East Asian Math. J.
Vol. 38 (2022), No. 1, pp. 013–020
http://dx.doi.org/10.7858/eamj.2022.002

MULTIPLICITY OF POSITIVE SOLUTIONS TO SCHRÖDINGER-TYPE POSITONE PROBLEMS

Eunkyung Ko

Abstract. We establish multiplicity results for positive solutions to the Schrödinger-type singular positone problem: $-\Delta u + V(x)u = \lambda f(u)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N, N > 2, \lambda$ is a positive parameter, $V \in L^\infty(\Omega)$ and $f : [0,\infty) \to (0,\infty)$ is a continuous function. In particular, when f is sublinear at infinity we discuss the existence of at least three positive solutions for a certain range of λ. The proofs are mainly based on the sub- and supersolution method.

1. Introduction and Main Results

We are concerned with the existence of multiple positive solutions of the following Schrödinger-type positone problems with Dirichlet boundary condition

$$\begin{cases} -\Delta u + V(x)u = \lambda f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $0 \in \Omega$ is a nonempty bounded domain in $\mathbb{R}^N, N > 2$, with a smooth boundary $\partial\Omega$, $V \in L^\infty(\Omega)$ and λ is a positive real parameter. We assume that $f \in C([0,\infty),(0,\infty))$ satisfies

- (F_1) $\displaystyle\lim_{s\to\infty} \frac{f(s)}{s} = 0,$
- (F_2) f is nondecreasing for all $s \geq 0$.

We further assume that $V \in L^\infty(\Omega)$ satisfies the following condition:

- (V_1) There exists $c_V > 0$ such that $V(x) \geq -c_V > -\frac{1}{\|e\|_\infty}$ for $x \in \Omega$, when e is the positive solution of

$$\begin{cases} -\Delta e = 1, \text{ in } \Omega, \\ e = 0, \text{ on } \partial\Omega. \end{cases}$$

Received December 24, 2021; Accepted January 3, 2022.
2010 *Mathematics Subject Classification.* 35J66, 35J20.
Key words and phrases. Schrödinger-type problem, positive solution, multiplicity.
This work was financially supported by the National Research Foundation of Korea (NRF) grant funded by the Korea Government (NRF-2020R1F1A1A01065912).

©2022 The Youngnam Mathematical Society
(pISSN 1226-6973, eISSN 2287-2833)

The equation (1) is derived based on the nonlinear Schrödinger equation, which is detailed in [9]. Nonlinear Schrödinger equations have been widely studied to investigate the existence of solutions that act according to V on the whole space $\mathbb{R}^N$ (see [2, 5, 7] or references therein) or on bounded domains with linear boundary conditions in [4]. In the case when $V \equiv 0$, studying existence of multiple positive solutions of (1) has a rich history for a long time (see [3] and [8] for $\beta = 0$ or references therein). In this paper, we establish the existence of positive solutions of a singular Schrödinger equation (1) for all $\lambda > 0$ and multiple positive solutions of (1) for a certain range of λ by the method of sub and supersolution when $V \not\equiv 0$ is bounded in Ω. We first state our main result.

Theorem 1.1. *Assume* (F_1) *and* (V_1). *Then, for each* $\lambda > 0$ *the problem* (1) *has a positive solution* $u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega})$ *Moreover, if* $\frac{s}{f(s)}$ *is nondecreasing, the solution is unique.*

Next, to state the multiplicity result, we define for any $0 < a < d$

$$Q(a,d) := \frac{a}{f(a)} \Big/ \frac{d}{f(d)}$$

and let

$$A = \frac{(N+1)^{N+1}}{N^N} \text{ and } B = \frac{R^2}{AN + \|V\|_\infty R^2},$$

where R is the radius of the largest inscribed ball B_R in Ω. We further let $K > 0$ be a constant such that

$$\frac{1}{K} \leq 1 - c_V \|e\|_\infty. \tag{2}$$

from the condition (V_1).

