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EXISTENCE OF POSITIVE SOLUTIONS FOR GENERALIZED LAPLACIAN PROBLEMS WITH A PARAMETER

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Abstract. In this paper, we study singular Dirichlet boundary value problems involving φ-Laplacian. Using fixed point index theory, the existence of positive solutions is established under the assumption that the nonlinearity $f = f(u)$ has a positive falling zero and is either superlinear or sublinear at $u = 0$.

1. Introduction

In this paper, we study the existence of positive solutions to the following boundary value problem

$$\begin{cases} (q(t)\varphi(u'(t)))' + \lambda h(t)f(u(t)) = 0, \ t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $q \in C([0,1],(0,\infty))$, $\lambda \in \mathbb{R}_+ := [0,\infty)$ is a parameter, $h \in C((0,1),\mathbb{R}_+)$ and $f \in C(\mathbb{R}_+,\mathbb{R})$.

By a solution u to problem (1), we mean $u \in C^1(0,1) \cap C[0,1]$ with $w\varphi(u') \in C^1(0,1)$ satisfies (1). Problem (1) arises naturally in studying radial solutions to the following quasilinear elliptic equation defined on an annular domain

$$\begin{cases} \operatorname{div}(w(|x|)A(|\nabla v|)\nabla v) + \lambda k(|x|)f_1(v) = 0 \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ with $N \geq 2$ and $0 < R_1 < R_2 < \infty$, $w \in C([R_1,R_2],(0,\infty))$, $k \in C((R_1,R_2),\mathbb{R}_+)$ and $f_1 \in C(\mathbb{R}_+,\mathbb{R})$. Indeed, applying change of variables

$$v(x) = (R_2 - R_1)u(t) \text{ and } |x| = (R_2 - R_1)t + R_1 =: r(t),$$

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problem (2) is transformed into problem (1) with $\varphi(s) = A(|s|)s$, $q(t) = w(r(t))r^{N-1}(t)$, $h(t) = (R_2 - R_1)r^{N-1}(t)k(r(t))$ and $f(u) = f_1((R_2 - R_1)u)$ (see, e.g., [4, 18]). Thus, the existence of positive solutions to problem (1) guarantees the existence of positive radial solutions to problem (2).

Throughout this paper, we assume that the odd increasing homeomorphism φ satisfies the following assumption

(F_1) there exist increasing homeomorphisms $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(x)\psi_1(y) \le \varphi(yx) \le \varphi(x)\psi_2(y) \text{ for all } x, y \in \mathbb{R}_+. \tag{3}$$

For example, define $\varphi : \mathbb{R} \to \mathbb{R}$ to be an odd function with

$$(i)\ \varphi(x) = \tfrac{x^2}{1+x} \text{ or } (ii)\ \varphi(x) = x + x^2 \text{ for } x \in \mathbb{R}_+.$$

Then it is easy to check that (F_1) is satisfied with

$$\psi_1(y) = \min\{y, y^2\} \text{ and } \psi_2(y) = \max\{y, y^2\}.$$

Let us introduce notations

$$f_0 := \lim_{s\to 0^+} \frac{f(s)}{\varphi(s)},\ f_\infty := \lim_{s\to\infty} \frac{f(s)}{\varphi(s)}$$

and, for an increasing homeomorphism ξ on $\mathbb{R}_+$,

$$\mathcal{H}_\xi := \left\{ g \in C((0,1), \mathbb{R}_+) : \int_0^1 \left| \xi^{-1} \left(\int_s^{\frac{1}{2}} g(\tau)d\tau \right) \right| ds < \infty \right\}.$$

Now we give a list of assumptions which will be used in this paper.

(F_2) $h \in \mathcal{H}_\varphi \setminus \{0\}$.
(F_2') $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$.
(F_3) there exists $M > 0$ such that $f(s) > 0$ for $s \in (0, M)$ and $f(M) = 0$.
(F_4) $f_0 = \infty$.
(F_4') $f_0 = 0$.

