$. From this and (2.30) with $k = 0$ we can verify that (cf. [11], [25]) $A^2X = \alpha AX + c(X - \eta(X)\xi)$ for any vector field X on M, which enables us to obtain

$$h_{(2)} = \alpha h + 2(n-1)c. \tag{4.25}$$

On the other hand, differentiating (4.24) covariantly along M and using the previously obtained formulas and the Ricci indentity for K, we can deduce that (for detail, see (4.20) and (4.22) of [22])

$$(h + 3\alpha)(h - \alpha) = 4(n-1)\{(n+1)\theta - 2c(n+2)\}, \tag{4.26}$$

$$(\theta - 3c)(h - \alpha) = 2(n-1)(\theta - 2c)\alpha. \tag{4.27}$$

Now, from (2.29) the Ricci tensor S of M is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X - K^2X - L^2X,$$

where we have used $k = l = 0$, which together with (2.36) and (4.24) gives

$$SX = \{c(2n+1) - 2(\theta - c)\}X + (2\theta - 5c)\eta(X)\xi + hAX - A^2X.$$

Therefore, the scalar curvature $\bar{r}$ of M is given by

$$\bar{r} = 2(n-1)(2n+1)c - 4(n-1)(\theta - c) + h(h - \alpha), \tag{4.28}$$

where we have used (4.25).

Lemma 4.3. *$\theta - c = 0$ if $\bar{r} - 2(n-1)c \le 0$.*

Proof. If we put $\delta = 4(n-1)\{(n+1)\theta - 2c(n+2)\}$, then $\delta \ne 0$ for $c < 0$, because $\theta - c$ is nonnegative. However, we also see that $\delta \ne 0$ for $c > 0$.

In fact, if not, then we have $\delta = 0$. So we have $\theta = 2(n-1)c/(n+1)$. Hence, if follows that $\theta - c = (n+3)c/(n+1)$. By the way, from (4.27) we see that $(h + 3\alpha)(h - \alpha) = 0$. Using the last relationships, we can write (4.28) as

$$\bar{r} - 2(n-1)c = 4c(n-1)(n^2 - 3)/(n+1) + \varepsilon^2,$$

where $\varepsilon^2 = 0$ or $12\alpha^2$, a contradiction because of $\bar{r} - 2(n-1)c \le 0$ was assumed. Consequently $\delta \ne 0$ on M is proved. Combining (4.26) to (4.27), we obtain

$$\delta\{(\theta - 3c)^2 - (\theta - 2c)\alpha^2\} = 0,$$

which enables us to obtain

$$(\theta - 3c)^2 = (\theta - 2)\alpha^2. \tag{4.28}$$

By the way, it is clear that $\theta - 3c \neq 0$ for $c < 0$ because $\theta - c$ is nonnegative. But, we also see that $\theta - 3c \neq 0$ for $c > 0$ provided that $\bar{r} - 2(n-1)c \leq 0$.

Indeed, if not, then we have $\theta - 3c = 0$. So we see from (4.28) that $\alpha = 0$ because $\theta - 2c \neq 0$ was assumed. Thus, (4.26) becomes $h^2 = 4(n-1)^2 c$. Using these facts, we can write (4.28) as

$$\bar{r} - 2(n-1)c = 4(n-1)(2n-3)c,$$

a contradiction because of $c > 0$. Therefore $\theta - 3c \neq 0$ on M is proved.

If we combine (4.28) to (4.27), then we find

$$\alpha(h - \alpha) = 2(n-1)(\theta - 3c).$$

Using this fact, (4.26) turns out to be

$$h(h - \alpha) = 2(n-1)(2n-1)(\theta - c) - 4n(n-1)c,$$

which together with (4.28) implies that

$$\bar{r} - 2(n-1)c = 2(n-1)(2n-3)(\theta - c).$$

Accordingly we have $\theta - c = 0$ if $\bar{r} - 2(n-1)c \leq 0$. This completes the proof of Lemma 4.3. □

According to Lemma 4.3 we have $K = L = 0$ because of (2.36) and (4.24). And hence the normal connection of M is flat.

