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WEIGHTED SOBOLEV REGULARITY OF VISCOSITY SOLUTIONS FOR FULLY NONLINEAR PARABOLIC EQUATIONS

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Abstract. We obtain interior regularity estimates in the weighted Orlicz spaces for viscosity solutions of fully nonlinear uniformly parabolic equations

$$u_t - F(D^2u, x, t) = f(x, t) \text{ in } Q_1$$

under relaxed structure conditions on the nonlinear operator F.

1. Introduction

The paper is devoted to studying interior regularity of viscosity solutions for the fully nonlinear parabolic equation

$$u_t - F(D^2u, x, t) = f(x, t) \text{ in } Q_1 := B_1 \times (-1, 0], \tag{1.1}$$

where F is an uniformly elliptic operator in $S(n) \times Q_1$ and f is a given datum. Here, $S(n)$ denotes the set of $n \times n$ real symmetric matrices with real entries. We prove the interior regularity estimates in the weighted Orlicz spaces for viscosity solutions of (1.1) under relaxed structure conditions imposed on the nonlinearity F. The integrability of Hessian of the solutions is relevant to the behavior of the nonlinear operator $F(X, x, t)$ in X variable near infinity. In this respect, we take the approximation argument using the notion of the *recession* operator $F^\star$ defined by

$$F^\star(X, x, t) = \lim_{\mu \to 0} \mu F(\mu^{-1}X, x, t)$$

which was introduced in the context of regularity theory for fully nonlinear elliptic equations in [18, 19] . In order to relax the structure conditions on the nonlinear operator F, we impose the convexity condition with respect to X variable and the small oscillation condition with respect to x, t variables on the recession operator $F^\star$ instead of the original operator F.

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L^p regularity theory for fully nonlinear equations has been extensively studied since Caffarelli proved interior L^p regularity estimates for the Hessian of solutions of fully nonlinear elliptic equations for all $p > n$ in [4]. Adapting the approach of Caffarelli, Wang [21] contributed to the development of L^p regularity theory for fully nonlinear parabolic equations. This approach is based on the perturbation argument by comparing the solutions of the original problem with ones of the corresponding limiting problem. Its crucial point is that it needs the $C^{2,1}$ regularity property for solutions of the limiting problem which is guaranteed under the convexity assumption of the problem, see [15]. Recently, various attempts have been made to relax this structure assumption on the operator F for the L^p regularity theory of fully nonlinear equations in for instance [3, 6, 14, 18]. In particular, Castillo and Pimentel [6] established the L^p estimates for the Hessian and time derivatives of solutions to fully nonlinear parabolic equations under asymptotic assumptions. Our results in the present paper extend their results to the settings of weighted Orlicz spaces. Similar problems have already studied for fully nonlinear elliptic equations by the author in [16].

The remainder of this paper is organized as follows. In Section 2, we present our main result providing definitions and properties related to weighted Orlicz spaces. The main result is proved in Section 3.

2. Preliminaries and Main result

2.1. Notations and Definitions

We denote by $B_r(y)$ the open ball in $\mathbb{R}^n$ centered at $y \in \mathbb{R}^n$ with radius $r > 0$ and $Q_r(y,s) = B_r(y) \times (s - r^2, s]$ as the parabolic cylinder in $\mathbb{R}^{n+1}$ for $(y,s) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$. We define the parabolic distance between the points $(x,t), (y,s) \in \mathbb{R}^n \times \mathbb{R}$ by

$$d_p((x,t),(y,s)) := \max\{|x-y|, \sqrt{|t-s|}\}$$

where $|\cdot|$ is the Euclidean norm. The parabolic boundary of $Q_r(y,s)$ is defined by

$$\partial_p Q_r(y,s) = (\partial B_r(y) \times (s-r^2, s)) \cup (B_r(y) \times \{t = s - r^2\}).$$

For simplicity, we denote $B_r \equiv B_r(0)$ and $Q_r \equiv Q_r(0,0)$.

Let U be a bounded domain in $\mathbb{R}^{n+1}$ with $n \geq 2$. For a function $g : U \to \mathbb{R}$, we denote its spatial gradient by Dg, its spatial Hessian by $D^2 g$ and its time derivative by g_t. The space $C^{2,1}(U)$ is the space of functions which are continuously differentiable twice with respect to space and once with respect to time. In other words, $g \in C^{2,1}(U)$ means that $g, Dg, D^2 g, g_t \in C(U)$. For simplicity, we write

$$\fint_U g(x,t)\, dxdt = \frac{1}{|U|}\int_U g(x,t)\, dxdt$$

for $g \in L^1_{loc}(U)$, where $|U|$ is the $n+1$-dimensional Lebesgue measure of U. From now on, the letter c denotes a positive universal constant that may vary at each appearance.

We now introduce the definition of weighted Orlicz spaces and their basic properties treated in this paper. A function $\Phi\colon [0,\infty) \to [0,\infty]$ is called an N-*function* if it is convex, continuous and increasing, and satisfies that $\Phi(0)=0$, $\Phi(\rho)>0$ for all $\rho>0$, $\lim_{\rho\to\infty}\Phi(\rho)=+\infty$, and $\lim_{\rho\to 0^+}\frac{\Phi(\rho)}{\rho}=\lim_{\rho\to\infty}\frac{\rho}{\Phi(\rho)}=0$.