From (F_1), it follows that

$$\varphi^{-1}(x)\psi_2^{-1}(y) \le \varphi^{-1}(xy) \le \varphi^{-1}(x)\psi_1^{-1}(y) \text{ for all } x, y \in \mathbb{R}_+ \tag{4}$$

and

$$L^1(0,1) \cap C(0,1) \subseteq \mathcal{H}_{\psi_1} \subseteq \mathcal{H}_\varphi \subseteq \mathcal{H}_{\psi_2}$$

(see, e.g., ([4], Remark 1)). Thus, the assumption (F_2') is stronger than the one (F_2). For example, let $\varphi(x) = x + x^2$ for $x \in \mathbb{R}_+$, and define $h : (0,1) \to \mathbb{R}_+$ by

$$h(t) = t^{-c}(\tfrac{3}{4} - t) \text{ for } t \in (0, \tfrac{3}{4}) \text{ and } h(t) = 0 \text{ for } t \in [\tfrac{3}{4}, 1] \text{ and}$$

From the facts that $\psi_1^{-1}(s) = s$ and $\varphi^{-1}(s) = \frac{-1+\sqrt{1+4s}}{2}$ for all $s \ge 1$, it follows that $h \in \mathcal{H}_{\psi_1} \setminus L^1(0,1)$ for any $c \in [1, 2)$ and $t^{-2} \in \mathcal{H}_\varphi \setminus \mathcal{H}_{\psi_1}$.

We introduce two assumptions on φ which are equivalent to (F_1):

$(H1)$ there exists an increasing homeomorphism $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(x)\psi_1(y) \le \varphi(xy) \text{ for all } x, y \in \mathbb{R}_+;$$

$(H1')$ there exist an increasing homeomorphism $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and a function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(x)\psi_1(y) \le \varphi(xy) \le \varphi(x)\chi(y) \text{ for all } x, y \in \mathbb{R}_+.$$

It looks like that the assumption $(H1)$ is more general than the ones $(H1')$ and (F_1), but all the assumptions on φ are identical (see [4]). To be more specific, if we assume that $(H1)$ holds, then we can define an increasing homeomorphism ψ_2 which satisfies the second inequality in the assumption (F_1). Indeed, let us define $\psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\psi_2(0) := 0 \text{ and } \psi_2(y) := \left(\psi_1(y^{-1})\right)^{-1} \text{ for } y > 0.$$

Then ψ_2 is an increasing homeomorphism on $\mathbb{R}_+$. From $(H1)$, it follows that

$$0 < \varphi(xy)\psi_1(y^{-1}) \leq \varphi(x) \text{ for } x, y > 0.$$

Consequently, $\varphi(xy) \leq \varphi(x) \left(\psi_1(y^{-1})\right)^{-1} = \varphi(x)\psi_2(y)$ for all $x, y \in \mathbb{R}_+$, and all the assumptions on φ above are identical.

Over recent decades, the existence of positive solutions to p-Laplacian or more generalized Laplacian problems have been extensively studied (see, e.g., [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). For example, for $\varphi(s) = |s|^{p-2}s$ with $p > 1$, $h \in \mathcal{H}_\varphi$ and $q \equiv 1$, the existence and multiplicity of positive solutions to problem (1) under various assumptions on f_0 and f_∞ were studied in [1]. When $f(s)$ satisfies $f(0) > 0$ and $f_\infty = \infty$, in [10], it was shown that there exists $\lambda_* > 0$ such that (1) has at least two positive solutions for $\lambda \in (0, \lambda_*)$, one positive solution for $\lambda = \lambda_*$ and no positive solution for $\lambda > \lambda_*$. Recently, under the assumptions that $h \in C^1((0, 1], (0, \infty))$ is strictly decreasing, $h(t) \leq Ct^{-\eta}$ for some $C > 0$ and $\eta \in (0, 1)$, in [16], the conditions on the nonlinearity $f = f(u)$ was investigated which ensure the uniqueness of positive solution to problem (1) for all large $\lambda > 0$.

For φ satisfying $(H1)$, in [6], the existence of positive solutions of the one dimensional differential equation with deviating arguments was studied. In [15], when φ satisfies $(H1')$, $q \equiv 1$, $h \in \mathcal{H}_{\psi_1}$ and either $f_0 = f_\infty = \infty$ or $f_0 = f_\infty = 0$, it was shown that there exist $\lambda^* \geq \lambda_* > 0$ such that (1) has at least two positive solutions for $\lambda \in (0, \lambda_*)$, one positive solution for $\lambda \in [\lambda_*, \lambda^*]$ and no positive solution for $\lambda > \lambda_*$ under the assumption that $h \in \mathcal{H}_{\psi_1}$ and $f_0 = f_\infty = \infty$,. In [4], for nonnegative nonlinearity $f = f(t, u)$ satisfying $f(t_0, 0) > 0$ for some $t_0 \in [0, 1]$ and $h \in \mathcal{H}_\varphi$, the existence of an unbounded solution component $\mathcal{C}$ was shown and, the existence, nonexistence and multiplicity of positive solutions were studied by investigating the shape of $\mathcal{C}$ depending on the behavior of $f = f(t, u)$ at $u = \infty$.

