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HARDY TYPE ESTIMATES FOR RIESZ TRANSFORMS ASSOCIATED WITH SCHRÖDINGER OPERATORS ON THE HEISENBERG GROUP

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Abstract. Let $\mathbb{H}^n$ be the Heisenberg group and $Q = 2n + 2$ be its homogeneous dimension. Let $\mathcal{L} = -\Delta_{\mathbb{H}^n} + V$ be the Schrödinger operator on $\mathbb{H}^n$, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian and the nonnegative potential V belongs to the reverse Hölder class B_{q_1} for $q_1 \geq Q/2$. Let $H^p_{\mathcal{L}}(\mathbb{H}^n)$ be the Hardy space associated with the Schrödinger operator $\mathcal{L}$ for $Q/(Q+\delta_0) < p \leq 1$, where $\delta_0 = \min\{1, 2 - Q/q_1\}$. In this paper, we consider the Hardy type estimates for the operator $T_\alpha = V^\alpha(-\Delta_{\mathbb{H}^n} + V)^{-\alpha}$, and the commutator $[b, T_\alpha]$, where $0 < \alpha < Q/2$. We prove that T_α is bounded from $H^p_{\mathcal{L}}(\mathbb{H}^n)$ into $L^p(\mathbb{H}^n)$. Suppose that $b \in BMO^\theta_{\mathcal{L}}(\mathbb{H}^n)$, which is larger than $BMO(\mathbb{H}^n)$. We show that the commutator $[b, T_\alpha]$ is bounded from $H^1_{\mathcal{L}}(\mathbb{H}^n)$ into weak $L^1(\mathbb{H}^n)$.

1. Introduction

Let $\mathbb{H}^n$ be the Heisenberg group, $Q = 2n+2$, be its homogeneous dimension. Let $\mathcal{L} = -\Delta_{\mathbb{H}^n} + V$ be the Schrödinger operator on $\mathbb{H}^n$, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian and the nonnegative potential V belongs to the reverse Hölder class B_{q_1} for $q_1 \geq Q/2$ and $Q \geq 3$. As generalizations of the classical Riesz transform $\nabla(-\Delta)^{-1/2}$, Riesz type operators associated to $\mathcal{L}$ were studied widely by many mathematicians. On $\mathbb{R}^n$, Shen [13] investigated the fundamental solution of $\mathcal{L}$ under the assumption that the potential belongs to the reverse Hölder's class. As an application, Shen [13] obtained the L^p-boundedness of the operators $T_1 = (-\Delta + V)^{-1}V$ and $T_2 = (-\Delta + V)^{-1/2}V^{1/2}$. By a different technology, for the operators

$$\begin{cases} T_{1,\mathbb{H}^n} = (-\Delta_{\mathbb{H}^n} + V)^{-1}V; \\ T_{2,\mathbb{H}^n} = (-\Delta_{\mathbb{H}^n} + V)^{-1/2}V^{1/2}, \end{cases}$$

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Li [5] further extended the results of [13] to the Heisenberg group $\mathbb{H}^n$. It is obvious that T_i, $i=1,2$, is the special cases of the following Riesz type operator:

$$T_\alpha^* := (-\Delta_{\mathbb{H}^n}+V)^{-\alpha}V^\alpha,\ \alpha>0.$$

In this paper, we consider the Hardy type estimates for the operator T_α, and the commutator $[b,T_\alpha]$, where $0<\alpha<Q/2$. The investigation of Riesz type operators T_α on $\mathbb{H}^n$ with nonnegative potentials has attracted the attention of many authors. Liu-Tang [10] proved that the dual operators $T^*_{1,\mathbb{H}^n}$ and $T^*_{2,\mathbb{H}^n}$ are bounded from $H^p_{\mathcal{L}}(\mathbb{H}^n)$ into $L^p(\mathbb{H}^n)$ for $Q/(Q+\delta_0)<p\le 1$. Tang-Liu [9] considered the Schrödinger operator $\mathcal{L}=-\Delta_{\mathbb{H}^n}+V$, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian and the nonnegative potential V belongs to the reverse Hölder class B_{q_1} for $q_1>Q/2$, and showed that the operator $T_\alpha^* := V^\alpha(-\Delta_{\mathbb{H}^n}+V)^{-\alpha}$ is bounded from $H^1_{\mathcal{L}}(\mathbb{H}^n)$ into $L^1(\mathbb{H}^n)$. For further information on this topic, we refer the reader to $[4,5,9,10,13]$ and the references therein.

In the study of harmonic analysis and the partial differential equations, the commutators related to singular integral operators play an important role. On $\mathbb{R}^n$, Bongioanni, Harboure and Salinas [1] introduced a new class of BMO type spaces associated with Schrödinger operators denoted by $BMO\sigma(\rho)$ as a generalization of $BMO(\mathbb{R}^n)$. In [1], the authors proved the $L^p(\mathbb{R}^n)$-boundedness of $[b,\nabla(-\Delta+V)^{-1/2}]$, where $b\in BMO\sigma(\rho)$. Suppose that $b\in BMO_\sigma(\rho)$, which is larger than $BMO(\mathbb{R}^n)$. Li-Wan [6] proved that the commutators $[b,T_\beta]$ and $[b,R_{\mathcal{L}}]$ are bounded on Herz spaces, where $R_{\mathcal{L}}=\nabla(-\Delta+V)^{-1/2}$ is the Riesz transform associated to $\mathcal{L}$. Hu-Wang [3] considered the Hardy type estimates for the operator $V^\alpha(-\Delta+V)^{-\alpha}$, $0<\alpha<n/2$, and proved that the commutator $[b,T_\alpha]$ is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into weak $L^1(\mathbb{R}^n)$, where b is a new BMO function. In the Heisenberg setting, Li-Peng [5] obtained the L^p-estimates for commutators $[b,T_i], i=1,2$, where $b\in BMO(\mathbb{H}^n)$. For further progress on commutators related with Schrödinger operator, we refer to Li-Wan [6], [7].

Inspired by the above results, we are interested in the boundedness of T_α and $[b,T_\alpha]$ on $H^p_{\mathcal{L}}(\mathbb{H}^n)$. The results of this paper are as follows.

Theorem 1.1. *Let $V\in B_{q_1}$ with $q_1>Q/2$, and let $0<\alpha<Q/2$. Suppose $Q/(Q+\delta_0)<p\le 1$. Then*

$$\|T_\alpha f\|_{L^p(\mathbb{H}^n)} \le C\,\|f\|_{H^p_{\mathcal{L}}(\mathbb{H}^n)},$$

where $\delta_0=\min\{1,2-Q/q_1\}$.

Theorem 1.2. *Let $V\in B_{q_1}$ with $q_1>Q/2$, and let $0<\alpha<Q/2$. Suppose $b\in BMO^\theta_{\mathcal{L}}(\mathbb{H}^n)$. Then the commutator $[b,T_\alpha]$ is bounded from $H^1_{\mathcal{L}}(\mathbb{H}^n)$ into weak $L^1(\mathbb{H}^n)$.*

This paper is organized as follows. In Section 2, we state some notations and known results which will play an important role in this paper. In Section 3, we prove that the operator $T_{\beta_1,\beta_2}=(-\Delta_{\mathbb{H}^n}+V)^{-\beta_1}V^{\beta_2}$ is bounded from $L^{p_1}(\mathbb{H}^n)$ into $L^{p_2}(\mathbb{H}^n)$, where $0\le\beta_2\le\beta_1<Q/2$, $1/p_2=1/p_1-(2\beta_1-2\beta_2)/Q$. In

Section 4, we prove that the commutator $[b, T_\alpha]$ is bounded on $L^p(\mathbb{H}^n)$ with $b \in BMO^\theta_{\mathcal{L}}(\mathbb{H}^n)$. In Section 5, we give the proofs of main results.

Throughout this paper, we use c and C to denote universal positive constants. The constants are independent of the functions and may different in different situations. If $c^{-1}A \leq B \leq cA$, we denote $A \approx B$.

Let's review some basic facts about the Heisenberg group. The Heisenberg group is a Lie group with basic manifold $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, and its multiplication is defined as

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2x'y - 2xy').$$

On $\mathbb{H}^n$, a basis for the Lie algebra of left-invariant vector fields is defined by

$$\mathrm{X}_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t},\ \mathrm{Y}_j = \frac{\partial}{\partial y_j} + 2x_j \frac{\partial}{\partial t},\ \mathrm{T} = \frac{\partial}{\partial t},\ j = 1, 2, \dots, n.$$

All non-trivial commutation relations are given by $[\mathrm{X}_j, \mathrm{Y}_j] = -4\mathrm{T}$, $j = 1, 2, \dots, n$. The sub-Laplacian $\Delta_{\mathbb{H}^n}$ is defined by $\Delta_{\mathbb{H}^n} = \sum_{j=1}^n (\mathrm{X}_j^2 + \mathrm{Y}_j^2)$ and the gradient operator $\nabla_{\mathbb{H}^n}$ is defined by $\nabla_{\mathbb{H}^n} = (\mathrm{X}_1, \mathrm{X}_2, \dots, \mathrm{X}_n, \mathrm{Y}_1, \mathrm{Y}_2, \dots, \mathrm{Y}_n)$. on $\mathbb{H}^n$, the dilations have the form $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$, $\lambda > 0$. The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. $|E|$ presents the measure of any measurable set E. Then $|\delta_\lambda E| = \lambda^Q |E|$, where $Q = 2n + 2$ is called the homogeneous dimension of $\mathbb{H}^n$.