In this paper, the N-function Φ considered are assumed to satisfy $\Delta_2\cap\nabla_2$-condition (denoted by $\Phi\in\Delta_2\cap\nabla_2$), which means that there exist constants $\kappa_1,\kappa_2>1$ such that

$$\Phi(2\rho)\le\kappa_1\Phi(\rho) \quad\text{and}\quad \Phi(\rho)\le\frac{1}{2\kappa_2}\Phi(\kappa_2\rho) \quad\text{for all } \rho>0.$$

Given a weight w and an N-function $\Phi\in\Delta_2\cap\nabla_2$, the *weighted Orlicz space* $L^\Phi_w(U)$ is defined as the set of all Lebesgue measurable functions g on U such that

$$\int_U \Phi\left(|g(x,t)|\right) w(x,t)\, dxdt < +\infty.$$

The space $L^\Phi_w(U)$ is a reflexive Banach space under the following *Luxemburg norm*

$$\|g\|_{L^\Phi_w(U)} = \inf\left\{ s>0 : \int_U \Phi\left(\frac{|g(x,t)|}{s}\right) w(x,t)\, dxdt \le 1\right\}$$

by the $\Delta_2\cap\nabla_2$-condition of Φ. In addition, for $\Phi\in\Delta_2\cap\nabla_2$, we note that there exist constants q_1,q_2 with $1<q_1\le q_2<\infty$ such that

$$\frac{1}{c}\min\{\nu^{q_1},\nu^{q_2}\}\,\Phi(\rho)\le\Phi(\nu\rho)\le c\,\max\{\nu^{q_1},\nu^{q_2}\}\,\Phi(\rho) \tag{2.1}$$

for $\nu,\rho\ge 0$, where the constant c is independent of ν and ρ, and

$$\int_U \Phi\left(\frac{|g(x,t)|}{\|g\|_{L^\Phi_w(U)}}\right) w(x,t)\, dxdt = 1,$$

for nonzero function $g\in L^\Phi_w(U)$. Then one can see that

$$\|g\|_{L^\Phi_w(U)} - 1 \le \int_U \Phi(|g(x,t)|) w(x,t)\, dxdt \le c\left(\|g\|^{q_2}_{L^\Phi_w(U)}+1\right) \tag{2.2}$$

where the constant $c>1$ is independent of g. We further have the unit ball property as follows:

$$\int_U \Phi(|g(x,t)|) w(x,t)\, dxdt \le 1 \iff \|g\|_{L^\Phi_w(U)}\le 1.$$

We refer to [10, 12, 13] for the properties and more details about the N-function Φ and the weighted Orlicz spaces.

Next we will present one of main assumptions which is imposed on the weight. A weight w is called *an A_q weight* with $1 < q < \infty$, denoted by $w \in A_q$, if

$$[w]_q := \sup_{Q \subset \mathbb{R}^{n+1}} \left(\fint_Q w(x,t)\, dxdt \right) \left(\fint_Q w(x,t)^{\frac{-1}{q-1}}\, dxdt \right)^{q-1} < \infty,$$

where the supremum is taken over all parabolic cylinder $Q \subset \mathbb{R}^{n+1}$. We use the notation $w(V)$ to denote $\displaystyle\int_V w(x,t)\, dxdt$ for a measurable set $V \subset \mathbb{R}^{n+1}$.

Basically, the A_q weights are invariant under translation, dilation and multiplication by a positive scalar and have doubling property and monotonicity, that is, $A_{q_1} \subset A_{q_2}$ for $q_1 \leq q_2$. Moreover, they have the following self-improving property: if $w \in A_q$, then $w \in A_{q-\epsilon}$ for some small constant $\epsilon = \epsilon(n, q, [w]_q) > 0$. In particular, the following property of the A_q weights is essential in the proof of our main result, see [9, 20] for its detail proof with further properties of A_q weights.

Lemma 2.1. *Let $w \in A_q$ where $1 < q < \infty$. There are constants $\gamma_1, \gamma_2 > 0$ depending only on n, q and $[w]_q$ such that for any parabolic cylinder Q in $\mathbb{R}^{n+1}$ and any measurable set $D \subset Q$,*

$$\frac{1}{[w]_q} \left(\frac{|D|}{|Q|} \right)^q \leq \frac{w(D)}{w(Q)} \leq \gamma_1 \left(\frac{|D|}{|Q|} \right)^{\gamma_2}.$$

Given the N-function Φ with the $\Delta_2 \cap \nabla_2$-condition, our main assumption on the weight w is that $w \in A_{i(\Phi)}$. Here, $i(\Phi)$ is the *lower index* of Φ defined by

$$i(\Phi) = \lim_{\nu \to 0^+} \frac{\log(h_\Phi(\nu))}{\log \nu} = \sup_{0<\nu<1} \frac{\log(h_\Phi(\nu))}{\log \nu},$$

where $h_\Phi(\nu) = \sup_{\rho > 0} \dfrac{\Phi(\nu\rho)}{\Phi(\rho)}$ for $\nu > 0$. When $\Phi(\rho) = \rho^q$ with $q > 1$, it is clear that $i(\Phi) = q$. If $w \in A_{i(\Phi)}$, we remark that the boundedness of the Hardy-Littlewood maximal function holds in the corresponding weighted Orlicz space L^Φ_w, see Lemma 3.2 in the next section. We further notice that $i(\Phi)$ is equal to the supremum of those q_1 in the above inequality (2.1) with $\nu \geq 1$, and then $i(\Phi) > 1$, see [8] for more details.