We give some examples of $f = f(u)$ satisfying either (F_3) and (F_4) or (F_3) and (F_4'). Define $f : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(u) = [\varphi(u)]^d (1 - u) \text{ for } u \in \mathbb{R}_+.$$

Then if $d \in (0, 1)$, then the nonlinearity $f = f(u)$ satisfies (F_3) and (F_4); if $d \in (1, \infty)$, then it satisfies (F_3) and (F_4').

To the author's knowledge, there is no existence results for positive solutions to generalized Laplacian problem (1) under the assumptions that the weight function h admits stronger singularity than $L^1(0, 1)$ at $t = 0$ and/or $t = 1$, and the nonlinearity $f = f(u)$ satisfies (F_3), i.e., it has a positive falling zero. The following is the main result in this paper.

Theorem 1.1. *Assume that* $(F1), (F_2')$ *and* (F_3) *hold.*

- (i) *Assume, in addition, that* (F_4) *holds. Then problem* (1) *has a positive solution* $u(\lambda)$ *for any* $\lambda \in (0, R_1(M))$ *satisfying* $\|u_\lambda\|_\infty \to 0$ *as* $\lambda \to 0^+$. *Here,* R_1 *is the function which will be defined in Section 3.*
- (ii) *Assume, in addition, that* (F_4') *is assumed. Then there exists* $\lambda_* > 0$ *such that problem* (1) *has a positive solution* $u(\lambda)$ *for any* $\lambda \in (\lambda_*, \infty)$ *satisfying* $\|u_\lambda\|_\infty \to 0$ *as* $\lambda \to \infty$.

The rest of this paper is organized as follows. In Section 2, we give preliminary results which are essential for proving the main result (Theorem 1.1) in this paper. In Section 3, the main result is proved.

2. Preliminaries

Throughout this section, we assume that (F_1), (F_2) and $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ hold. The usual maximum norm in a Banach space $C[0,1]$ is denoted by

$$\|u\|_\infty := \max_{t\in[0,1]} |u(t)| \text{ for } u \in C[0,1].$$

$$\alpha_h := \inf\{x \in (0,1) : h(x) > 0\},\ \beta_h := \sup\{x \in (0,1) : h(x) > 0\},$$
$$\bar{\alpha}_h := \sup\{x \in (0,1) : h(y) > 0 \text{ for all } y \in (\alpha_h, x)\},$$
$$\bar{\beta}_h := \inf\{x \in (0,1) : h(y) > 0 \text{ for all } y \in (x, \beta_h)\},$$
$$\gamma_h^1 := \frac{1}{4}(3\alpha_h + \bar{\alpha}_h), \text{ and } \gamma_h^2 := \frac{1}{4}(\bar{\beta}_h + 3\beta_h).$$

Then, since $h \in C((0,1), \mathbb{R}_+) \setminus \{0\}$, we have two cases: either

$$(i)\ 0 \le \alpha_h < \bar{\alpha}_h \le \bar{\beta}_h < \beta_h \le 1$$

or

$$(ii)\ 0 \le \alpha_h = \bar{\beta}_h < \beta_h \le 1 \text{ and } 0 \le \alpha_h < \bar{\alpha}_h = \beta_h \le 1.$$

Consequently,

$$h(t) > 0 \text{ for } t \in (\alpha_h, \bar{\alpha}_h) \cup (\bar{\beta}_h, \beta_h), \text{ and } 0 \le \alpha_h < \gamma_h^1 < \gamma_h^2 < \beta_h \le 1. \tag{5}$$

Let $\rho_h := \rho_1 \min\{\gamma_h^1, 1-\gamma_h^2\} \in (0,1)$, where

$$q_0 := \min_{t\in[0,1]} q(t) > 0 \text{ and } \rho_1 := \psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{q_0}\right)\right]^{-1} \in (0,1].$$