Theorem 1.2. *Assume* $(F_1), (F_2)$ *and* (V_1). *If there exist* a, d *and* b *with* $0 < a < d < \frac{2}{A} b$ *such that* $Q(a,d) > \frac{K\|e\|_\infty}{B}$,

$$\tilde{f}(s) := f(s) - \frac{f(d)}{d} B \|V\|_\infty s > 0 \ \forall s \in [0, b]$$

and is nondecreasing on $[a, b]$, *then the problem* (1) *has at least three positive solutions* $u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega})$ *for all* $\lambda_* < \lambda < \lambda^*$, *where*

$$\lambda_* = \frac{d}{f(d)} \frac{1}{B}, \ \ \lambda^* = \min\{\frac{a}{f(a)} \frac{1}{K\|e\|_\infty}, \frac{2b}{f(d)AB}\}.$$

To obtain multiple positive solutions for a certain range of λ, it is important to construct a pair of sub and supersolution (ψ_2, Z_2) of (1) with the property that $\psi_1 \leq \psi_2 \leq Z_1$, $\psi_1 \leq Z_2 \leq Z_1$ and $\psi_2 \not\leq Z_2$ when (ψ_1, Z_1) is a pair of sub and supersolution of (1). However, the term $V(x)$ acting on u gives a difficulty on the construction of a second pair of sub and supersolution (ψ_2, Z_2). Furthermore, since we allow V to be negative on Ω, the operator $-\Delta + V(x)$ does not satisfy the maximum principle. To overcome this issues, we manipulate the first equation of (1) in such a way that we define a new function $\tilde{f} : [0, \infty) \to \mathbb{R}$

by $\tilde{f}(s) = f(s) - \frac{f(d)}{d}B\|V\|_\infty s$ so that the function $V(x)$ is related to the reaction term. Here we also emphasize that $\tilde{f}(s)$ is negative for s large since $f(s)$ is sublinear near infinity. In this case, it is not easy to construct the second pair of sub and supersolution (ψ_2, Z_2) satisfying the above property in the region at which $\tilde{f}$ is positive.

Remark 1. A simple example satisfying the assumptions of Theorem 1.2 is

$$\begin{cases} -\Delta u + V(x)u = \lambda e^{\frac{\alpha u}{\alpha+u}}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

Clearly, $f(u) = e^{\frac{\alpha u}{\alpha+u}}$ satisfies assumptions (F_1) and (F_2). Choosing $d = \alpha$, we can see that $\tilde{f}(d) = (1 - B\|V\|_\infty)f(\alpha) > 0$ for all $\alpha > 0$ and $\tilde{f}'(d) = e^{\frac{\alpha}{2}}[\frac{1}{4} - \frac{B\|V\|_\infty}{\alpha}] \to \frac{1}{4}e^{\frac{\alpha}{2}} > 0$ as $\alpha \to \infty$. Hence, there exists $\alpha_1 > 0$ such that

$$\tilde{f}'\left((1 - \frac{1}{\sqrt{\alpha_1}})\alpha_1\right) > 0,$$

which implies that $\tilde{f}'(\alpha) > 0$ for all $\alpha > (1 - \frac{1}{\sqrt{\alpha_1}})\alpha_1$. Noting that $Q(c\alpha, \alpha) = ce^{\frac{1}{2} - \frac{c}{c+1}\alpha}$ for a constant c, there exists $\alpha_2 > 0$ such that

$$Q\left((1 - \frac{1}{\sqrt{\alpha_2}})\alpha_2, \alpha_2\right) = \left(1 - \frac{1}{\sqrt{\alpha_2}}\right)e^{\frac{\alpha_2}{4\sqrt{\alpha_2}-2}} > \frac{K\|e\|_\infty}{B},$$

which also yields that $Q\left((1 - \frac{1}{\sqrt{\alpha}})\alpha, \alpha\right) > \frac{K\|e\|_\infty}{B}$ for all $\alpha > \alpha_2$. Observing that for a constant c

$$\tilde{f}'(c\alpha) = \frac{e^{\frac{c}{c+1}\alpha}}{(c+1)^2} - \frac{e^{\frac{\alpha}{2}}}{\alpha}B\|V\|_\infty,$$

there exist $\bar{c} > \frac{A}{2}$ and $\alpha_3 > 0$ large enough such that

$$\tilde{f}'(\bar{c}\alpha_3) \geq 0$$

as $\frac{c}{c+1} > \frac{1}{2}$ for $c > 1$. It also holds $\tilde{f}'(\bar{c}\alpha) \geq 0$ for all $\alpha \geq \alpha_3$. Now, letting $\alpha^* = \max\{\alpha_1, \alpha_2, \alpha_3\}$ and choosing $a = (1 - \frac{1}{\sqrt{\alpha^*}})\alpha^*, d = \alpha^*$ and $b = \bar{c}\alpha^*$, we have