On $\mathbb{H}^n$, a homogeneous norm function is defined by $|g| = ((|x|^2 + |y|^2)^2 + |t|^2)^{1/4}$, $g = (x, y, t) \in \mathbb{H}^n$. This norm satisfies the triangular inequality and leads to a left-invariant distant function $d(g, h) = |g^{-1}h|$. Then the ball of radius r centered at g is defined by $B(g, r) = \{h \in \mathbb{H}^n : \left|g^{-1}h\right| < r\}$. The ball $B(g, r)$ is the left translation by g of $B(0, r)$ and we have $|B(g, r)| = \alpha^1 r^Q$, where $\alpha^1 = |B(0, 1)|$, but it is not important for us.

2. Preliminaries

2.1. Schrödinger operator and the auxiliary function

In this paper, we consider the Schrödinger differential operator $\mathcal{L} = -\Delta_{\mathbb{H}^n} + V$ on $\mathbb{H}^n$, where the potential $V \in B_{q_1}$, $q_1 \geq Q/2$, is defined as follows.

Definition. A nonnegative locally L^{q_1} integrable function V on $\mathbb{H}^n$ is said to belong to $B_{q_1}, 1 < q_1 < \infty$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\Big(\frac{1}{|B|} \int_B V(g)^{q_1} dg\Big)^{1/q_1} \leq \frac{C}{|B|} \int_B V(g) dg$$

holds for every ball B in $\mathbb{H}^n$.

Suppose that $V \in B_{q_1}$ for some $q_1 > Q/2$. The definition of the auxiliary function $m(g, V)$ is given as follows.

Definition. For $g \in \mathbb{H}^n$, the auxiliary function $m(g,V)$ is defined by

$$\rho(g) = \frac{1}{m(g,V)} = \sup_{r>0}\Big\{r : \frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)dh \le 1\Big\}.$$

Next we give some related lemmas about the auxiliary function. We assume that the potential V is nonnegative and belongs to B_{q_1} for $q_1 \ge Q/2$. Lemmas 2.1-2.5 below have been proved in [11].

Lemma 2.1. *There exists a constant $C > 0$ such that for $0 < r < R < \infty$,*

$$\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)dh \le C\Big(\frac{R}{r}\Big)^{Q/q_1-2} \frac{1}{R^{Q-2}} \int_{B(g,R)} V(h)dh.$$

Lemma 2.2. *If $r = \rho(g)$, then $\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)dh = 1$. $\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)dh \sim 1$ if and only if $r \sim \rho(g)$.*

Lemma 2.3. *There exist $C > 0$ and $k_0 > 0$ such that*

$$\begin{aligned}\frac{1}{C}\Big(1 + m(g,V)|g^{-1}h|\Big)^{-k_0} &\le \frac{m(h,V)}{m(g,V)} \\ &\le C\Big(1 + |g^{-1}h| m(g,V)\Big)^{k_0/(k_0+1)}.\end{aligned} \tag{2.1}$$

In particular, $\rho(g) \approx \rho(h)$ if $|g^{-1}h| < C\rho(g)$.

A ball centered at g and with radius $\rho(g)$ is called critical. In this paper, we use the symbol $B(g,\rho(g))$ to denote the critical ball. The inequality (2.1) implies that if $g, h \in \mathcal{Q} = B(g,\rho(g))$, then $\rho(g) \le C_0\rho(h)$, where the constant C_0 depends on the constants C and k_0 in (2.1).

Lemma 2.4. *There exist $C > 0$ and $l_0 > 0$ such that*

$$\int_{B(g,R)} \frac{V(h)}{|g^{-1}h|^{Q-2}} dh \le \frac{C}{R^{Q-2}} \int_{B(g,R)} V(h)dh \le C\left(1 + Rm(g,V)\right)^{l_0}.$$

Lemma 2.5. *The measure $V(h)dh$ is a doubling measure. Namely, there exists a constant C such that*

$$\int_{B(g,2r)} V(h)dh \le C \int_{B(g,r)} V(h)dh$$

for all balls $B(g,r)$ in $\mathbb{H}^n$.

Lemma 2.6. *Suppose that $V \in B_{q_1}$ with $q_1 > Q/2$. Let $k \in \mathbb{N}$ and $g \in 2^{k+1}B(g_0,r) \setminus 2^kB(g_0,r)$. Then*

$$\frac{1}{\left(1 + 2^kr/\rho(g)\right)^N} \le \frac{C}{\left(1 + 2^kr/\rho(g_0)\right)^{N/(k_0+1)}}.$$

Proof. Via a simple computation, for $g \in 2^{k+1}B(g_0,r) \setminus 2^kB(g_0,r)$ and $h \in B(g_0,r)$, we can deduce that $|g^{-1}h| \approx 2^kr$. By (2.1), we can get

$$\rho(g) \le C\rho(g_0)\Big(1 + |g^{-1}h|/\rho(g_0)\Big)^{k_0/(k_0+1)} \le C\rho(g_0)\Big(1 + 2^kr/\rho(g_0)\Big)^{k_0/(k_0+1)}.$$

This gives

$$\frac{1}{(1+2^k r/\rho(g))^N} \le \frac{C}{\left(1+\frac{2^k r}{\rho(g_0)\left(1+2^k r/\rho(g_0)\right)^{k_0/(k_0+1)}}\right)^N}$$
$$\le \frac{C}{\left(1+2^k r/\rho(g_0)\right)^{N/(k_0+1)}}.$$

This completes the proof of Lemma 2.6. □

2.2. New BMO type spaces $BMO^{\theta}_{\mathcal{L}}(\mathbb{H}^n)$

According to [1], we define the new BMO type space $BMO^{\theta}_{\mathcal{L}}(\mathbb{H}^n)$ on $\mathbb{H}^n$.

Definition. The new BMO type space $BMO^{\theta}_{\mathcal{L}}(\mathbb{H}^n)$ with $0 \le \theta < \infty$ is defined as the set of all locally integrable functions b such that

$$\frac{1}{|B(g,r)|}\int_{B(g,r)} |b(h) - b_B|\, dh \le C\Big(1+\frac{r}{\rho(g)}\Big)^{\theta} \tag{2.2}$$

for all $g \in \mathbb{H}^n$ and $r > 0$, where $b_B = \frac{1}{|B|}\int_B b(h)dh$. A norm for $b \in BMO^{\theta}_{\mathcal{L}}(\mathbb{H}^n)$, denoted by $[b]_\theta$, is given by the infimum of the constants in (2.2). Clearly, $BMO(\mathbb{H}^n) \subseteq BMO^{\theta}_{\mathcal{L}}(\mathbb{H}^n)$.

We give some lemmas concerning the function b which will play an important role to obtain the main results.

Lemma 2.7. *Let $\theta > 0$ and $1 \le s < \infty$. If $b \in BMO^{\theta}_{\mathcal{L}}(\mathbb{H}^n)$, then*

$$\left(\frac{1}{|B|}\int_B |b(h) - b_B|^s dh\right)^{1/s} \le C[b]_\theta \left(1+\frac{r}{\rho(g)}\right)^{\theta'} \tag{2.3}$$

for all $B = B(g,r)$ with $g \in \mathbb{H}^n$ and $r > 0$, where $\theta' = (k_0+1)\theta$ and k_0 is the constant appearing in Lemma 2.3.

Proof. From the standard John-Nirenberg inequality, given a ball B_0 and a function $f \in BMO(B_0)$, for each $1 \le s < \infty$, we have

$$\Big(\frac{1}{|B|}\int_B |f(h) - f_B|^s dh\Big)^{1/s} \le C\|f\|_{BMO(B_0)} \tag{2.4}$$

for every ball $B \subset B_0$, where the constant C is independent of the ball B_0.

Therefore, to prove (2.3), we only need to show the claim: if $R \ge 1$ and $\mathcal{Q}$ is a critical ball, then we have $b \in BMO(R\mathcal{Q})$ and

$$\|b\|_{BMO(R\mathcal{Q})} \le C[b]_\theta (1+R)^{(k_0+1)\theta}.$$

If this is true, an application of (2.4) implies that for any ball $B \subset R\mathcal{Q}$,

$$\left(\frac{1}{|B|}\int_B |b(h) - b_B|^s dh\right)^{1/s} \le C[b]_\theta (1+R)^{(k_0+1)\theta}. \tag{2.5}$$

Now, let $B = B(g,r)$ and $\mathcal{Q} = B(g,\rho(g))$ with $g \in \mathbb{H}^n$ and $r > 0$. If $r \le \rho(g)$, choose $R = 1$. We may apply (2.5) to get (2.3). In the case $r > \rho(g)$, we notice that $B = (r/\rho(g))\mathcal{Q}$. Then we apply (2.5) with $R = r/\rho(g)$ which yields (2.3).