Then

$$\mathcal{K} := \{u \in C([0,1], \mathbb{R}_+) : u(t) \ge \rho_h \|u\|_\infty \text{ for } t \in [\gamma_h^1, \gamma_h^2]\}$$

is a cone in $C[0,1]$. For $r > 0$, let $\mathcal{K}_r := \{u \in \mathcal{K} : \|u\|_\infty < r\}$, $\partial\mathcal{K}_r := \{u \in \mathcal{K} : \|u\|_\infty = r\}$ and $\overline{\mathcal{K}}_r := \mathcal{K}_r \cup \partial\mathcal{K}_r$.

For $g \in \mathcal{H}_\varphi$, consider the following problem

$$\begin{cases} (q(t)\varphi(u'(t)))' + g(t) = 0,\ t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{6}$$

Define a function $T : \mathcal{H}_\varphi \to C[0,1]$ by $T(0) = 0$ and, for $g \in \mathcal{H}_\varphi \setminus \{0\}$,

$$T(g)(t) = \begin{cases} \int_0^t \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma g(\tau)d\tau\right) ds, & \text{if } 0 \le t \le \sigma, \\ \int_t^1 \varphi^{-1}\left(\frac{1}{q(s)}\int_\sigma^s g(\tau)d\tau\right) ds, & \text{if } \sigma \le t \le 1, \end{cases} \tag{7}$$

where $\sigma = \sigma(g)$ is a constant satisfying

$$\int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma g(\tau)d\tau\right) ds = \int_\sigma^1 \varphi^{-1}\left(\frac{1}{q(s)}\int_\sigma^s g(\tau)d\tau\right) ds. \tag{8}$$

For any $g \in \mathcal{H}_\varphi$ and any σ satisfying (8), $T(g)$ is monotone increasing on $[0,\sigma)$ and monotone decreasing on $(\sigma, 1]$. We notice that $\sigma = \sigma(g)$ is not necessarily unique, but $T(g)$ is independent of the choice of σ satisfying (8) (see, e.g., [18] or [5, Remark 2]).

Lemma 2.1. ([4, Lemma 1 and Lemma 2]) *Assume that* $(F_1), (F_2)$ *and* $g \in \mathcal{H}_\varphi$ *hold. Then* $T(g)$ *is a unique solution to problem* (6) *satisfying the following properties:*

(*i*) $T(g)(t) \ge 0$ *for* $t \in [0,1]$*;*
(*ii*) σ *is a constant satisfying* (8) *if and only if* $T(g)(\sigma) = \|T(g)\|_\infty$*;*
(*iii*) $T(g)(t) \ge \rho_1 \min\{t, 1-t\}\|T(g)\|_\infty$ *for* $t \in [0,1]$ *and* $T(g) \in \mathcal{K}$.

Define a function $F : \mathbb{R}_+ \times \mathcal{K} \to C(0,1)$ by

$$F(\lambda, u)(t) := \lambda h(t) f(u(t)) \text{ for } (\lambda, u) \in \mathbb{R}_+ \times \mathcal{K} \text{ and } t \in (0,1).$$

Clearly, $F(\lambda, u) \in \mathcal{H}_\varphi$ for any $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$, since $h \in \mathcal{H}_\varphi$. Let us define an operator $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ by

$$H(\lambda, u) := T(F(\lambda, u)) \text{ for } (\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}.$$

By Lemma 2.1 (*iii*), $H(\mathbb{R}_+ \times \mathcal{K}) \subseteq \mathcal{K}$ and consequently, H is well defined.

$$H(\lambda, u)(\sigma) = \|H(\lambda, u)\|_\infty > 0 \text{ for any } (\lambda, u) \in (0,\infty) \times \mathcal{K}.$$

Moreover, u is a positive solution to problem (1) if and only if $H(\lambda, u) = u$ for some $(\lambda, u) \in (0, \infty) \times \mathcal{K}$.

By the argument similar to those in the proof of [1, Lemma 3] or [11, Lemma 2.4], one can prove the following lemma.

Lemma 2.2. ([4, Lemma 4]) *Assume that* $(F_1), (F_2)$ *and* $f \in C([0,1]\times\mathbb{R}_+, \mathbb{R}_+)$ *hold. Then the operator* $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ *is completely continuous.*

Finally, we recall a well-known theorem of the fixed point index theory.