$$\tilde{f}'(s) \geq 0 \text{ in } \ [a, b] \text{ and } Q(a, d) > \frac{K\|e\|_\infty}{B}$$

and $\tilde{f}(s) = f(s) - \frac{f(d)}{d}B\|V\|_\infty s > 0$ on $[0, b]$ for sufficiently small value of $\|V\|_\infty$.

This paper is organized as follows: In the next Section 2, we introduce a method of sub and supersolutions for (1) and a three solution theorem for problem (1). Section 3 is devoted to the proofs of Theorem 1.1 and Theorem 1.2.

2. Preliminary

In this section, we define sub and supersoluton of (1) and introduce the method of obtaining sub- and supersolutions and three solution theorem for (1).

A subsolution of (1) is defined as a function $\psi : \bar{\Omega} \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta\psi + V(x)\psi \leq \lambda f(\psi), & x \in \Omega, \\ \psi \leq 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

while a supersolution of (1) is defined as a function $\phi : \bar{\Omega} \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta\phi + V(x)\phi \geq \lambda f(\phi), & x \in \Omega, \\ \phi \geq 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

Now we introduce the theorem of sub and supersolution and three solution theorem.

Lemma 2.1. *(Theorem for sub and supersolution in [1]). If a subsolution ψ and a supersolution ϕ of (1) exist such that $\psi \leq \phi$ on $\bar{\Omega}$, then (1) has at least one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying $\psi \leq u \leq \phi$ on $\bar{\Omega}$.*

Lemma 2.2. *(Three solution Theorem in [1] and [10]). Suppose there exists two pairs of ordered sub and supersolutions $(\psi_1, Z1)$ and (ψ_2, Z_2) of (1) with the property that $\psi_1 \leq \psi_2 \leq Z_1$, $\psi_1 \leq Z_2 \leq Z_1$ and $\psi_2 \not\leq Z_2$. Additionally assume that ψ_2, Z_2 are not solutions of (1). Then there exists at least three solutions $u_i, i = 1, 2, 3$ for (1) where $u_1 \in [\psi_1, Z_2], u_2 \in [\psi_2, Z_1]$ and $u_3 \in [\psi_1, Z_1] \setminus ([\psi_1, Z_2] \cup [\psi_2, Z_1])$.*

Lemma 2.3. *(see [6]). Assume (V_1). Then the problem*

$$\begin{cases} -\Delta w + V(x)w = 1, \ \textit{in } \Omega \\ w = 0, \ \textit{on } \partial\Omega \end{cases} \tag{W}$$

has a solution w such that $w(x) > 0$ for $x \in \Omega$ and $\frac{\partial w}{\partial \eta} < 0$ on $\partial\Omega$.

3. Proof of Main Theorems

3.1. Proof of Theorem 1.1

Proof. It is easy to see that $\psi_1 \equiv 0$ is a strict subsolution of (1). Now, we construct a supersolution. Let us define $\bar{f}(s) := \max_{t \leq s} f(t)$. Then, it follows that $f(s) \leq \bar{f}(s)$, $\bar{f}$ is monotone increasing and $\lim_{s\to\infty} \frac{\bar{f}(s)}{s} = 0$. This implies that there exists $M_\lambda \gg 1$ such that

$$\frac{\bar{f}(M_\lambda \|w\|_\infty)}{M_\lambda \|w\|_\infty} \leq \frac{1}{\lambda \|w\|_\infty}. \tag{6}$$

Let $Z_1 = M_\lambda w$. Then, using (6) and the definition of $\bar{f}$, we can find

$$\begin{aligned}-\Delta Z_1 + V(x)Z_1 = M_\lambda(-\Delta w + V(x)w) &= M_\lambda \\ &\geq \lambda\bar{f}(M_\lambda\|w\|_\infty) \\ &\geq \lambda\bar{f}(M_\lambda w) \\ &\geq \lambda f(M_\lambda w) = \lambda f(Z_1).\end{aligned}$$

Also we easily get $Z_1 = 0$ on $\partial\Omega$, which implies that Z_1 is a supersolution of (1). By Lemma 2.1, there exists a solution u_λ such that $0 \leq u_\lambda \leq Z_1$ for each $\lambda > 0$.