Next we prove the claim. Let $B = B(z,r) \subset R\mathcal{Q}$ with $z \in \mathbb{H}^n$ and $r > 0$. Due to (2.1), we have $\rho(g)(1+R)^{-k_0} \le C\rho(z)$. Since $r < R\rho(g)$, then $r/\rho(z) \le C(1+R)^{k_0+1}$. By $b \in BMO^{\theta}_{\mathcal{L}}(\mathbb{H}^n)$, it leads to

$$\frac{1}{|B|}\int_B |b(h) - b_B|dh \le C[b]_\theta (1+R)^{(k_0+1)\theta}.$$

This completes the proof of Lemma 2.7. □

Lemma 2.8. *Let $b \in BMO^{\theta}_{\mathcal{L}}(\mathbb{H}^n)$, $B = B(g,r)$ and $s \ge 1$. Then*

$$\Big(\frac{1}{|2^kB|}\int_{2^kB} |b(h) - b_B|^s dh\Big)^{1/s} \le C[b]_\theta k\Big(1 + \frac{2^k r}{\rho(g)}\Big)^{\theta'}$$

for all $k \in \mathbb{N}$ with $r > 0$, where $\theta' = (k_0+1)\theta$ and k_0 is the constant appearing in Lemma 2.3.

Proof. Following standard arguments and Lemma 2.7, we have

$$\begin{aligned}
&\Big(\frac{1}{|2^kB|}\int_{2^kB} |b(h) - b_B|^s dh\Big)^{1/s} \\
&\le C\Big(\frac{1}{|2^kB|}\int_{2^kB} |b(h) - b_{2^kB}|^s dh\Big)^{1/s} + \sum_{j=1}^{k} |b_{2^jB} - b_{2^{j-1}B}| \\
&\le C[b]_\theta \sum_{j=1}^{k}\Big(1 + \frac{2^j r}{\rho(g)}\Big)^{\theta'} \le C[b]_\theta k\Big(1 + \frac{2^k r}{\rho(g)}\Big)^{\theta'}.
\end{aligned}$$

This completes the proof of Lemma 2.8. □

2.3. Hardy space $H^p_{\mathcal{L}}(\mathbb{H}^n)$ associated with the Schrödinger operator $\mathcal{L}$

We recall the Hardy space $H^p_{\mathcal{L}}(\mathbb{H}^n)$ associated with the Schrödinger operator $\mathcal{L} = -\Delta_{\mathbb{H}^n} + V$ on $\mathbb{H}^n$ established in [10]. When $p = 1$, $H^1_{\mathcal{L}}(\mathbb{H}^n)$ has been studied in [8].

Let $\{T_s : s > 0\} = \{e^{s\triangle_{\mathbb{H}^n}} : s > 0\}$ be the heat semigroup with the convolution kernel $H_s(g)$. We know $0 < H_s(g) \le Cs^{-\frac{Q}{2}}e^{-A_0 s^{-1}|g|^2}$, where A_0 is a positive constant. The Schrödinger operator $\mathcal{L}$ generates a (C_0) contraction semigroup. Let $K^{\mathcal{L}}_s(g,h)$ denote the kernel of $T^{\mathcal{L}}_s$, $0 \le K^{\mathcal{L}}_s(g,h) \le H_s(g,h) = H_s(h^{-1}g)$.

Let us consider the maximal functions with respect to the semigroups $\{T_s : s > 0\}$ and $\{T^{\mathcal{L}}_s : s > 0\}$ defined by $Mf(g) = \sup_{s>0}|T_s f(g)|$, $M^{\mathcal{L}}f(g) = \sup_{s>0}|T^{\mathcal{L}}_s f(g)|$. It is well known that the maximal function Mf characterizes the Hardy space $H^1(\mathbb{H}^n)$, that is, $f \in H^1(\mathbb{H}^n)$ if and only if $Mf \in L^1(\mathbb{H}^n)$, and $\|f\|_{H^1} \sim \|Mf\|_{L^1}$.

In [8], the Hardy space $H^1_{\mathcal{L}}(\mathbb{H}^n)$ associated with the Schrödinger operator $\mathcal{L}$ is defined as follows.

Definition. A function $f \in L^1(\mathbb{H}^n)$ is said to be in $H^1_{\mathcal{L}}(\mathbb{H}^n)$ if the maximal function $M^{\mathcal{L}}f$ belongs to $L^1(\mathbb{H}^n)$. The norm of such a function is defined by $\|f\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)} = \left\|M^{\mathcal{L}}f\right\|_{L^1(\mathbb{H}^n)}$.

In [10], Liu and Tang introduced the Campanato type space on $\mathbb{H}^n$ in order to define the dual space of the Hardy space associated with the Schrödinger operator $\mathcal{L}$. Let f be a locally integrable function on $\mathbb{H}^n$. Set

$$f_B = \frac{1}{|B|}\int_B f(h)dh$$

and

$$f(B,V) = \begin{cases} f_B, & r < \rho(g), \\ 0, & r \geq \rho(g), \end{cases}$$

where $B = B(g,r)$. Let $\delta_0 = \min\{1, 2 - Q/q_1\}$, $Q/(Q+\delta_0) < p < 1$ and $1 \leq q' \leq \infty$. A locally integrable function f is said to be in the Campanato type space $\Lambda^{\mathcal{L}}_{1/p-1,q'}(\mathbb{H}^n)$ if

$$\|f\|_{\Lambda^{\mathcal{L}}_{1/p-1,q'}(\mathbb{H}^n)} = \sup_{B\subseteq\mathbb{H}^n}\left\{|B|^{1-1/p}\Big(\int_B |f - f(B,V)|^{q'}\frac{dh}{|B|}\Big)^{1/q'}\right\} < \infty.$$

The spaces $\Lambda^{\mathcal{L}}_{1/p-1,q'}(\mathbb{H}^n)$ are mutually coincident with equivalent norms and they are the dual space of $H^p_{\mathcal{L}}(\mathbb{H}^n)$. Thus it will be simply denoted by

$$\Lambda^{\mathcal{L}}_{1/p-1}(\mathbb{H}^n).$$

They concluded that, for every $t > 0$,

$$\sup_{h\subseteq\mathbb{H}^n}\left\|H^{\mathcal{L}}_t(\cdot,h)\right\|_{\Lambda^{\mathcal{L}}_{1/p-1}(\mathbb{H}^n)} < Ct^{-Q/2p}. \tag{2.6}$$

Thus by (2.6), Liu and Tang concluded that the maximal function $M^{\mathcal{L}}f$ is well defined for $f \in \left(\Lambda^{\mathcal{L}}_{1/p-1}(\mathbb{H}^n)\right)^*$. The Hardy space $H^p_{\mathcal{L}}(\mathbb{H}^n)$ is defined as follows for $Q/(Q+\delta_0) < p < 1$.

Definition. Let $\delta_0 = \min\{1, 2 - Q/q_1\}$. For $Q/(Q+\delta_0) < p < 1$, we say that $f \in \left(\Lambda^{\mathcal{L}}_{1/p-1}(\mathbb{H}^n)\right)^*$ is an element of $H^p_{\mathcal{L}}(\mathbb{H}^n)$ if the maximal function $M^{\mathcal{L}}f$ belongs to $L^p(\mathbb{H}^n)$. The quasi-norm of f is defined by $\|f\|^p_{H^p_{\mathcal{L}}(\mathbb{H}^n)} = \left\|M^{\mathcal{L}}f\right\|^p_{L^p(\mathbb{H}^n)}$.

Definition. Let $Q/(Q+\delta_0) < p \leq 1 \leq q \leq \infty$ and $p \neq q$. A function a is called an $H^{p,q}_{\mathcal{L}}(\mathbb{H}^n)$-*atom* associated to a ball $B(g_0,r)$ if $r < \rho(g_0)$ and a satisfies the following conditions:

(i) supp$a \subset B(g_0,r)$,

(ii) $\|a\|_{L^q(\mathbb{H}^n)} \leq |B(g_0,r)|^{1/q-1/p}$,
(iii) If $r < \rho(g_0)/4$, then $\int_{B(g_0,r)} a(g)dg = 0$.
The atomic quasi-norm is defined by

$$\|f\|_{H^{p,q}_{\mathcal{L}}(\mathbb{H}^n)} \sim \inf\{(\sum_j |\lambda_j|^p)^{1/p}\},$$

where the infimum is taken over all atomic decompositions $f = \sum_j \lambda_j a_j$, where a_j are $H^{p,q}_{\mathcal{L}}(\mathbb{H}^n)$-atoms and λ_j are scalars.