2.2. Main result

Let U be a bounded domain in $\mathbb{R}^{n+1}$ with $n \geq 2$. We consider the fully nonlinear parabolic equations

$$u_t - F(D^2u, x, t) = 0 \quad \text{in } U, \tag{2.3}$$

where the nonlinearity $F = F(X, x, t)$ is a Carathéodory function defined on $S(n) \times U$, that is, $X \mapsto F(X, x, t)$ is continuous for a.e. $(x, t) \in U$ and $(x, t) \mapsto F(X, x, t)$ is measurable for all $X \in S(n)$.

In this paper, we assume that the operator F is uniformly elliptic (i.e. (2.3) is uniformly parabolic), that is, there exist constants λ and Λ with $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda\|Y\| \leq F(X+Y,x,t) - F(X,x,t) \leq \Lambda\|Y\|$$

for all $X, Y \in S(n), Y \geq 0$ and almost all $(x,t) \in U$, where $\|Y\| := \sup\limits_{|x|=1} |Yx|$.

Definition 1. We say that $u \in C(U)$ is *an L^q-viscosity solution* of (2.3) if the following two conditions hold:

(1) for all φ such that $\varphi, \varphi_t, D^2\varphi \in L^q_{\mathrm{loc}}(U)$, whenever $\epsilon > 0$, $\mathcal{O} \subset U$ is open and
$$\varphi_t - F(D^2\varphi, x, t) \geq \epsilon \ \text{ a.e. in } \mathcal{O},$$
$u - \varphi$ cannot attain a local maximum in $\mathcal{O}$,

(2) for all φ such that $\varphi, \varphi_t, D^2\varphi \in L^q_{\mathrm{loc}}(U)$, whenever $\epsilon > 0$, $\mathcal{O} \subset U$ is open and
$$\varphi_t - F(D^2\varphi, x, t) \leq -\epsilon \ \text{ a.e. in } \mathcal{O},$$
$u - \varphi$ cannot attain a local minimum in $\mathcal{O}$.

Whenever F is continuous in all variables, $u \in C(U)$ is called *a C-viscosity solution* of (2.3) if $\varphi \in C^2(U)$ in Definition 1. It is seen that C-viscosity solutions of (2.3) are L^q-viscosity solutions whenever F is continuous in all variables, see [5, Proposition 2.9].

Remark 1. For $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_{i(\Phi)}$, $L^\Phi_w(U)$ is continuously embedded in $L^p(U)$ for some constant $p = p(\Phi, w)$ satisfying $1 < p < i(\Phi)$ by the self-improving property of w, see [1, Lemma 2.5] for its proof. In this regard, we can deal with $L^{p(n+1)}$-viscosity solutions of the equation (1.1) provided that $|f|^{n+1} \in L^\Phi_w(Q_1)$.

As mentioned in the introduction, we will use the *recession* operator $F^\star$ associated with the nonlinear operator F which is given by

$$F^\star(X,x,t) := \lim_{\mu \to 0} F_\mu(X,x,t)$$

assuming its existence for any $X \in S(n)$ and $(x,t) \in Q_1$, where $F_\mu(X,x,t) := \mu F(\mu^{-1}X, x, t)$ for any $\mu > 0$. Note that F_μ and $F^\star$ are uniformly elliptic with the same ellipticity constants as F. See [18, 19] for an overview for the recession operator $F^\star$.

In this paper, we suppose that the recession operator $F^\star$ associated with the original operator F exists and $v_t - F^\star(D^2v, x_0, t_0) = 0$ has $C^{2,1}$ interior estimates with constant $c_\star$ for any $(x_0, t_0) \in Q_1$, that is, for any $v_0 \in C(\overline{Q}_1)$ there exists a C-viscosity solution $v \in C^{2,1}(Q_1) \cap C(\overline{Q}_1)$ of

$$\begin{cases} v_t - F^\star(D^2v, x_0, t_0) &= 0 \quad \text{in} \quad Q_1, \\ v &= v_0 \quad \text{on} \quad \partial_p Q_1, \end{cases}$$

with

$$\sup_{Q_{1/2}} (|v_t| + |D^2 v|) \le c_\star \|v_0\|_{L^\infty(\partial_p Q_1)}$$

for some universal constant $c_\star > 0$. We remark that if $F^\star(X, x, t)$ has the uniform convexity with respect to X variable, the solutions of the fully nonlinear parabolic equation $v_t - F(D^2 v, x_0, t_0) = 0$ are in $C^{2,1}$, see for instance [15].