Theorem 2.3. ([2, 3]) *Assume that, for some* $r > 0$, $H : \overline{\mathcal{K}}_r \to \mathcal{K}$ *is completely continuous. Then the following assertions are true.*

(*i*) $i(H, \mathcal{K}_r, \mathcal{K}) = 1$ *if* $\|H(u)\|_\infty < \|u\|_\infty$ *for* $u \in \partial\mathcal{K}_r$*;*
(*ii*) $i(H, \mathcal{K}_r, \mathcal{K}) = 0$ *if* $\|H(u)\|_\infty > \|u\|_\infty$ *for* $u \in \partial\mathcal{K}_r$.

3. Main results

Throughout this section, we assume that (F_1) and (F_2) hold, and that

(F_3') $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and there exists $M > 0$ such that $f(s) > 0$ for $s \in (0, M)$,

unless otherwise stated. Let

$$\mathcal{C}_1 := \psi_1^{-1}\left(\frac{1}{q_0}\right)\max\left\{\int_0^{\gamma_h}\psi_1^{-1}\left(\int_s^{\gamma_h}h(\tau)d\tau\right)ds, \int_{\gamma_h}^1\psi_1^{-1}\left(\int_{\gamma_h}^s h(\tau)d\tau\right)ds\right\}$$

and

$$\mathcal{C}_2 := \psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right)\min\left\{\int_{\gamma_h^1}^{\gamma_h}\psi_2^{-1}\left(\int_s^{\gamma_h}h(\tau)d\tau\right)ds, \int_{\gamma_h}^{\gamma_h^2}\psi_2^{-1}\left(\int_{\gamma_h}^s h(\tau)d\tau\right)ds\right\}.$$

Here, $\gamma_h := \dfrac{\gamma_h^1 + \gamma_h^2}{2}$. Clearly, by (5), $\mathcal{C}_1 > 0$ and $\mathcal{C}_2 > 0$.

Define continuous functions $R_1, R_2 : (0, M) \to (0, \infty)$ by

$$R_1(m) := \psi_1(\mathcal{C}_1^{-1})\frac{\varphi(m)}{f^*(m)} \text{ and } R_2(m) := \psi_2(\mathcal{C}_2^{-1})\frac{\varphi(m)}{f_*(m)} \text{ for } m \in (0, \infty).$$

Here, $f^*(m) := \max\{f(x) : 0 \le x \le m\}$ and $f_*(m) := \min\{f(x) : \rho_h m \le x \le m\}$.

Remark 1. (1) By (3) and (4),

$$\psi_1(x) \le \psi_2(x) \text{ and } \psi_2^{-1}(x) \le \psi_1^{-1}(x) \text{ for all } x \in \mathbb{R}_+.$$

Consequently,

$$0 < \mathcal{C}_2 < \mathcal{C}_1 \text{ and } 0 < R_1(m) < R_2(m) \text{ for all } m \in (0, M).$$

Assume that (F_3) holds. Then

$$\lim_{m\to M^-} R_1(m) = \frac{1}{\hat{f}}\varphi\left(\frac{M}{\mathcal{C}_1}\right) =: R_1(M) \text{ and } \lim_{m\to M^-} R_2(m) = \infty.$$

Here, $\hat{f} := \max\{f(s) : 0 \le s \le M\}$.

(2) It is well known that

(i) $\displaystyle\lim_{m\to 0^+}\frac{f^*(m)}{\varphi(m)} = \lim_{m\to 0^+}\frac{f_*(m)}{\varphi(m)} = 0$ if $f_0 = 0$,

(ii) $\displaystyle\lim_{m\to 0^+}\frac{f^*(m)}{\varphi(m)} = \lim_{m\to 0^+}\frac{f_*(m)}{\varphi(m)} = \infty$ if $f_0 = \infty$

(see. e.g., [12, Remark 2]). Consequently, for $i \in \{1, 2\}$,

$$\lim_{m\to 0^+} R_i(m) = \infty \text{ if } f_0 = 0, \text{ and } \lim_{m\to 0^+} R_i(m) = 0 \text{ if } f_0 = \infty.$$

The following lemma can be proved by the argument similar to those in the proofs of [12, Lemma 3 and Lemma 4]. For the reader's convenience, we give the proof of it in details.