Proposition 2.9 ([10, Proposition 1]). *Let $\delta_0 = \min\{1, 2 - Q/q_1\}$, $Q/(Q + \delta_0) < p \leq 1 \leq q \leq \infty$, $p \neq q$. Then $f \in H^p_{\mathcal{L}}(\mathbb{H}^n)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where a_j are $H^{p,q}_{\mathcal{L}}(\mathbb{H}^n)$-atoms and $\sum_j |\lambda_j| < \infty$, and the sum converges in the $H^p_{\mathcal{L}}(\mathbb{H}^n)$ quasi-norm.*

2.4. Estimates of fundamental solutions for the Schrödinger operator

We recall the estimates of fundamental solutions of the operator $-\Delta_{\mathbb{H}^n} + V + \lambda$ and the estimates of the kernels of Riesz transforms. Let $\Gamma(g,h,\lambda)$ be the fundamental solution of the operator $-\Delta_{\mathbb{H}^n} + V + \lambda$, where $\lambda \in [0,\infty)$. Obviously, $\Gamma(g,h,\lambda) = \Gamma(h,g,\lambda)$.

Lemma 2.10 ([10, Lemma 5]). *Suppose $V \in B_{q_1}, q_1 > Q/2$. For any integer $N > 0$, there exists $C_N > 0$ such that for $g \neq h$,*

$$|\Gamma(g,h,\lambda)| \leq \frac{C_N}{\left\{1 + |g^{-1}h||\lambda|^{1/2}\right\}^N \left\{1 + |g^{-1}h|\rho(g)^{-1}\right\}^N} \cdot \frac{1}{|g^{-1}h|^{Q-2}}.$$

Let K_α be the kernel of the operator $(-\Delta_{\mathbb{H}^n} + V)^{-\alpha}$. The operator $T_\alpha = V^\alpha(-\Delta_{\mathbb{H}^n} + V)^{-\alpha}$ is defined by

$$T_\alpha f(g) = \int_{\mathbb{H}^n} V^\alpha(g) K_\alpha(g,h) f(h) dh.$$

Lemma 2.11 ([9, Lemma 3.2]). *Suppose $V \in B_{q_1}, q_1 > Q/2$. For any integer $N > 0$, there exists $C_N > 0$ such that*

$$|K_\alpha(g,h)| \leq \frac{C_N}{\{1 + |g^{-1}h|\rho(g)^{-1}\}^N} \cdot \frac{1}{|g^{-1}h|^{Q-2\alpha}}$$

and

$$|K_\alpha(g,\xi h) - K_\alpha(g,h)| \leq \frac{C_N}{\left\{1 + |g^{-1}h|\rho(g)^{-1}\right\}^N} \cdot \frac{|\xi|^\delta}{|g^{-1}h|^{Q-2\alpha+\delta}}$$

for any $g, h \in \mathbb{H}^n$, $|\xi| \leq |g^{-1}h|/2$ and for some $\delta > 0$.

3. Schrödinger type operators on $L^p(\mathbb{H}^n)$

In this section, we consider the Schrödinger type operator T_{β_1,β_2} and its duality on $\mathbb{H}^n$, where the potential $V \in B_{q_1}$, $q_1 \geq Q/2$, $T_{\beta_1,\beta_2} = (-\Delta_{\mathbb{H}^n} + V)^{-\beta_1} V^{\beta_2}$. We show that the operator T_{β_1,β_2} is bounded from $L^{p_1}(\mathbb{H}^n)$ into

$L^{p_2}(\mathbb{H}^n)$. Moreover, when $\beta_1 = \beta_2 = \alpha$, we obtain the $L^p(\mathbb{H}^n)$-boundedness of $T_\alpha = V^\alpha(-\Delta_{\mathbb{H}^n} + V)^{-\alpha}$.

Definition. Let $f \in L^q_{loc}(\mathbb{H}^n)$. The Hardy-Littlewood maximal function Mf and its variant $M_{\sigma,\gamma} f$ are defined by

$$\begin{cases} Mf(g) = \sup\limits_{g\in B} \frac{1}{|B|} \int_B |f(h)|dh, \\ M_{\sigma,\gamma} f(g) = \sup\limits_{g\in B} \left(\frac{1}{|B|^{1-\sigma\gamma/Q}} \int_B |f(h)|^\gamma dh \right)^{1/\gamma}. \end{cases}$$

If $\sigma = 0$, then $M_{0,\gamma} f(g)$ will be denoted by $M_\gamma f(g)$.

Lemma 3.1 ([2]). *Suppose $1 < \gamma < p_1 < Q/\sigma$ and $1/p_2 = 1/p_1 - \sigma/Q$. Then*

$$\|M_{\sigma,\gamma} f\|_{L^{p_2}(\mathbb{H}^n)} \leq C \|f\|_{L^{p_1}(\mathbb{H}^n)}.$$

Theorem 3.2. *Suppose that $V \in B_{q_1}$ for $q_1 > Q/2$. Let $0 \leq \beta_2 \leq \beta_1 < Q/2$. Then*

$$|(-\Delta_{\mathbb{H}^n} + V)^{-\beta_1}(V^{\beta_2} f)(g)| \leq C M_{2(\beta_1-\beta_2),(q_1/\beta_2)'} f(g),$$

where $(q_1/\beta_2)'$ is the conjugate of (q_1/β_2).

Proof. Let $r = \rho(g)$, $B = B(g_0, r)$. With the help of Lemma 2.11, we use Hölder's inequality to deduce that

$$\begin{aligned} I &:= |(-\Delta_{\mathbb{H}^n} + V)^{-\beta_1}(V^{\beta_2} f)(g)| \\ &\leq C \sum_{k=-\infty}^{+\infty} \int_{2^k B \setminus 2^{k-1}B} \frac{1}{(1 + |g^{-1}h|\rho(g)^{-1})^N} \cdot \frac{1}{|g^{-1}h|^{Q-2\beta_1}} \cdot V^{\beta_2}(h)|f(h)|dh \\ &\leq C \sum_{k=-\infty}^{+\infty} \frac{(2^k r)^{2\beta_2}}{(1+2^k)^N} \cdot \Big(\frac{1}{|2^k B|} \int_{2^k B} |V(h)|dh \Big)^{\beta_2} \cdot M_{2(\beta_1-\beta_2),(q_1/(q_1-\beta_2))} f(g). \end{aligned}$$

For $k \geq 1$, because $V(h)dh$ is a doubling measure, we have

$$\frac{(2^k r)^2}{|2^k B|} \int_{2^k B} V(h)dh \leq C \frac{(2^k r)^2}{(2^k r)^Q} \int_{2^k B} V(h)dh \leq C(1 + 2^k)^{l_0}.$$

Taking N large enough such that $N - l_0\beta > 0$, we can get

$$\sum_{k=1}^{+\infty} \frac{(1+2^k)^{l_0\beta}}{(1+2^k)^N} \cdot M_{2(\beta_1-\beta_2),q_1/(q_1-\beta_2)} f(g) \leq C M_{2(\beta_1-\beta_2),q_1/(q_1-\beta_2)} f(g).$$

For $k \leq 0$, Lemma 2.1 implies that

$$\frac{(2^k r)^2}{|2^k B|} \int_{2^k B} V(h)dh \leq C \frac{1}{(2^k r)^{Q-2}} \int_{2^k B} V(h)dh \leq C(2^k)^{2-Q/q_1}.$$

Take N large enough, we obtain

$$\sum_{k=-\infty}^{0} \frac{(2^k)^{2-Q/q_1}}{(1+2^k)^N} \cdot M_{2(\beta_1-\beta_2),q_1/(q_1-\beta_2)} f(g) \leq C M_{2(\beta_1-\beta_2),q_1/(q_1-\beta_2)} f(g).$$

Finally, it holds

$$|(-\Delta_{\mathbb{H}^n}+V)^{-\beta_1}(V^{\beta_2}f)(g)| \le CM_{2(\beta_1-\beta_2),(q_1/\beta_2)'}f(g). \qquad \square$$

By Theorem 3.2 and the duality, we can obtain:

Corollary 3.3. *Suppose $V \in B_{q_1}$ for $q_1 > Q/2$.*

(1) *If $1 < (q_1/\beta_2)' < p_1 < Q/(2\beta_1 - 2\beta_2)$ and $1/p_2 = 1/p_1 - (2\beta_1 - 2\beta_2)/Q$, then*

$$\left\|(-\Delta_{\mathbb{H}^n}+V)^{-\beta_1}V^{\beta_2}f\right\|_{L^{p_2}(\mathbb{H}^n)} \le C\left\|f\right\|_{L^{p_1}(\mathbb{H}^n)},$$

where $q_1/\beta_2 + (q_1/\beta_2)' = 1$.

(2) *If $1 < p_2 < q_1/\beta_2$ and $1/p_2 = 1/p_1 - (2\beta_1 - 2\beta_2)/Q$, then*

$$\left\|V^{\beta_2}(-\Delta_{\mathbb{H}^n}+V)^{-\beta_1}f\right\|_{L^{p_2}(\mathbb{H}^n)} \le C\left\|f\right\|_{L^{p_1}(\mathbb{H}^n)}.$$

Let $\beta_1 = \beta_2$. By Theorem 3.2, we can obtain the following results.