For any $(x, t), (y, s) \in U$, we denote

$$\theta_{F^\star}((x,t),(y,s)) := \sup_{X \in S(n)\setminus\{0\}} \frac{|F^\star(X,x,t) - F^\star(X,y,s)|}{\|X\|}.$$

This will be used when measuring the oscillation of the operator $F^\star(X, x, t)$ with respect to (x, t). Without loss of generality, we further assume that $F(0, \cdot, \cdot) \equiv 0$ in Q_1.

The main result of this paper is following:

Theorem 2.2 (Main Theorem). *Assume that $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_{i(\Phi)}$. Let u be an $L^{p(n+1)}$-viscosity solution of (1.1), where p is given in Remark 1. Suppose that $F^\star(X, x, t)$ exists and $v_t - F^\star(D^2 v, x_0, t_0) = 0$ has $C^{2,1}$ interior estimates with constant $c_\star$ for any $(x_0, t_0) \in Q_1$. Suppose that $f \in L^\Psi_w(Q_1)$ with $\Psi(\rho) := \Phi(\rho^{n+1})$. Then there exists a constant $\delta = \delta(n, \lambda, \Lambda, \Phi, w, c_\star) > 0$ such that if*

$$\left(\fint_{Q_r(x_0,t_0)} \theta_{F^\star}((x,t),(x_0,t_0))^{n+1}\, dxdt \right)^{\frac{1}{n+1}} \le \delta$$

for any parabolic cylinder $Q_r(x_0, t_0) \subset Q_1$ with $r > 0$, then $u_t, D^2 u \in L^\Psi_w(Q_{\frac{1}{2}})$ with the estimate

$$\|u_t\|_{L^\Psi_w(Q_{\frac{1}{2}})} + \|D^2 u\|_{L^\Psi_w(Q_{\frac{1}{2}})} \le c \left(\|f\|_{L^\Psi_w(Q_1)} + \|u\|_{L^\infty(Q_1)} \right) \tag{2.4}$$

for some $c = c(n, \lambda, \Lambda, \Phi, w, c_\star) > 0$.

3. Proof of main theorem

Let U be a bounded domain in $\mathbb{R}^{n+1}$. For a constant $M > 0$, we say that P is a paraboloid of aperture M if

$$P(x,t) = a + bx + c\left(\frac{|x|^2}{2} + t \right)$$

where $|a| + |b| + |c| \le M$. For a continuous function $v \colon U \to \mathbb{R}$, we define

$$\underline{\mathcal{G}}_M(v, U) := \left\{ (x_0,t_0) \in U \colon \begin{array}{c} \text{there is a paraboloid } P \text{ of aperture } M \\ \text{such that } P(x_0,t_0) = v(x_0,t_0), \\ P(x,t) \le v(x,t) \text{ for any } (x,t) \in U \end{array} \right\}$$

and

$$\overline{\mathcal{G}}_M(v, U) := \left\{ (x_0,t_0) \in U \colon \begin{array}{c} \text{there is a paraboloid } P \text{ of aperture } M \\ \text{such that } P(x_0,t_0) = v(x_0,t_0), \\ P(x,t) \ge v(x,t) \text{ for any } (x,t) \in U \end{array} \right\}.$$

We define $\underline{\mathcal{A}}_M(v,U) := U \setminus \underline{\mathcal{G}}_M(v,U)$ and $\overline{\mathcal{A}}_M(v,U) := U \setminus \overline{\mathcal{G}}_M(v,U)$. We denote

$$\mathcal{G}_M(v,U) := \underline{\mathcal{G}}_M(v,U) \cap \overline{\mathcal{G}}_M(v,U) \text{ and } \mathcal{A}_M(v,U) := \underline{\mathcal{A}}_M(v,U) \cap \overline{\mathcal{A}}_M(v,U).$$

We further denote

$$\Theta(v,U)(x,t) := \sup\{\underline{\Theta}(v,U)(x,t), \overline{\Theta}(v,U)(x,t)\},$$

where

$$\begin{aligned} \underline{\Theta}(v,U)(x,t) &:= \inf\{M > 0 \colon (x,t) \in \underline{\mathcal{G}}_M(v,U)\}, \\ \overline{\Theta}(v,U)(x,t) &:= \inf\{M > 0 \colon (x,t) \in \overline{\mathcal{G}}_M(v,U)\}. \end{aligned}$$

Then we see that both integrabilities of v_t and $D^2 v$ depend on the function Θ. The following lemma is the modified parabolic version of [1, Lemma 3.4]. This can be proved by the same way as in the proof of [1, Lemma 3.4].

Lemma 3.1. *Assume that $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_{i(\Phi)}$. Let u be a continuous function in U. For $r > 0$, we define*

$$\Theta(u,r)(x,t) := \Theta(u, U \cap Q_r(x,t))(x,t) \quad \text{for } (x,t) \in U.$$

If $\Theta(u,r) \in L^{\Phi}_w(U)$, then we have $u_t, D^2 u \in L^{\Phi}_w(U)$ with the estimate

$$\|u_t\|_{L^{\Phi}_w(U)} + \|D^2 u\|_{L^{\Phi}_w(U)} \le c \|\Theta(u,r)\|_{L^{\Phi}_w(U)}$$

for some $c = c(\Phi) > 0$.