Lemma 3.1. *Assume that (F_1), (F_2') and (F_3') hold. Let $m \in (0, M)$ be fixed.*

(i) *For any $\lambda \in (0, R_1(m))$, $\|H(\lambda, v)\|_\infty < \|v\|_\infty$ for all $v \in \partial\mathcal{K}_m$ and*

$$i(H(\lambda, \cdot), \mathcal{K}_m, \mathcal{K}) = 1. \tag{9}$$

(ii) *For any* $\lambda \in (R_2(m), \infty)$, $\|H(\lambda, v)\|_\infty > \|v\|$ *for all* $v \in \partial\mathcal{K}_m$ *and*

$$i(H(\lambda,\cdot), \mathcal{K}_m, \mathcal{K}) = 0. \tag{10}$$

Proof. (i) Let $\lambda \in (0, R_1(m))$ and $v \in \partial\mathcal{K}_m$ be fixed. Then, for $t \in [0,1]$,

$$0 \le \lambda f(v(t)) \le \lambda f^*(m) = \frac{\lambda}{R_1(m)}\varphi(m)\psi_1(\mathcal{C}_1^{-1}) < \varphi(m)\psi_1(\mathcal{C}_1^{-1}). \tag{11}$$

Let σ be a number satisfying $H(\lambda, v)(\sigma) = \|H(\lambda, v)\|_\infty$. We have two cases: either (i) $\sigma \in (0, \gamma_h)$ or (ii) $\sigma \in [\gamma_h, 1)$. We only consider the case (i) since the case (ii) can be proved in a similar manner. From (4),(11) and the definition of $\mathcal{C}_1$, it follows that

$$\begin{aligned}
\|H(\lambda, v)\|_\infty &= \int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma \lambda h(\tau) f(v(\tau))d\tau\right) ds \\
&< \int_0^{\gamma_h} \varphi^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau q_0^{-1}\varphi(m)\psi_1(\mathcal{C}_1^{-1})\right) ds \\
&\le \int_0^{\gamma_h} \psi_1^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right) ds\varphi^{-1}\left(q_0^{-1}\varphi(m)\psi_1(\mathcal{C}_1^{-1})\right) \\
&\le \int_0^{\gamma_h} \psi_1^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right) ds\psi_1^{-1}\left(q_0^{-1}\right)\varphi^{-1}\left(\varphi(m)\psi_1(\mathcal{C}_1^{-1})\right) \\
&\le \mathcal{C}_1 m \mathcal{C}_1^{-1} = m = \|v\|_\infty.
\end{aligned}$$

By Theorem 2.3, (9) holds for any $\lambda \in (0, R_1(m))$.

(ii) Let $\lambda \in (R_2(m), \infty)$ and $v \in \partial\mathcal{K}_m$ be fixed. Then

$$\rho_h m \le v(t) \le m \text{ for } t \in [\gamma_h^1, \gamma_h^2],$$

and

$$\lambda f(v(t)) \ge \frac{\lambda}{R_2(m)}\varphi(m)\psi_2(\mathcal{C}_2^{-1}) > \varphi(m)\psi_2(\mathcal{C}_2^{-1}) \text{ for } t \in [\gamma_h^1, \gamma_h^2]. \tag{12}$$

Let σ be a constant satisfying $H(\lambda, v)(\sigma) = \|H(\lambda, v)\|_\infty$. Then we have two cases: either (i) $\sigma \in [\gamma_h, 1)$ or (ii) $\sigma \in (0, \gamma_h)$. We only consider the case (i) since the case (ii) can be dealt in a similar manner. It follows from (4), (12) and the definition of C_2 that

$$\begin{aligned}
\|H(\lambda, v)\|_\infty &= \int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma \lambda h(\tau) f(v(\tau))d\tau\right) ds \\
&> \int_{\gamma_h^1}^{\gamma_h} \varphi^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau \|q\|_\infty^{-1}\varphi(m)\psi_2(\mathcal{C}_2^{-1})\right) ds \\
&\ge \int_{\gamma_h^1}^{\gamma_h} \psi_2^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right) ds\varphi^{-1}\left(\|q\|_\infty^{-1}\varphi(m)\psi_2(\mathcal{C}_2^{-1})\right) \\
&\ge \int_{\gamma_h^1}^{\gamma_h} \psi_2^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right) ds\psi_2^{-1}\left(\|q\|_\infty^{-1}\right)\varphi^{-1}\left(\varphi(m)\psi_2(\mathcal{C}_2^{-1})\right) \\
&\ge \mathcal{C}_2 m \mathcal{C}_2^{-1} = m = \|v\|_\infty.
\end{aligned}$$

By Theorem 2.3, (10) holds for any $\lambda \in (R_2(m), \infty)$.