Corollary 3.4. *Suppose that $V \in B_{q_1}$ with $q_1 > Q/2$. Let $1 < \alpha < Q/2$.*

(1) *For $q_1/(q_1-\alpha) < p \le \infty, T_\alpha^*$ is bounded on $L^p(\mathbb{H}^n)$.*

(2) *For $1 \le p < q_1/\alpha, T_\alpha$ is bounded on $L^p(\mathbb{H}^n)$.*

4. Boundedness of the commutators on $L^p(\mathbb{H}^n)$

In this section, let $b \in BMO_{\mathcal{L}}^{\theta}(\mathbb{H}^n)$. We consider the boundedness of the commutator $[b, T_\beta]$ and its duality on $L^p(\mathbb{H}^n)$, where $T_\beta = (-\Delta_{\mathbb{H}^n}+V)^{-\beta}V^\beta$, $\beta > 0$.

Proposition 4.1 ([8])**.** *There exists a sequence of points $\{g_k\}_{k=1}^{\infty} \subset \mathbb{H}^n$ such that the family of critical balls $\{\mathcal{Q}_k = B(g_k, \rho(g_k))\}_{k=1}^{\infty}$ satisfies:*

(i) $\mathbb{H}^n = \cup_k \mathcal{Q}_k$.

(ii) *There exists N such that for every $k \in \mathbb{N}$, card $\{j : 4\mathcal{Q}_j \cap 4\mathcal{Q}_k \neq \emptyset\} \le N$.*

Definition. Let $\gamma > 0$ and $\mathcal{B}_{\rho,\gamma} = \{B(h,r) : h \in \mathbb{H}^n, r \le \gamma\rho(h)\}$. For $f \in L^1_{loc}(\mathbb{H}^n)$ and $g \in \mathbb{H}^n$, define the following two maximal functions:

$$\begin{cases} M_{\rho,\gamma}(f)(g) = \sup\limits_{g\in B\in\mathcal{B}_{\rho,\gamma}} \frac{1}{|B|}\int_B |f(h)|dh, \\ M^{\sharp}_{\rho,\gamma}(f)(g) = \sup\limits_{g\in B\in\mathcal{B}_{\rho,\gamma}} \frac{1}{|B|}\int_B |f(h)-f_B|dh. \end{cases}$$

Definition. Let $\mathcal{S}(\mathrm{Q}) = \{B(h,r) : h \in \mathrm{Q}, r > 0\}$ and let Q be a ball in $\mathbb{H}^n$. For $f \in L^1_{loc}(\mathbb{H}^n)$ and $h \in \mathrm{Q}$, define

$$\begin{cases} M_{\mathrm{Q}}(f)(g) = \sup\limits_{g\in B\in\mathcal{S}(\mathrm{Q})} \frac{1}{|B\cap\mathrm{Q}|}\int_{B\cap\mathrm{Q}} |f(h)|dh, \\ M^{\sharp}_{\mathrm{Q}}(f)(g) = \sup\limits_{g\in B\in\mathcal{S}(\mathrm{Q})} \frac{1}{|B\cap\mathrm{Q}|}\int_{B\cap\mathrm{Q}} |f(h)-f_{B\cap\mathrm{Q}}|dh. \end{cases}$$

Lemma 4.2 (Fefferman-Stein type inequality)**.** *For $1 < p < \infty$, there exist ξ and γ such that if $\{\mathcal{Q}_k\}_{k=1}^{\infty}$ is a sequence of the balls as those in Proposition 4.1, then for all $f \in L^1_{loc}(\mathbb{H}^n)$,*

$$\int_{\mathbb{H}^n} |M_{\rho,\xi}(f)(g)|^p dg \leq C\Big\{ \int_{\mathbb{H}^n} |M^{\sharp}_{\rho,\gamma}(f)(g)|^p dg + \sum_k |\mathcal{Q}_k| \Big(\frac{1}{|\mathcal{Q}_k|} \int_{2\mathcal{Q}_k} |f(g)| dg \Big)^p \Big\}.$$

Proof. If $\mathcal{Q} = (g, \rho(g))$ is a critical ball and $g \in \mathcal{Q}$, we can see that

$$M_{\rho,\xi} f(g) \leq M_{2\mathcal{Q}}(f\chi_{2\mathcal{Q}})(g) \tag{4.1}$$

with $\xi = 1/2C_0^2$ (C_0 is the constant appearing in Lemma 2.3), and for $g \in 2\mathcal{Q}$,

$$M^{\sharp}_{2\mathcal{Q}}(f\chi_{2\mathcal{Q}})(g) \leq C M^{\sharp}_{\rho,2} f(g). \tag{4.2}$$

Now we prove (4.2). In fact, given a ball $B = B(h, r) \in \mathcal{S}(2\mathcal{Q})$, we divide the argument according to r greater or less than $3^{-k_0/(k_0+1)}\rho(g_0)/C$, where C and k_0 are the constants appearing in Lemma 2.3. In the first case, $r > 3^{-k_0/(k_0+1)}\rho(g_0)/C$, $B \cap 2\mathcal{Q}$ has the measure comparable to $2\mathcal{Q}$ which belongs to $\mathcal{B}_{\rho,2}$. In the other case, $r < 3^{-k_0/(k_0+1)}\rho(g_0)/C$, we apply $B \in \mathcal{B}_{\rho,1} \subset \mathcal{B}_{\rho,2}$ to deduce that $|B \cap 2\mathcal{Q}|$ is comparable with $|B|$.

Next we use the decomposition of $\mathbb{H}^n$ given by Proposition 4.1, Proposition 3.4 in [12], and inequalities (4.1) and (4.2) to obtain

$$\begin{aligned}
\int_{\mathbb{H}^n} |M_{\rho,\xi}(f)|^p \, dh &\leq \sum_k \int_{\mathcal{Q}_k} |M_{2\mathcal{Q}_k}(f\chi_{2\mathcal{Q}_k})|^p \, dh \\
&\leq C \sum_k \int_{2\mathcal{Q}_k} \Big| M^{\sharp}_{2\mathcal{Q}_k}(f\chi_{2\mathcal{Q}_k}) \Big|^p \, dh \\
&\quad + C \sum_k |2\mathcal{Q}_k| \Big(\frac{1}{|2\mathcal{Q}_k|} \int_{2\mathcal{Q}_k} |f(h)| dh \Big)^p \\
&\leq C \int_{\mathbb{H}^n} \Big| M^{\sharp}_{\rho,4}(f) \Big|^p \, dh + \sum_k |\mathcal{Q}_k| \Big(\frac{1}{|\mathcal{Q}_k|} \int_{2\mathcal{Q}_k} |f(h)| dh \Big)^p.
\end{aligned}$$

This completes the proof of Lemma 4.2. □

Theorem 4.3. *Suppose that $V \in B_{q_1}$, $q_1 \geq Q/2$, and $b \in BMO^{\theta}_{\mathcal{L}}(\mathbb{H}^n)$, $0 < \theta < \infty$. If $q_1/(q_1 - \beta) < p < \infty$, there exists a constant $C > 0$ such that, for all $f \in L^p_{loc}(\mathbb{H}^n)$ and every critical ball $\mathcal{Q} = B(g_0, \rho(g_0))$,*

$$\frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} |[b, T_\beta] f(g)| dg \leq C[b]_\theta \Big\{ \inf_{g \in \mathcal{Q}} M_p f(g) + \inf_{g \in \mathcal{Q}} M_p(T_\beta f)(g) \Big\}.$$

Proof. For any constant a, $b(g) - b(h) = (b(g) - a) - (b(h) - a)$. Then we have $[b, T_\beta] f(g) := I + II$, where

$$\begin{cases} I := (b(g) - a) T_\beta f(g), \\ II := T_\beta((b(g) - a) f)(g). \end{cases}$$

Let $f \in L^p(\mathbb{H}^n)$ and $\mathcal{Q} = B(g_0, \rho(g_0))$ with $a = b_{2\mathcal{Q}}$. We deal with the average of I and II on $\mathcal{Q}$, respectively. For I, using Lemma 2.7 and Hölder's inequality with $p > q_1/(q_1 - \beta)$, we can get

$$\begin{aligned}\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}}|I|dg &\le \Big(\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}}|b(g)-b_{2\mathcal{Q}}|^{p'}dg\Big)^{1/p'}\Big(\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}}|T_\beta f(g)|^p dg\Big)^{1/p}\\ &\le C[b]_\theta \inf_{g\in\mathcal{Q}} M_p(T_\beta f)(g).\end{aligned}$$

For II, we set $f = f_1 + f_2$, where $f_1(g) = f(g)\chi_{2\mathcal{Q}}(g)$ and $f_2(g) = f(g)\chi_{(2\mathcal{Q})^c}(g)$. Take $p_1 \in (q_1/(q_1-\beta), p)$ and denote $m = p/(p-p_1)$. By Corollary 3.4, Lemma 2.7 and Hölder's inequality, we have

$$\begin{aligned}&\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}}|T_\beta((b-b_{2\mathcal{Q}})f_1)(g)|dg\\ &\le C\Big(\frac{1}{|\mathcal{Q}|}\int_{2\mathcal{Q}}|b(g)-b_{2\mathcal{Q}}|^{mp_1}dg\Big)^{1/mp_1}\Big(\frac{1}{|\mathcal{Q}|}\int_{2\mathcal{Q}}|f(g)|^p dg\Big)^{1/p}\\ &\le C[b]_\theta \inf_{g\in\mathcal{Q}} M_p(f)(g).\end{aligned}$$