Let $N_0(p) = \{\upsilon \in T_p^{\perp}(M) : A_\upsilon = 0\}$ and $H_0(p)$ be the maximal J-invariant subspace of $N_0(p)$. Since $K = L = 0$, the orthogonal complement of $H_0(p)$ is invariant under parallel translation with respect to the normal connection because of $\nabla^{\perp}\mathcal{C} = 0$. Thus, by the reduction theorem in [10], [28] we see that M is a real hypersurface in a complex space form $M_n(c)$.

Since we have $\nabla^{\perp}\mathcal{C} = 0$ and $k = 0$, we can write (2.30) and (3.3) as

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},$$

$$\alpha(\phi A - A\phi)X - g(A\xi, X)U - g(U, X)A\xi = 0$$

respectively. Making use of (2.4) and (2.5), and above two equations, it is proved in [25] that $g(U, U) = 0$, that is, M is a Hopf hypersurface. Hence, we conclude that $\alpha(A\phi - \phi A) = 0$ and thus $A\xi = 0$ or $A\phi = \phi A$. Here, we note that the case $\alpha = 0$ correspond to the case of radius $\pi/4$ in complex projective space $P_n\mathbb{C}$ ([3], [18]). But, in the case complex hyperbolic space $H_n\mathbb{C}$ it is known that α never vanishes for Hopf hypersurfaces (cf.[5]). Thus, owing to Theorem O-MR, we have

Theorem 4.4. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ with constant holomorphic sectional curvature $4c$ such that R_ξ is $\phi\nabla_\xi\xi$-parallel and the third fundamental form t satisfies $dt(X, Y) = 2\theta(\phi X, Y)$ for a scalar $\theta(\neq 2c)$*

and any vector fields X and Y on M. Then $R_\xi\phi = \phi R_\xi$ holds on M if and only if $A\xi = 0$ or M is locally congruent to one of the following hypersurfaces provided that the scalar curvature $\bar{r}$ of M satisfies $\bar{r} - 2(n-1)c \leq 0$:

(I) *in case that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*
  (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
  (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, ..., n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$;*

(II) *in case that $M_n(c) = H_n\mathbb{C}$,*
  (A_0) *a horosphere,*
  (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,*
  (A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, ..., n-2\}$.*

References

[1] A. Bejancu, CR-submanifolds of a *Kähler* manifold I, *Proc. Amer. Math. Soc.* 69(1978), 135-142.

[2] J. Berndt, Real hypersurfaces with constant principal curvatures in a complex hyperbolic space, *J. Reine Angew. Math.* 395(1989), 132-141.

[3] J. Berndt and H. Tamaru, Cohomogeneity one actions on non compact symmetric spaces of rank one, *Trans. Amer. Math. Soc.* 359 (2007), 3425-3438.

[4] D. E. Blair, G. D. Ludden and K. Yano, Semi-invariant immersions, *Kodai Math. Sem. Rep.* 27(1976), 313-319.

[5] T. E. Cecil and P. J. Ryan, Focal sets and real hypersurfaces in complex projective space, *Trans. Amer. Math. Soc.* 269(1982), 481-499.

[6] J. T. Cho and U-H. Ki, Real hypersurfaces of a complex projective space in terms of the Jacobi operators. *Acta Math. Hungar.* 80(1998), 155-167.

[7] J. T. Cho and U-H. Ki, Real hypersurfaces in complex space forms with Reeb flow symmetric Jacobi operator, *Canadian Math. Bull.* 51(2008), 359-371.

[8] J. Erbacher, Reduction of the codimension of an isometric immersion, *J. Diff. Geom.* 3(1971), 333-340.

[9] J. I. Her, U-H. Ki and S.-B. Lee, Semi-invariant submanifolds of codimension 3 of a complex projective space in terms of the Jacobi operator, *Bull. Korean Math. Soc.* 42(2005), 93-119.

[10] S. Kawamoto, Codimension reduction for real submanifolds of a complex hyperbolic space. *Rev. math. univ. comput. madrid.* 7(1994), 119-128.

[11] U-H. Ki, Cyclic-parallel real hypersurfaces of a complex space form, *Tsukuba J. Math.* 12(1988), 259-268.

[12] U-H. Ki, Commuting Jacobi operators for a semi-invariant submanifold of codimension 3 in complex space forms, *J. Nat. Acad. Sci. ROK, Sci. Ser.* 58-2(2019), 271-306.