$\square$

The following theorem can be proved easily in view of Lemma 3.1 and the fixed point index theory. Thus, we omit the proof of it.

Theorem 3.2. *Assume that* (F_1), (F_2') *and* (F_3') *hold. Assume, in addition, that there exist* m_1 *and* m_2 *such that* $0 < m_1 < m_2 < M$ *(resp.,* $0 < m_2 < m_1 < M$*) and* $R_2(m_2) < R_1(m_1)$. *Then* (1) *has a positive solution* $u = u(\lambda)$ *satisfying* $m_1 < \|u\|_\infty < m_2$ *(resp.,* $m_2 < \|u\|_\infty < m_1$*) for any* $\lambda \in (R_2(m_2), R_1(m_1))$.

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Consider the following modified problem

$$\begin{cases} (q(t)\varphi(u'(t)))' + \lambda h(t)\bar{f}(u(t)) = 0, \ t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{13}$$

where

$$\bar{f}(s) = \begin{cases} 0, & \text{for } (s) \in [M, \infty), \\ f(s), & \text{for } (s) \in [0, M). \end{cases}$$

Then, by (F_3), f_1 satisfies the assumption (F_3').

(i) Assume that $f_0 = \infty$. By Remark 1, for $i = 1, 2$, $R_i(m) \to 0$ as $m \to 0^+$, and $R_2(m) \to \infty$ and $R_1(m) \to R_1(M)$ as $m \to M^-$. Let $\lambda \in (0, R_1(M))$ be fixed. Then there exist $m_1(\lambda)$ and $m_2(\lambda)$ such that

$$0 < m_2(\lambda) < m_1(\lambda) < M \text{ and } R_2(m_2(\lambda)) < \lambda < R_1(m_1(\lambda)).$$

By Theorem 3.2, there exists a positive solution u_λ to problem (13) (equivalently, problem (1)) satisfying $m_2(\lambda) < \|u_\lambda\|_\infty < m_1(\lambda) < M$. Since $0 < R_1(m) < R_2(m)$ for all $m \in (0, M)$ and $R_1(m) \to 0$ as $m \to 0$, we may choose $m_1(\lambda)$ and $m_2(\lambda)$ such that $m_i(\lambda) \to 0$ as $\lambda \to 0^+$ for $i = 1, 2$. Consequently, we can choose positive solutions u_λ to problem (1) for small $\lambda > 0$ satisfying $\|u_\lambda\|_\infty \to 0$ as $\lambda \to 0^+$.

(ii) Assume that $f_0 = 0$. By Remark 1, $R_i(m) \to \infty$ as $m \to 0^+$ for $i = 1, 2$ and $R_2(m) \to \infty$ as $m \to M^-$. Consequently, there exists $\lambda_* := \min\{R_1(m) : m \in (0, M)\} \in (0, \infty)$. Let $\lambda \in (\lambda_*, \infty)$ be fixed. Then there exist $m_1(\lambda)$ and $m_2(\lambda)$ such that

$$0 < m_1(\lambda) < m_2(\lambda) < M \text{ and } R_2(m_2(\lambda)) < \lambda < R_1(m_1(\lambda)).$$

By Theorem 3.2, there exists a positive solution u_λ to problem (13) (equivalently, problem (1)) satisfying $m_1(\lambda) < \|u_\lambda\|_\infty < m_2(\lambda) < M$. Since $R_i(m) \to \infty$ as $m \to 0^+$ for $i = 1, 2$, we may choose $m_1(\lambda)$ and $m_2(\lambda)$ so that $m_i(\lambda) \to 0$ as $\lambda \to \infty$. Consequently, we can choose positive solutions u_λ to problem (1) for large $\lambda > 0$ satisfying $\|u_\lambda\|_\infty \to 0$ as $\lambda \to \infty$.

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