Now we consider the term

$$\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}}|T_\beta((b-b_{2\mathcal{Q}})f_2)(g)|dg.$$

Since $g \in B(g_0, \rho(g_0))$ and $z \in 2^{j+1}B \setminus 2^j B$, then $|g^{-1}z| \approx |g_0^{-1}z| \approx 2^j\rho(g_0)$. With the help of Lemma 2.6 and Lemma 2.8, we use Hölder's inequality to obtain

$$\begin{aligned}&\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}}|T_\beta((b-b_{2\mathcal{Q}})f_2)(g)|dg\\ &\le C\sum_{j=1}^{\infty}\frac{1}{(1+2^j)^{N/(k_0+1)}}\frac{|2^{j+1}\rho(g_0)|^Q}{|2^j\rho(g_0)|^{Q-2\beta}}\left(\frac{1}{|2^{j+1}\mathcal{Q}|}\int_{2^{j+1}\mathcal{Q}}|V(z)|^{q_1}dz\right)^{\beta/q_1}\\ &\quad\times\left(\frac{1}{|2^{j+1}\mathcal{Q}|}\int_{2^{j+1}\mathcal{Q}}|f(z)|^p dz\right)^{1/p}\left(\frac{1}{|2^{j+1}\mathcal{Q}|}\int_{2^{j+1}\mathcal{Q}}|b(z)-b_{2\mathcal{Q}}|^{p_0}dz\right)^{1/p_0}\\ &\le C\sum_{j=1}^{\infty}\frac{j}{(1+2^j)^{N/(k_0+1)-l_0\beta-\theta'}}[b]_\theta \inf_{h\in\mathcal{Q}} M_p(f)(h)\\ &\le C[b]_\theta \inf_{h\in\mathcal{Q}} M_p(f)(h),\end{aligned}$$

where $\beta/q_1 + 1/p + 1/p_0 = 1$. Taking N large enough such that $N/(k_0+1) - l_0\beta - \theta' > 0$, we complete the proof of Theorem 4.3. $\square$

Remark 4.4. It is easy to check that if the critical ball $\mathcal{Q}$ is replaced by $2\mathcal{Q}$, Theorem 4.3 also holds.

Lemma 4.5. *Let $V \in B_{q_1}$, $q_1 \ge Q/2$, and $b \in BMO_{\mathcal{L}}^{\theta}(\mathbb{H}^n)$, $0 < \theta < \infty$. Then for any $p > q_1/(q_1-\beta)$ and $\gamma \ge 1$, there exists a constant C such that for all*

f and $g, h \in B = B(g_0, r)$ with $r < \gamma\rho(g_0)$,

$$\int_{(2B)^c} |K(g,z) - k(h,z)| V^{\beta}(z)|b(z) - b_B||f(z)|dz \leq C[b]_\theta \inf_{u \in B} M_p f(u).$$

Proof. Because $g \in B(g_0, r)$ and $z \in 2^{j+1}B \setminus 2^j B$, we deduce that $|g^{-1}z| \approx |g_0^{-1}z|$. Using Lemmas 2.6, 2.8, and 2.11, we have

$$\begin{aligned}
&\int_{(2B)^c} |K(g,z) - k(h,z)| V^{\beta}(z)|b(z) - b_B||f(z)|dz \\
&\leq C \sum_{j=1}^{\infty} \frac{1}{\left(1 + \frac{2^j r}{\rho(g_0)}\right)^{N/(k_0+1)}} \frac{|r|^{\delta}}{(2^j r)^{Q-2\beta+\delta}} \int_{2^{j+1}B} V^{\beta}(z)|b(z) - b_B||f(z)|dz \\
&\leq C \sum_{j=1}^{\infty} \frac{[b]_\theta \inf_{u \in \mathcal{Q}} M_p(f)(u)}{\left(1 + \frac{2^j r}{\rho(g_0)}\right)^{N/(k_0+1) - l_0\beta - \theta'}} \frac{j}{2^{j\delta}} \leq C[b]_\theta \inf_{u \in \mathcal{Q}} M_p(f)(u),
\end{aligned}$$

where we take N sufficiently large. Thus this completes the proof of Lemma 4.5. □

Theorem 4.6. *Let $V \in B_{q_1}$, $q_1 \geq Q/2$, and let $b \in BMO_{\mathcal{L}}^{\theta}(\mathbb{H}^n)$, $0 < \theta < \infty$.*

(i) *If $1 < \beta < Q/2$, $q_1/(q_1 - \beta) < p < \infty$, then*

$$\|[b, T_\beta]f\|_{L^p(\mathbb{H}^n)} \leq C[b]_\theta \|f\|_{L^p(\mathbb{H}^n)}.$$

(ii) *If $1 < \beta < Q/2$, $1 < p < q_1/\beta$, then*

$$\|[b, T_\beta^*]f\|_{L^p(\mathbb{H}^n)} \leq C[b]_\theta \|f\|_{L^p(\mathbb{H}^n)}.$$

Proof. We only prove (i), and (ii) follows by duality. For $f \in L^p(\mathbb{H}^n)$, $q_1/(q_1 - \beta) < p < \infty$, by Theorem 4.3, we can see that $[b, T_\beta]f \in L^1_{loc}(\mathbb{H}^n)$. By Theorem 4.3, Lemma 4.5, and Remark 4.4, we get

$$\begin{aligned}
\|[b, T_\beta]f\|^q_{L^q(\mathbb{H}^n)} &\leq C \int_{\mathbb{H}^n} |M^{\sharp}_{\rho,\gamma}([b, T_\beta]f)(g)|^q dg \\
&\quad + \sum_k |Q_k| \Big(\frac{1}{|\mathcal{Q}_k|} \int_{2\mathcal{Q}_k} |[b, T_\beta]f(g)| dg \Big)^q \\
&\leq C \int_{\mathbb{H}^n} |M^{\sharp}_{\rho,\gamma}([b, T_\beta]f)(g)|^q dg \\
&\quad + [b]_\theta \Big(\sum_k \int_{2\mathcal{Q}_k} |M_p f(g)|^q dg + \sum_k \int_{2\mathcal{Q}_k} |M_p(T_\beta f)(g)|^q dg \Big) \\
&\leq C \int_{\mathbb{H}^n} |M^{\sharp}_{\rho,\gamma}([b, T_\beta]f)(g)|^q dg + [b]_\theta \|f\|^q_{L^q(\mathbb{H}^n)}.
\end{aligned}$$

Now we consider the term

$$\int_{\mathbb{H}^n} |M^{\sharp}_{\rho,\gamma}\Big([b, T_\beta]f\Big)(g)|^q dg.$$

We write $[b, T_\beta]f(g) = B_1(g) - B_2(g)$, where

$$\begin{cases} B_1(g) := (b(g) - b_B)T_\beta f(g), \\ B_2(g) := T_\beta((b - b_B)f)(g). \end{cases}$$

This gives

$$\frac{1}{|B|}\int_B \Big|[b, T_\beta]f(g) - \Big([b, T_\beta]f\Big)_B\Big| dg \leq CI + II,$$

where

$$\begin{cases} I := \frac{1}{|B|}\int_B |B_1(g) - (B_1)_B|\, dg, \\ II := \frac{1}{|B|}\int_B |B_2(g) - (B_2)_B|\, dg. \end{cases}$$

For I, let $p > q_1/(q_1 - \beta)$. Because $r < \gamma\rho(g_0)$, it follows from Hölder's inequality and Lemma 2.7 that

$$I \leq C\Big(\frac{1}{|B|}|b(g) - b_B|^{p'}\Big)^{1/p'}\Big(\frac{1}{|B|}|T_\beta f(g)|^p\Big)^{1/p'} \leq C[b]_\theta M_p(T_\beta f)(g).$$

For II, let $g \in \mathbb{H}^n$ and $B = B(g_0, r)$ with $r < \gamma\rho(g_0)$ such that $g \in B$. We split $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$. Hence we can divide II into two parts as $II = II_1 + II_2$, where

$$\begin{cases} II_1 := \frac{1}{|B|}\int_B |T_\beta((b - b_B)f_1)(g) - (T_\beta(b - b_B)f_1)_B|\, dg, \\ II_2 := \frac{1}{|B|}\int_B |T_\beta((b - b_B)f_2)(g) - (T_\beta(b - b_B)f_2)_B|\, dg. \end{cases}$$

For II_1, take $p_1 \in (q_1/(q_1 - \beta), p)$ and let $m = p/(p - p_1)$. By Corollary 3.4, we apply Hölder's inequality to obtain

$$\begin{aligned} II_1 &\leq C\Big(\frac{1}{|B|}\int_{2B} |b - b_B|^{mp_1} dg\Big)^{1/mp_1}\Big(\frac{1}{|B|}\int_{2B} |f(g)|^p dg\Big)^{1/p} \\ &\leq C[b]_\theta M_p(f)(g). \end{aligned}$$

For II_2, we can use Lemma 4.5 to get

$$\begin{aligned} II_2 &\leq \frac{C}{|B|^2}\int_B\int_B |T_\beta((b - b_B)f_2)(h) - (T_\beta(b - b_B)f_2)(u)|\, dh du \\ &\leq C[b]_\theta M_p(f)(g). \end{aligned}$$

Finally, the $L^p(\mathbb{H}^n)$ boundedness of M_p implies that

$$|M^\sharp_{\rho,\gamma}([b, T_\beta]f)(g)| \leq C[b]_\theta\Big(M_p(T_\beta f)(g) + M_p(f)(g)\Big) \leq C[b]_\theta \|f\|_{L^q(\mathbb{H}^n)}.$$

This completes the proof of Theorem 4.6. □

By Theorem 4.6, we get the following proposition.