[13] Ki, U-H, The Ricci tensor and structure Jacobi operator of submanifolds of codimension 3 in complex space forms. *J. Nat. Acad. Sci. ROK, Sci. Ser.* 59-2(2020), 367-394.

[14] Ki, U-H. and Kim, S. J, Jacobi operators along the structure flow on semi-invariant subminifolds of codimension 3 in a complex space form. *J. Nat. Acad. Sci. ROK. Sci. Ser.* 59-1(2020), 19-44.

[15] U-H. Ki and S. J. Kim, Structure Jacobi operators of semi-invariant submanifolds in a complex space form, *East Asian Math. J.* 36-3(2020), 389-415.

[16] U-H. Ki, S. J. Kim and H. Kurihara, Jacobi operators with respect to the Reeb vector fields on real hypersurfaces in a nonflat complex space form, *Kyungpook Math J.* 56(2016), 541-575.

[17] U-H. Ki and H. Kurihara, Semi-invariant submanifolds of codimension 3 in a complex space form in terms of the structure Jacobi operators, to appear in *Comm. Korean Math. Soc.*
[18] U-H. Ki, S. Nagai and R. Takagi, The structure vector filed and structure Jacobi operator of real hypersurfaces in nonflat complex space forms, *Geom. Dedicata* 149(2010), 161-176.
[19] U-H. Ki and H. Song, Jacobi operators on a semi-invariant submanifolds of codimension 3 in a complex projective space, *Nihonkai Math J.* 14(2003), 1-16.
[20] U-H. Ki and H. Song, Submanifold of codimension 3 with almost contact metric structure in a complex space form, *J. Nat. Acad. Sci. ROK. Sci. Ser.* 58-1(2019), 37-63.
[21] U-H. Ki and H. Song, Semi-invariant submanifolds of condimension 3 in a complex space form with commuting structure Jacobi operator, to appear in *Kyungpook. Math. J.*
[22] U-H. Ki, H. Song and R. Takagi, Submanifolds of codimension 3 admitting almost contact metric structure in a complex projective space, *Nihonkai Math J.* 11(2000), 57-86.
[23] M. Kimura, Real hypersurfaces and complex submanifolds in complex projective space, *Trans. Amer. Math. Soc.* 296(1986), 137-149.
[24] S. Montiel and A.Romero, On some real hypersurfaces of a complex hyperbolic space, *Geom. Dedicata* 20(1986), 245-261.
[25] R. Niebergall and P. J. Ryan, Real hypersurfaces in complex space form, in Tight and Taut submanifolds, *Cambridge University Press* : (1998(T. E. Cecil and S.-S. Chern eds.)), 233-305.
[26] M. Okumura, On some real hypersurfaces of a complex projective space, *Trans. Amer. Math. Soc.* 212(1973), 355-364.
[27] M. Okumura, Normal curvature and real submanifold of the complex projective space. *Geom. Dedicata* 7(1978), 509-517.
[28] M. Okumura, Codimension reduction problem for real submanifolds of complex projective space. *Colloq. Math János Bolyai.* 56(1989), 574-585.
[29] M. Ortega, J. D. Pérez and F. G. Santos, Non-existence of real hypersurface with parallel structure Jacobi operator in nonflat complex space forms, *Rocky Mountain J. Math.* 36(2006), 1603-1613.
[30] H. Song, Some differential-geometric properties of R-spaces, *Tsukuba J. Math.* 25(2001), 279-298.
[31] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, *Osaka J. Math.* 19(1973), 495-506.
[32] R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures I,II, *J. Math. Soc. Japan* 27(1975), 43-53, 507-516.
[33] Y. Tashiro, Relations between the theory of almost complex spaces and that of almost contact spaces (in Japanese), *Sũgaku* 16(1964), 34-61.
[34] K. Yano, and U-H. Ki, On $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$, *Kodai Math. Sem. Rep.* 29(1978), 285-307.
[35] K. Yano and M. Kon, CR submanifolds of Kaehlerian and Sasakian manifolds, *Birkhäuser* (1983).

U-HANG KI
The National Academy of Sciences, Seoul 06579, Korea
E-mail address: uhangki2005@naver.com

SOO JIN KIM
Department of Mathematics Chosun University Gwangju, 61452, Korea
E-mail address: ccamzzicee@naver.com