Next, let us show that this solution is unique for any $\lambda > 0$ provided $\frac{s}{f(s)}$ is nondecreasing on $(0,\infty)$. Since $\psi_1 \equiv 0$ is a subsolution of (1), there exists a minimal solution of (1). Let u_1 be a minimal solution and u_2 any other solution of (1). Then $u_1 \leq u_2$ in Ω. It follows from (1) that

$$\begin{aligned}0 = \int_\Omega (u_1\Delta u_2 - u_2\Delta u_1)dx &= \int_\Omega u_1(-\lambda f(u_2) + V(x)u_2) + u_2(\lambda f(u_1) - V(x)u_1)dx \\ &= \int_\Omega \lambda f(u_1)f(u_2)\left[\frac{u_2}{f(u_2)} - \frac{u_1}{f(u_1)}\right]dx \geq 0,\end{aligned}$$

which yields $u_1 = u_2$. Hence, the solution is unique. □

3.2. Proof of Theorem 1.2

Proof. We first construct a supersolution for $\lambda < \lambda^*$. From Assumption (V_1) there exists $c_V > 0$ such that $V(x) \geq -c_V > -\frac{1}{\|e\|_\infty}$ Let $Z_2 = a\frac{e}{\|e\|_\infty}$. Then in Ω, we have

$$\begin{aligned}-\Delta Z_2 + V(x)Z_2 &= \frac{a}{\|e\|_\infty}(-\Delta e + V(x)e) \\ &> \lambda K f(a)(1 + V(x)e) \\ &\geq \lambda K f(a\frac{e}{\|e\|_\infty})(1 - c_V\|e\|_\infty) \geq \lambda f(Z_2)\end{aligned}$$

where we used the fact $\lambda < \frac{a}{f(a)K\|e\|_\infty}$ at the first inequality and $1 - c_V\|e\|_\infty \geq \frac{1}{K}$ at the second inequality. Clearly, $Z_2 = 0$ on $\partial\Omega$. Hence, Z_2 is supersolution for $\lambda < \lambda^*$.

Now we construct a positive subsolution ψ_2 of the following problem

$$\begin{cases}-\Delta u + \|V\|_\infty u = \lambda f(u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega.\end{cases} \tag{7}$$

when $\lambda > \lambda_*$. Then, ψ_2 is a subsolution of (1) since

$$-\Delta\psi_2 + V(x)\psi_2 \leq -\Delta\psi_2 + \|V\|_\infty\psi_2 \leq \lambda f(\psi_2).$$

In order to construct the positive subsolution ψ_2, we recall $\tilde{f}(u) = f(u) - \frac{f(d)}{d}B\|V\|_\infty u > 0$ and consider the following problem

$$\begin{cases} -\Delta u = \lambda \tilde{f}(u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases} \tag{8}$$

Recall that R is the radius of the biggest inscribed ball in Ω. For $0 < \epsilon < R$ and $\delta, \mu > 1$ let us define $\rho : [0, R] \to [0, 1]$ by

$$\rho(r) = \begin{cases} 1, \ 0 \le r \le \epsilon, \\ 1 - (1 - (\frac{R-r}{R-\epsilon})^{\mu})^{\delta}, \ \epsilon < r \le R. \end{cases}$$

Then we have

$$\rho'(r) = \begin{cases} 0, \ 0 \le r \le \epsilon, \\ -\frac{\delta\mu}{R-\epsilon}(1 - (\frac{R-r}{R-\epsilon})^{\mu})^{\delta-1}(\frac{R-r}{R-\epsilon})^{\mu-1}, \ \epsilon < r \le R. \end{cases}$$

Let $v(r) = d\rho(r)$. Note that $|v'(r)| \le d\frac{\delta\mu}{R-\epsilon}$. Define ψ as the radially symmetric solution of

$$\begin{cases} -\Delta \psi = \lambda \tilde{f}(v(|x|)), \text{ in } B_R(0), \\ \psi = 0, \text{ on } \partial B_R(0). \end{cases}$$