One of our main tools is the Hardy-Littlewood maximal function which is defined on the Lebesgue space $L^1_{loc}(\mathbb{R}^{n+1})$ by

$$\mathcal{M}g(y,s) = \sup_{r>0} \fint_{K_r(y,s)} |g(x,t)| \, dxdt.$$

for $(y,s) \in \mathbb{R}^{n+1}$. Here, $K_r(y,s) := \prod_{i=1}^n \left(y_i - \frac{r}{2}, y_i + \frac{r}{2}\right) \times (s - r^2, s]$ is the open parabolic cube in $\mathbb{R}^{n+1}$ centered at $(y,s) = (y_1, \dots, y_n, s) \in \mathbb{R}^n \times \mathbb{R}$ with side-length $r > 0$. For simplicity, we denote $K_r \equiv K_r(0,0)$ from now on.

As mentioned before, we consider the following boundedness of the maximal function $\mathcal{M}$ on the weighted Orlicz spaces, see [11, 12] for its proof and more details.

Lemma 3.2. *Assume that $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_{i(\Phi)}$. Then for any $g \in L^{\Phi}_w(\mathbb{R}^{n+1})$, we have*

$$\int_{\mathbb{R}^{n+1}} \Phi(|g|) w(x,t) \, dxdt \le \int_{\mathbb{R}^{n+1}} \Phi\left(\mathcal{M}g\right) w(x,t) \, dxdt \le c \int_{\mathbb{R}^{n+1}} \Phi(|g|) w(x,t) \, dxdt$$

where a constant $c > 0$ is independent of g.

Classical measure theory and basic properties of the N-function Φ imply the following lemma, see [2, Lemma 4.6] for its proof and more details.

Lemma 3.3. *Assume that $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_q$ for some $1 < q < \infty$. Let $\eta > 0$ and $S > 1$ be constants. Then for any nonnegative measurable function g in $U \subset \mathbb{R}^{n+1}$, we have that*

$$g \in L^{\Phi}_w(U) \text{ if and only if } \mathcal{T} := \sum_{j\geq 1} \Phi(S^j) w\left(\{(x,t) \in U \colon g(x,t) > \eta S^j\}\right) < \infty$$

and moreover,

$$\frac{1}{c}\mathcal{T} \leq \int_U \Phi(|g|) w(x,t)\, dxdt \leq c\,(w(U) + \mathcal{T}),$$

where $c > 0$ is a constant depending only on $\eta, S, \Phi(1), q,$ and $[w]_q$.

To prove our main result, the following power decay estimates are needed for $\mathcal{A}_M$, see [6, Proposition 4.5] for its proof and more details.

Lemma 3.4. *Let U be a bounded domain with $Q_{8\sqrt{n}} \subset U$. Let $u \in C(U)$ be a C-viscosity solution of*

$$u_t - F_\mu(D^2u, x, t) = f(x,t) \text{ in } Q_{8\sqrt{n}}$$

with $\|u\|_{L^\infty(Q_{8\sqrt{n}})} \leq 1$. Suppose that $F^\star(X,x,t)$ exists and $u_t - F^\star(D^2v, x_0, t_0) = 0$ has $C^{2,1}$ interior estimates with constant $c_\star$ for any $(x_0,t_0) \in Q_{8\sqrt{n}}$. For any $\epsilon \in (0,1)$, there exist $M = M(n, c_\star) > 1$ and $\delta = \delta(n, \lambda, \Lambda, c_\star, \epsilon) > 0$ such that if $\mu + \|f\|_{L^{n+1}(Q_{8\sqrt{n}})} \leq \delta$ and

$$\left(\fint_{Q_r(x_0,t_0)} \theta_{F^\star}((x,t),(x_0,t_0))^{n+1}\, dxdt\right)^{\frac{1}{n+1}} \leq \delta$$

for any ball $Q_r(x_0,t_0) \subset Q_{8\sqrt{n}}$ with $r > 0$, then extending f by zero outside $Q_{8\sqrt{n}}$, for $j = 0, 1, 2, \ldots,$ we have

$$\begin{aligned}&\left|\mathcal{A}_{M^{j+1}}(u, Q_{8\sqrt{n}}) \cap K_1\right| \\ &\quad \leq \epsilon\left|\left(\mathcal{A}_{M^j}(u, Q_{8\sqrt{n}}) \cap K_1\right) \cup \left\{(x,t) \in K_1 \colon \mathcal{M}(f^{n+1})(x,t) \geq (\eta M^i)^{n+1}\right\}\right|\end{aligned}$$

for some constant $\eta = \eta(n, \lambda, \Lambda, c_\star, \epsilon) > 0$.

We obtain the weighted measure version of Lemma 3.4 by applying the above lemma 3.4 and taking account into the properties of the A_q weight.