Proposition 4.7. *Suppose that $V \in B_{q_1}$ with $q_1 > Q/2$. Let $1 < \alpha < Q/2$, and $b \in BMO^\theta_{\mathcal{L}}(\mathbb{H}^n)$. Then for $1 < p < \infty$, $\|[b, T_\alpha](f)\|_{L^p(\mathbb{H}^n)} \leq C[b]_\theta \|f\|_{L^p(\mathbb{H}^n)}$.*

5. The proof of main results

In this section, we prove that T_α is bounded from $H^p_{\mathcal{L}}(\mathbb{H}^n)$ into $L^p(\mathbb{H}^n)$ for $Q/(Q+\delta_0) < p \leq 1$. Moreover, we also prove that the commutator $[b, T_\alpha]$ is bounded from $H^1_{\mathcal{L}}(\mathbb{H}^n)$ into weak $L^1(\mathbb{H}^n)$.

In order to prove Theorem 1.1, we only need to prove the following lemma.

Lemma 5.1. *Let $q_1 > Q/2$. There is a number q with $1 < q < q_1/\alpha$ such that $\|T_\alpha a\|_{L^p(\mathbb{H}^n)} \leq C$ holds for any $H^{p,q}_{\mathcal{L}}(\mathbb{H}^n)$-atom a, where the constant $C > 0$ is independent of a.*

Proof. Since $0 < \alpha < Q/2 < q_1$, we select q such that $1 < q < q_1/\alpha$. Assume that supp$a \subset B(g_0, r)$, $r < \rho(g_0)$. Then $\|T_\alpha a(g)\|_{L^p(\mathbb{H}^n)} \leq I_1 + I_2$, where

$$\begin{cases} I_1 := \|\chi_{2B} T_\alpha a(g)\|_{L^p(\mathbb{H}^n)}, \\ I_2 := \left\|\chi_{(2B)^c} T_\alpha a(g)\right\|_{L^p(\mathbb{H}^n)}. \end{cases}$$

For I_1, the Hölder inequality implies that

$$I_1 \leq |2B|^{1/p-1/q} \cdot \Big(\int_{2B} |T_\alpha a(g)|^q dg\Big)^{1/q} \leq |2B|^{1/p-1/q} \cdot \|a(g)\|_{L^q(\mathbb{H}^n)} \leq C.$$

For I_2, we divide into two case: $r \geq \rho(g_0)/4$ and $r < \rho(g_0)/4$.

Case 1: in this case, $r \geq \rho(g_0)/4$, then $r \approx \rho(g_0)$. Applying Lemma 2.11 and Lemma 2.6, we can get

$$\begin{aligned} I_2 &\leq C\Big(\sum_{k\geq 1} \int_{2^{k+1}B\backslash 2^k B} V(g)^{\alpha p} \Big(\int_B \frac{(2^k r)^{2\alpha - Q}}{(1+(2^k r)\rho(g)^{-1})^N} \cdot |a(h)| dh\Big)^p dg\Big)^{1/p} \\ &\leq C\Big(\sum_{k\geq 1} \frac{\int_{2^{k+1}B} V(g)^{\alpha p} dg}{\Big(1+2^k r/\rho(g_0)\Big)^{Np/(k_0+1)} (2^k r)^{(Q-2\alpha)p}} \Big(\int_B |a(h)| dh\Big)^p\Big)^{1/p}. \end{aligned}$$

We pick a number s such that $Q/2 < s < q_1$, then $\alpha p < s$. With the help of Lemma 2.4, we use Hölder's inequality to deduce that

$$\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} V(g)^{\alpha p} dg \leq C(2^k r)^{-2\alpha p} \left(1 + 2^k r/\rho(g_0)\right)^{l_0 \alpha p}. \tag{5.1}$$

Notice $r \approx \rho(g_0)$ and

$$\int_B |a(h)| dh \leq r^{Q-Q/p}. \tag{5.2}$$

Thus, we have the estimate of I_2.

$$I_2 \leq C\Big(\sum_{k\geq 1} \frac{1}{(2^k)^{Np/(k_0+1) - l_0\alpha p - Q + Qp}}\Big)^{1/p} \leq C,$$

where we take N large enough such that $Np/(k_0+1) - l_0\alpha p - Q + Qp > 0$.

Case 2: in this case, $r < \rho(g_0)/4$.

When $p = 1$, by Lemma 2.11, (5.1), (5.2) and the vanishing condition of a, we have

$$\begin{aligned} I_2 &\le C \sum_{k\ge 1} \frac{1}{(1+2^k r/\rho(g_0))^{N/(k_0+1)}} \cdot \frac{r^\delta}{(2^k r)^{Q-2\alpha+\delta}} \int_{2^{k+1}B} V(g)^\alpha dg \int_B |a(h)|dh \\ &\le C \sum_{k\ge 1} 2^{-k\delta} \frac{1}{(1+2^k r/\rho(g_0))^{N/(k_0+1)-l_0\alpha}}. \end{aligned}$$

Take N large enough, we get $I_2 \le C \sum_{k\ge 1} 2^{-k\delta} \le C$.

When $Q/(Q+\delta_0) < p < 1$, for any p_0 such that $Q/(Q+\delta_0) < p_0 < p < 1$. Using Lemma 2.11, (5.1) and (5.2), we can get

$$\begin{aligned} I_2 &\le C\Big(\sum_{k\ge 1} \frac{r^{pQ-Q+\delta p}}{(1+2^k r/\rho(g_0))^{Np/(k_0+1)-l_0 p}} \cdot \frac{(2^k r)^Q}{(2^k r)^{(Q+\delta)p}} \Big)^{1/p} \\ &\le C\Big(\sum_{k\ge 1} \frac{2^{kQ}}{2^{k(Q+\delta)p}} \Big)^{1/p} \le C, \end{aligned}$$

where we choose N sufficiently large. Thus this completes the proof of Lemma 5.1. □

Next we give the proof of Theorem 1.2.

Proof. Let $f \in H^1_{\mathcal{L}}(\mathbb{H}^n)$. We write $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where each a_j is an $H^{1,q}_{\mathcal{L}}-$*atom*, $1 < q < q_1/\alpha$ and $\sum_{j=-\infty}^{\infty} |\lambda_j| \le C \|f\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)}$. Suppose that supp$a_j \subset B_j = B(g_j, r_j)$ with $r_j < \rho(g_j)$. Write $[b, T_\alpha] f(g) = \sum_{i=1}^{4} \sum_{j=-\infty}^{\infty} \lambda_j A_{ij}(g)$, where

$$\begin{cases} A_{1j} := [b, T_\alpha] a_j(g) \chi_{8B_j}(g), \\ A_{2j} := (b(g) - b_{B_j}) T_\alpha a_j(g) \chi_{(8B_j)^c}(g), \\ A_{3j} := (b(g) - b_{B_j}) T_\alpha a_j(g) \chi_{(8B_j)^c}(g), \\ A_{4j} := T_\alpha((b - b_{B_j}) a_j)(g) \chi_{(8B_j)^c}(g). \end{cases}$$

Note that $\left(\int_{B_j} |a_j(g)|^q dg \right)^{1/q} \le |B_j|^{1/q-1}$. We use Hölder's inequality and Proposition 4.7 to obtain

$$\begin{aligned} \|A_{1j}(g)\|_{L^1(\mathbb{H}^n)} &\le C|B_j|^{1/q'} \cdot \Big(\int_{8B_j} |[b, T_\alpha] a_j(g)|^q dg \Big)^{1/q} \\ &\le C[b]_\theta |B_j|^{1/q'+1/q-1} \le C[b]_\theta, \end{aligned}$$

which leads to

$$\begin{aligned} \Big\| \sum_{j=-\infty}^{\infty} \lambda_j A_{1j}(g) \Big\|_{L^1(\mathbb{H}^n)} &\le \sum_{j=-\infty}^{\infty} |\lambda_j| \Big\| A_{1j}(g) \Big\|_{L^1(\mathbb{H}^n)} \\ &\le C \sum_{j=-\infty}^{\infty} |\lambda_j| [b]_\theta \le C[b]_\theta \|f\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)}. \end{aligned}$$