Then ψ satisfies

$$\begin{cases} -(r^{N-1}(\psi'(r))' = \lambda r^{N-1}\tilde{f}(v(r)), \\ \psi'(0) = 0, \ \psi(R) = 0. \end{cases} \tag{9}$$

Integrating (9), for $0 < r < R$, we have

$$-\psi'(r) = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1}\tilde{f}(v(s)) \, ds. \tag{10}$$

Here we claim that

$$\psi(r) \ge v(r), \ \forall \ 0 \le r \le R \tag{11}$$

and

$$\|\psi\|_\infty \le b \tag{12}$$

when $\dfrac{d}{f(d)}\dfrac{1}{B} < \lambda < \dfrac{2b}{f(d)AB}$.

In order to prove (11), it is enough to show that

$$-\psi'(r) \ge -v'(r), \ \forall \ 0 \le r \le R \tag{13}$$

as $\psi(R) = 0 = v(R)$. Notice that for $0 \le r \le \epsilon, \ \psi'(r) \le 0 = v'(r)$. Hence, for $r > \epsilon$ we get from (10)

$$\begin{aligned} -\psi'(r) &= \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1}\tilde{f}(v(s)) \, ds \\ &> \frac{\lambda}{R^{N-1}} \int_0^\epsilon s^{N-1}\tilde{f}(v(s)) \, ds \\ &= \frac{\lambda}{R^{N-1}} \frac{\epsilon^N}{N} \tilde{f}(d). \end{aligned}$$

If $\lambda > \frac{d}{\tilde{f}(d)} \frac{NR^{N-1}}{(R-\epsilon)\epsilon^N} \delta\mu$, then we conclude (13). Note that

$$\inf_{\epsilon} \frac{d}{\tilde{f}(d)} \frac{NR^{N-1}}{(R-\epsilon)\epsilon^N} \delta\mu = \frac{d}{\tilde{f}(d)} \frac{(N+1)^{N+1}}{R^2 N^{N-1}} \delta\mu$$

and is achieved at $\epsilon = \frac{NR}{N+1}$. Hence, if $\lambda > \frac{d}{\tilde{f}(d)} \frac{(N+1)^{N+1}}{R^2 N^{N-1}}$, then in the definition of ρ we can choose $\epsilon = \frac{NR}{N+1}$ and the values of δ and μ so that $\lambda \geq \frac{d}{\tilde{f}(d)} \frac{NR^{N-1}}{(R-\epsilon)\epsilon^N} \delta\mu$, and hence (13) holds. Note that it is clear that

$$\tilde{f}(d) = (1 - B\|V\|_\infty) f(d) \tag{14}$$

from the definition of $\tilde{f}(u)$. Hence the range of λ is written as

$$\lambda > \frac{d}{\tilde{f}(d)} \frac{(N+1)^{N+1}}{R^2 N^{N-1}} = \frac{d}{f(d)} \frac{1}{B}.$$

Now to show (12), we integrate (10) from t to R, we obtain that for $0 \leq r \leq R$

$$\begin{aligned} \psi(t) &= \int_t^R \frac{\lambda}{r^{N-1}} \left(\int_0^r s^{N-1} \tilde{f}(v(s))\, ds \right) dr \\ &\leq \int_t^R \frac{\lambda}{r^{N-1}} \tilde{f}(d) \left(\int_0^r s^{N-1}\, ds \right) dr \\ &\leq \lambda \frac{\tilde{f}(d)}{N} \int_0^R r\, dr = \lambda \frac{\tilde{f}(d)}{2N} R^2 \end{aligned}$$

Hence, if $\lambda < \dfrac{b}{\tilde{f}(d)} \dfrac{2N}{R^2}$, then we get $\|\psi\|_\infty \leq b$. Again, from (14), the range of λ is written as $\lambda < \dfrac{2b}{f(d)AB}$. Hence, we find that $v(r) \leq \psi(r) \leq b, \forall\ 0 \leq r \leq R$ when $\dfrac{d}{f(d)} \dfrac{1}{B} < \lambda < \dfrac{2b}{f(d)AB}$. From $v(r) \leq \psi(r) \leq b, \forall\ 0 \leq r \leq R$, we see

$$-\Delta\psi = \lambda \tilde{f}(v) \leq \lambda \tilde{f}(\psi), \quad \text{in } B_R(0) \text{ and } \psi = 0 \text{ on } \partial B_R(0).$$