Lemma 3.5. *Under the same assumptions as in Lemma 3.4, we further suppose that $w \in A_q$ for some $q > 1$. For any $\epsilon \in (0,1)$, there exist $M = M(n, c_\star) > 1$ and $\delta = \delta(n, \lambda, \Lambda, q, c_\star, w, \epsilon) \in (0,1)$ such that if $\mu + \|f\|_{L^{n+1}(Q_{8\sqrt{n}})} \leq \delta$ and*

$$\left(\fint_{Q_r(x_0,t_0)} \theta_{F^\star}((x,t),(x_0,t_0))^{n+1}\, dxdt\right)^{\frac{1}{n+1}} \leq \delta$$

for any $Q_r(x_0,t_0) \subset Q_{8\sqrt{n}}$ *with* $r>0$, *then extending* f *by zero outside* $Q_{8\sqrt{n}}$, *for* $j=0,1,2,\dots,$ *we have*

$$\begin{aligned} & w\left(\mathcal{A}_{M^j}(u,Q_{8\sqrt{n}}) \cap K_1\right) \\ & \quad \le \epsilon^j w(K_1) + \sum_{i=0}^{j-1} \epsilon^{j-i} w\left(\left\{(x,t)\in K_1 \colon \mathcal{M}(f^{n+1})(x,t) \ge (\eta M^i)^{n+1}\right\}\right) \end{aligned}$$

for some constant $\eta = \eta(n,\lambda,\Lambda,c_\star,\epsilon) > 0$.

Proof. Let $\epsilon \in (0,1)$ and choose $\delta = \delta(n,\lambda,\Lambda,c_\star,\epsilon)>0$ as in Lemma 3.4 with ϵ replaced by $\left(\frac{\epsilon}{\gamma_1}\right)^{\frac{1}{\gamma_2}}$ where γ_1,γ_2 are the constants depending only on n,q and $[w]_q$ in Lemma 2.1. Setting $D_1 := \mathcal{A}_{M^{j+1}}(u,Q_{8\sqrt{n}}) \cap K_1$ and

$$D_2 := \left(\mathcal{A}_{M^j}(u,Q_{8\sqrt{n}}) \cap K_1\right) \cup \left\{(x,t)\in K_1 \colon \mathcal{M}(f^{n+1})(x,t) \ge (\eta M^j)^{n+1}\right\}$$

for $j=0,1,2,\dots,$ we then have that $|D_1| < \left(\frac{\epsilon}{\gamma_1}\right)^{\frac{1}{\gamma_2}} |D_2|$ from Lemma 3.4. Then Lemma 2.1 yields that

$$\frac{w(D_1)}{w(D_2)} \le \gamma_1 \left(\frac{|D_1|}{|D_2|}\right)^{\gamma_2} < \gamma_1 \left(\frac{\epsilon}{\gamma_1}\right)^{\frac{\gamma_2}{\gamma_2}} = \epsilon,$$

which directly implies that

$$\begin{aligned} & w\left(\mathcal{A}_{M^{j+1}}(u,Q_{8\sqrt{n}}) \cap K_1\right) \\ & \le \epsilon\, w\left(\mathcal{A}_{M^j}(u,Q_{8\sqrt{n}}) \cap K_1\right) + \epsilon w\left(\left\{(x,t)\in K_1 \colon \mathcal{M}(f^{n+1})(x,t) \ge (\eta M^j)^{n+1}\right\}\right) \end{aligned}$$

for $j=0,1,2\dots$. Hence, by iterating these estimates, the desired estimates hold. □

We are now ready to prove the main result of this paper.

Proof of Theorem 2.2. It suffices to obtain the desired estimates (2.4) for C-viscosity solutions u of (1.1) assuming that F and f are continuous in all variables by using the same approximation procedure as in the proofs of [7, Theorems 4.1, 4.5].

Given the N-function Φ, we denote $\Psi(t) = \Phi(t^{n+1})$ for $t\in[0,\infty)$. Then it is clear that Ψ is also an N-function satisfying $\Delta_2 \cap \nabla_2$-condition and $i(\Psi) = (n+1)\, i(\Phi)$. By the monotonicity property, we further see that $w \in A_{i(\Phi)} \subset A_{i(\Psi)}$.

Without loss of generality, we fix $(x_0,t_0)=(0,0)$. Choose a small constant $r \in \left(0, \frac{1}{16\sqrt{n}}\right)$ which will be selected later and set

$$L := \frac{1}{\delta}\|f\|_{L^\Psi_w(Q_{8r\sqrt{n}})} + \frac{1}{r^2}\|u\|_{L^\infty(Q_{8r\sqrt{n}})}$$

where $\delta = \delta(n,\lambda,\Lambda,c_\star,\Phi,w,\epsilon) \in (0,1)$ is the same as in Lemma 3.5. Here ϵ will be determined later.