Then we have

$$\Big|\Big\{g\in\mathbb{H}^n:\Big|\sum_{j=-\infty}^{\infty}\lambda_jA_{1j}(g)\Big|>\frac{\lambda}{4}\Big\}\Big|\le\frac{C}{\lambda}\Big\|\sum_{j=-\infty}^{\infty}\lambda_jA_{1j}(g)\Big\|_{L^1(\mathbb{H}^n)}$$
$$\le\frac{C[b]_\theta}{\lambda}\|f\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)}.$$

For $A_{2j}(g)$, note that $h\in B_j(g_j,r_j)$, $g\in 2^{k+1}B_j\setminus 2^kB_j$, then $|g^{-1}h|\approx 2^kr_j$. Applying Lemma 2.6, we can get

$$\|A_{2j}(g)\|_{L^1(\mathbb{H}^n)}$$
$$\le C\sum_{k\ge3}\int_{2^{k+1}B_j\setminus 2^kB_j}|b(g)-b_{B_j}|V^\alpha(g)\int_{B_j}\frac{|g^{-1}h|^{2\alpha-Q}}{(1+|g^{-1}h|\rho(g)^{-1})^N}|a_j(h)|dhdg$$
$$\le C\sum_{k\ge3}\frac{(2^kr_j)^{2\alpha-Q}}{(1+2^kr_j/\rho(g_j))^{N/(k_0+1)}}\int_{2^{k+1}B_j}|b(g)-b_{B_j}|V^\alpha(g)dg\int_{B_j}|a_j(h)|dh.$$

Since $\alpha<Q/2<q_1$, we choose a number s such that $\alpha<Q/2<s<q_1$. Then

$$\frac{1}{|2^{k+1}B_j|}\int_{2^{k+1}B_j}|b(g)-b_{B_j}|V^\alpha(g)dg$$
$$\le Ck[b]_\theta(2^kr_j)^{-2\alpha}\Big(1+2^kr_j/\rho(g_j)\Big)^{\theta'+l_0\alpha}.$$

Note that $\int_{B_j}|a_j(h)|\le C$, and $r_j/\rho(g_j)\ge 1/4$. A direct computation gives

$$\|A_{2j}(g)\|_{L^1(\mathbb{H}^n)}\le C\sum_{k\ge3}k[b]_\theta\frac{1}{\Big(1+2^kr_j/\rho(g_j)\Big)^{N/(k_0+1)-\theta'-l_0\alpha}}$$
$$\le C\sum_{k\ge1}[b]_\theta\frac{k}{(2^k)^{N/(k_0+1)-\theta'-l_0\alpha}}\le C[b]_\theta,$$

which implies that

$$\Big\|\sum_{j:r_j\ge\rho(g_j)/4}\lambda_jA_{2j}(g)\Big\|_{L^1(\mathbb{H}^n)}\le\sum_{j:r_j\ge\rho(g_j)/4}|\lambda_j|\,\|A_{2j}(g)\|_{L^1(\mathbb{H}^n)}$$
$$\le C\sum_{j=-\infty}^{\infty}|\lambda_j|[b]_\theta\le C[b]_\theta\|f\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)},$$

and subsequently, we obtain

$$\Big|\Big\{g\in\mathbb{H}^n:\Big|\sum_{j:r_j\ge\rho(g_j)/4}\lambda_jA_{2j}(g)\Big|>\frac{\lambda}{4}\Big\}\Big|\le\frac{C}{\lambda}\Big\|\sum_{j:r_j\ge\rho(g_j)/4}\lambda_jA_{2j}(g)\Big\|_{L^1(\mathbb{H}^n)}$$
$$\le\frac{C[b]_\theta}{\lambda}\|f\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)}.$$

By the vanishing condition of a_j and Lemma 2.11, we have

$$\|A_{3j}(g)\|_{L^1(\mathbb{H}^n)} \leq C\sum_{k\geq 3}\int_{2^{k+1}B_j\setminus 2^kB_j}|b(g)-b_{B_j}|V^{\alpha}(g) \times\int_{B_j}\frac{|a_j(h)|}{(1+|g^{-1}h|\rho(g)^{-1})^N}\cdot\frac{|h^{-1}g_j|^{\delta}}{|g^{-1}h|^{Q-2\alpha+\delta}}dhdg.$$

Since $h\in B_j(g_j,r_j)$ and $g\in 2^{k+1}B_j\setminus 2^kB_j$, then $|g^{-1}h|\approx 2^kr_j$. With the help of Lemma 2.6, we can deduce that

$$\begin{aligned}\|A_{3j}(g)\|_{L^1(\mathbb{H}^n)} &\leq C\sum_{k\geq 3}\frac{\int_{2^{k+1}B_j}|b(g)-b_{B_j}|V^{\alpha}(g)dg}{(1+2^kr_j/\rho(g_j))^{N/(k_0+1)}}\cdot\frac{r_j^{\delta}\int_{B_j}|a_j(h)|dh}{(2^kr_j)^{Q-2\alpha+\delta}}\\ &\leq C\sum_{k\geq 3}[b]_{\theta}\frac{k}{2^{k\delta}}\leq C[b]_{\theta},\end{aligned}$$

which gives

$$\begin{aligned}\Big\|\sum_{j:r_j<\rho(g_j)/4}\lambda_jA_{3j}(g)\Big\|_{L^1(\mathbb{H}^n)} &\leq \sum_{j:r_j<\rho(g_j)/4}|\lambda_j|\Big\|A_{3j}(g)\Big\|_{L^1(\mathbb{H}^n)}\\ &\leq C\sum_{j=-\infty}^{\infty}|\lambda_j|[b]_{\theta}\leq C[b]_{\theta}\,\|f\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)}.\end{aligned}$$

The above estimate implies that

$$\begin{aligned}&\Big|\Big\{g\in\mathbb{H}^n:\Big|\sum_{j:r_j<\rho(g_j)/4}\lambda_jA_{3j}(g)\Big|>\frac{\lambda}{4}\Big\}\Big|\\ &\leq\frac{C}{\lambda}\Big\|\sum_{j:r_j<\rho(g_j)/4}\lambda_jA_{3j}(g)\Big\|_{L^1(\mathbb{H}^n)}\leq\frac{C[b]_{\theta}}{\lambda}\,\|f\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)}.\end{aligned}$$

For $\sum_{j=-\infty}^{\infty}\lambda_jA_{4j}(g)$, we have

$$\Big|\sum_{j=-\infty}^{\infty}\lambda_jA_{4j}(g)\Big|\leq\Big|T_{\alpha}\big(\sum_{j=-\infty}^{+\infty}\Big|\lambda_j(b(g)-b_{B_j})a_j(g)\Big|\chi_{(8B_j)^c}(g)\big)\Big|.$$

Thus

$$\begin{aligned}&\Big|\Big\{g\in\mathbb{H}^n:\Big|\sum_{j=-\infty}^{+\infty}\lambda_jA_{4j}(g)\Big|>\frac{\lambda}{4}\Big\}\Big|\\ &\leq\Big|\Big\{g\in\mathbb{H}^n:\Big|T_{\alpha}\big(\sum_{j=-\infty}^{+\infty}\Big|\lambda_j(b(g)-b_{B_j})a_j(g)\Big|\chi_{(8B_j)^c}(g)\big)\Big|>\frac{\lambda}{4}\Big\}\Big|\\ &\leq\frac{C}{\lambda}\sum_{k=-\infty}^{\infty}\Big|\lambda_j\Big|\Big\|\Big(b(g)-b_{B_j}\Big)a_j(g)\Big\|_{L^1(\mathbb{H}^n)}.\end{aligned}$$

Note that $r_j \leq \rho(g_j)$. Applying Lemma 2.7, we use Hölder's inequality to deduce that

$$\begin{aligned}\left\|\Big(b(g)-b_{B_j}\Big)a_j(g)\right\|_{L^1(\mathbb{H}^n)} &\leq \Big(\int_{B_j}\left|b(g)-b_{B_j}\right|^{q'}dg\Big)^{1/q'}\Big(\int_{B_j}|a_j(g)|^q\,dg\Big)^{1/q}\\ &\leq C[b]_\theta,\end{aligned}$$

which implies that

$$\left|\Big\{g\in\mathbb{H}^n:\Big|\sum_{j=-\infty}^{\infty}\lambda_j A_{4j}(g)\Big|>\frac{\lambda}{4}\Big\}\right| \leq \frac{C}{\lambda}\sum_{k=-\infty}^{\infty}|\lambda_j|\,[b]_\theta \leq \frac{C[b]_\theta}{\lambda}\|f\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)}.$$

Finally, it holds

$$\begin{aligned}&\left|\Big\{g\in\mathbb{H}^n:\Big|\sum_{i=1}^{4}\sum_{j=-\infty}^{\infty}\lambda_j A_{ij}(g)\Big|>\frac{\lambda}{4}\Big\}\right|\\ &\leq C\sum_{i=1}^{4}\left|\Big\{g\in\mathbb{H}^n:\Big|\sum_{j=-\infty}^{\infty}\lambda_j A_{ij}(g)\Big|>\frac{\lambda}{4}\Big\}\right| \leq \frac{C[b]_\theta}{\lambda}\|f\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)}.\end{aligned}$$

This completes the proof of Theorem 1.2. □

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