Now we let $\psi_2(x) = \psi(x)$ if $x \in B_R(0)$ and $\psi_2(x) = 0$ if $x \in \Omega \setminus B_R(0)$. Then ψ_2 is a positive subsolution of (8) for $\lambda_* = \frac{d}{f(d)} \frac{1}{B} < \lambda < \frac{2b}{f(d)AB}$. Finally, we find that for $\lambda > \lambda_*$

$$\begin{aligned} -\Delta\psi_2 \leq \lambda \tilde{f}(\psi_2) &= \lambda[f(\psi_2) - \frac{1}{\lambda_*} \|V\|_\infty \psi_2] \\ &< \lambda[f(\psi_2) - \frac{1}{\lambda} \|V\|_\infty \psi_2] \\ &= \lambda f(\psi_2) - \|V\|_\infty \psi_2, \end{aligned}$$

which implies that ψ_2 is a nonnegative subsolution of (7). Finally, we obtain the subsolution ψ_2 of (1) satisfying $\psi_2 \not\leq Z_2$ for $\lambda_* < \lambda < \lambda^*$.

From the proof of Theorem 1.1 we have a subsolution $\psi_1 \equiv 0$ such that $\psi_1 \leq Z_2$ and a sufficiently large supersolution $Z_1 = M_\lambda w$ such that $\psi_2 \leq Z_1$.

Hence, there exist a positive solutions u_1 and u_2 of (1) such that $\psi_1 \leq u_1 \leq Z_2$ and $\psi_2 \leq u_2 \leq Z_1$. Note that $u_1 \neq u_2$ as $\psi_2 \not\leq Z_2$. By three solution theorem 2.2, there exists a positive solution u_3 such that $u_3 \in [\psi_1, Z_1] \setminus ([\psi_1, Z_2] \cup [\psi_2, Z_1])$. □

References

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. 18 (1976), no. 4, 620- 709.

[2] T. Bartsch and Z.-Q. Wang, *Sign changing solutions of nonlinear Schrdinger equations*, Topol. Methods Nonlinear Anal. **13** (1999), no. 2, 191198.

[3] K.J. Brown, M.M.A. Ibrahim and R. Shivaji, *S - shaphed bifurcation curves*, Nonlinear Analysis, 5 (1981), 475-486.

[4] G.M. Figueiredo, J.R. Santos Júnior and A. Suárez, *Structure of the set of positive solutions of a non-linear Schrdinger equation*, (English summary) Israel J. Math. **227** (2018), no. 1, 485505.

[5] Y. Guo, Z.-Q. Wang, X. Zeng and H.-S. Zhou, *Properties of ground states of attractive Gross-Pitaevskii equations with multi-well potentials*, Nonlinearity **31** (2018), no. 3, 957979.

[6] E. Ko, E.K. Lee and I. Sim, *Existence of positive solution to Schrdinger-type semipositone problems with mixed nonlinear boundary conditions*, Taiwanese J. Math. 25 (2021), no. 1, 107124.

[7] J.-Q. Liu, Y.-Q. Wang and Z.-Q. Wang, *Solutions for quasilinear Schrdinger equations via the Nehari method*, Comm. Partial Differential Equations **29** (2004), no. 5-6, 879901.

[8] M. Ramaswamy and R. Shivaji, *Multiple positive solutions for classes of p-laplacian equations*, Differential Integral Equations, 17 (2004), no. 11-12,1255 – 1261.

[9] P.H. Rabinowitz, *On a class of nonlinear Schrdinger equations,* Z. Angew. Math. Phys. **43** (1992), no. 2, 270291.

[10] R. Shivaji, *A remark on the existence of three solutions via sub-super solutions. Nonlinear analysis and applications*, (Arlington, Tex., 1986), 561566, Lecture Notes in Pure and Appl. Math., 109, Dekker, New York, 1987.

Eunkyung Ko

Major in Mathematics, College of Natural Science, Keimyung University, Daegu 42601, South Korea

E-mail address: ekko@kmu.ac.kr