We define $\tilde{w}(x,t) := w(rx, r^2t)$ and $\tilde{u}(x,t) := \frac{\mu}{Lr^2}u(rx, r^2t)$. It is clear that $\tilde{w} \in A_{i(\Phi)}$. We observe that $\tilde{u}$ is a solution to

$$\tilde{u}_t - \tilde{F}_\mu(D^2\tilde{u}, x, t) = \tilde{f}(x,t) \text{ in } Q_{8\sqrt{n}}$$

where

$$\tilde{F}_\mu(X,x,t) := \frac{1}{L}F_\mu(LX, rx, r^2t) = \frac{\mu}{L}F\big(\frac{L}{\mu}X, rx, r^2t\big) \text{ and } \tilde{f}(x,t) := \frac{\mu}{L}f(rx, r^2t).$$

We set $\mu := \frac{\delta}{2}$ and then note that $\|\tilde{u}\|_{L^\infty(Q_{8\sqrt{n}})} \le \mu < \delta \le 1$ and $\|\tilde{f}\|_{L^\Psi_w(Q_{8\sqrt{n}})} \le 1$.

We recall $w \in A_p$ where p is given in Remark 1. By the same argument as in the proof of Theorem 2.8 in [17] (or see [2, Lemma 4.1]) with the fact that $\|\frac{\mu f}{\delta L}\|_{L^\Psi_w(Q_{8\sqrt{n}})} \le 1$, we have that

$$\begin{aligned}
\|\tilde{f}\|^{n+1}_{L^{n+1}(Q_{8\sqrt{n}})} &\le c\,\delta^{n+1}\left(\fint_{Q_{8r\sqrt{n}}}\left|\frac{\mu f}{\delta L}\right|^{p(n+1)} w\,dxdt\right)^{\frac{1}{p}}\left(\fint_{Q_{8r\sqrt{n}}} w^{-\frac{1}{p-1}}\,dxdt\right)^{\frac{p-1}{p}}\\
&\le c\,\delta^{n+1}[w]_p^{\frac{1}{p}}\left(\frac{1}{w(Q_{8r\sqrt{n}})}\int_{Q_{8r\sqrt{n}}}\left|\frac{\mu f}{\delta L}\right|^{p(n+1)} w\,dxdt\right)^{\frac{1}{p}}\\
&\le c\,\mu\delta^{n+1}\Phi^{-1}\left(\frac{1}{w(Q_{8r\sqrt{n}})}\int_{Q_{8r\sqrt{n}}}\Phi\Big(\left|\frac{\mu f}{\delta L}\right|^{n+1}\Big) w\,dxdt\right)\\
&\le c\,\delta^{n+1}
\end{aligned}$$

for some constant $c = c(n, \Phi, w) > 0$. One can check that the operator $\tilde{F}$ satisfies all the hypotheses in Lemma 3.5. Therefore, applying Lemma 3.5, we obtain

$$\begin{aligned}
&\tilde{w}\left(\mathcal{A}_{M^j}(\tilde{u}, Q_{8\sqrt{n}}) \cap K_1\right)\\
&\quad\le \epsilon^j\tilde{w}(K_1) + \sum_{i=0}^{j-1}\epsilon^{j-i}\tilde{w}\left(\{(x,t) \in K_1 \colon \mathcal{M}(\tilde{f}^{n+1})(x,t) \ge (\eta M^i)^{n+1}\}\right)
\end{aligned} \tag{3.1}$$

for some universal constants $M > 1$ and $\eta > 0$.

By $\Phi \in \Delta_2$, we see $\Phi(M^{n+1}\rho) \le \kappa\,\Phi(\rho)$ for any $\rho > 0$ and for some constant $\kappa = \kappa(n, M) > 0$. We iterate this inequality to discover $\Phi(M^{\ell(n+1)}) \le \kappa^\ell\Phi(1)$ for each $\ell \ge 1$. We also see that $\Phi(M^{\ell(n+1)}) \le \kappa^{\ell-m}\Phi(M^{m(n+1)})$ for any $0 \le m \le \ell - 1$. Therefore it follows from (3.1) that

$$\begin{aligned}
&\sum_{\ell\ge1}\Psi(M^\ell)\tilde{w}\left(\mathcal{A}_{M^\ell}(\tilde{u}, Q_{8\sqrt{n}}) \cap K_1\right) = \sum_{\ell\ge1}\Phi(M^{\ell(n+1)})\tilde{w}\left(\mathcal{A}_{M^\ell}(\tilde{u}, Q_{8\sqrt{n}}) \cap K_1\right)\\
&\le \Phi(1)\tilde{w}(K_1)\sum_{\ell\ge1}(\kappa\epsilon)^\ell\\
&+ \sum_{\ell\ge1}(\kappa\epsilon)^\ell\sum_{m\ge0}\Phi(M^{m(n+1)})\tilde{w}\left(\{(x,t) \in K_1 \colon \mathcal{M}(\tilde{f}^{n+1})(x,t) \ge (\eta M^m)^{n+1}\}\right).
\end{aligned} \tag{3.2}$$

By the assumption that $f \in L^{\Psi}_{w}(Q_1)$, we note that $|\tilde{f}|^{n+1} \in L^{\Phi}_{\tilde{w}}(Q_{8\sqrt{n}})$. Then we deduce that $\mathcal{M}(\tilde{f}^{n+1}) \in L^{\Phi}_{\tilde{w}}(Q_{8\sqrt{n}})$ and

$$\begin{aligned}
&\int_{Q_{8\sqrt{n}}} \Phi\left(\mathcal{M}(\tilde{f}^{n+1})(x,t)\right) \tilde{w}(x,t)\, dxdt \\
&\le c\int_{Q_{8\sqrt{n}}} \Phi\left(|\tilde{f}(x,t)|^{n+1}\right) \tilde{w}(x,t)\, dxdt = c\int_{Q_{8\sqrt{n}}} \Psi\left(|\tilde{f}(x,t)|\right) \tilde{w}(x,t)\, dxdt \le c
\end{aligned}$$

by taking into account (2.2), Lemma 3.2, and unit ball property with the fact that $\|\tilde{f}\|_{L^{\Psi}_{\tilde{w}}(Q_{8\sqrt{n}})} \le 1$. In turn, if follows from Lemma 3.3 that

$$\begin{aligned}
&\sum_{i\ge 0} \Phi(M^{m(n+1)})\tilde{w}\left(\{(x,t)\in K_1 \colon \mathcal{M}(\tilde{f}^{n+1})(x,t) \ge (\eta M^m)^{n+1}\}\right) \\
&\le c\int_{K_1} \Phi\left(\mathcal{M}(\tilde{f}^{n+1})(x,t)\right) \tilde{w}(x,t)\, dxdt \\
&\le c\int_{Q_{8\sqrt{n}}} \Phi\left(\mathcal{M}(\tilde{f}^{n+1})(x,t)\right) \tilde{w}(x,t)\, dxdt \le c.
\end{aligned} \tag{3.3}$$

Putting (3.3) into (3.2) and taking ϵ such that $\kappa\epsilon \le \frac{1}{2}$, we obtain that

$$\sum_{\ell\ge 1} \Psi\left(M^{\ell}\right) \tilde{w}\left(\mathcal{A}_{M^{\ell}}(\tilde{u}, Q_{8\sqrt{n}}) \cap K_1\right) \le \left(\Phi(1)\tilde{w}(K_1) + \frac{c}{r^{n+2}}\right) \sum_{\ell\ge 1} (\kappa\epsilon)^{\ell} \le c.$$

At this stage the small constant $\delta = \delta(n, \lambda, \Lambda, c_\star, \Phi, w)$ is determined. From the definition of Θ, we then obtain

$$\begin{aligned}
&\sum_{\ell\ge 1} \Psi(M^{\ell})\, \tilde{w}\left(\{(x,t)\in Q_{\frac{1}{2}} \colon \Theta(\tilde{u}, Q_{\frac{1}{2}})(x,t) > M^{\ell}\}\right) \\
&\le \sum_{\ell\ge 1} \Psi(M^{\ell})\, \tilde{w}\left(\mathcal{A}_{M^{\ell}}(\tilde{u}, Q_{\frac{1}{2}})\right) \le \sum_{\ell\ge 1} \Psi(M^{\ell})\, \tilde{w}\left(\mathcal{A}_{M^{\ell}}(\tilde{u}, Q_{8\sqrt{n}}) \cap K_1\right) \le c
\end{aligned}$$

for some constant $c = c(n, \lambda, \Lambda, c_\star, \Phi, w) > 0$. Therefore Lemma 3.3 allows us to discover

$$\begin{aligned}
&\int_{Q_{\frac{1}{2}}} \Psi\left(\Theta(\tilde{u}, Q_{\frac{1}{2}})(x,t)\right) \tilde{w}(x,t)\, dxdt \\
&\le c\left(\tilde{w}(Q_{\frac{1}{2}}) + \sum_{\ell\ge 1} \Psi(M^{\ell})\, \tilde{w}\left(\{(x,t)\in Q_{\frac{1}{2}} \colon \Theta(\tilde{u}, Q_{\frac{1}{2}})(x,t) > M^{\ell}\}\right)\right) \le c
\end{aligned}$$

for some constant $c = c(n, \lambda, \Lambda, c_\star, \Phi, w) > 0$. By virtue of Lemma 3.1, we have $\|\tilde{u}_t\|_{L^{\Psi}_{\tilde{w}}(Q_{\frac{1}{2}})} + \|D^2\tilde{u}\|_{L^{\Psi}_{\tilde{w}}(Q_{\frac{1}{2}})} \le c\|\Theta(\tilde{u}, Q_{\frac{1}{2}})\|_{L^{\Psi}_{\tilde{w}}(Q_{\frac{1}{2}})} \le c$, which implies

$$\|u_t\|_{L^{\Psi}_{w}(Q_{\frac{r}{2}})} + \|D^2 u\|_{L^{\Psi}_{w}(Q_{\frac{r}{2}})} \le c\left(\|f\|_{L^{\Psi}_{w}(Q_{8r\sqrt{n}})} + \frac{1}{r^2}\|u\|_{L^{\infty}(Q_{8r\sqrt{n}})}\right).$$

Consequently, we apply the standard covering argument to obtain the desired estimates (2.4) by choosing r sufficiently small so that $Q_{\frac{1}{2}}$ is covered by finite number of cylinders $Q_r(x_0, t_0)$ for $(x_0, t_0) \in Q_{\frac{1}{2}}$. □

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