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WIJSMAN ASYMPTOTICAL $\mathcal{I}_2$-LACUNARY STATISTICAL EQUIVALENCE OF ORDER η FOR DOUBLE SET SEQUENCES

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ABSTRACT. In this paper, for double set sequences, as a new approach to the notion of Wijsman asymptotical lacunary statistical equivalence of order η, we introduce new concepts which are called Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalence of order η and Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalence of order η where $0 < \eta \leq 1$. Also, some properties of these new concepts are investigated, and the existence of some relations between these and some previously studied asymptotical equivalence concepts for double set sequences is examined.

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1. Introduction

The notion of convergence for double sequences was introduced by Pringsheim [31]. Then, this notion was extended to the concept of statistical convergence by Mursaleen and Edely [18], the concept of lacunary statistical convergence by Patterson and Savaş [29] and the concept of $\mathcal{I}$-convergence by Das et al. [7]. Particularly, more studies in different settings on the concept of $\mathcal{I}$-convergence were done by many authors, especially Mursaleen [19, 20, 21, 22, 23]. Recently, new notions of convergence of order α for double sequences were studied by Bhunia et al. [4], Çolak and Altın [6], Savaş [33] and Altın et al.[1].

For double sequences, the notion of asymptotical equivalence was introduced by Patterson [28]. Then, this notion was extended to the concept of asymptotical double lacunary statistical equivalence by Esi [10], the concept of asymptotical $\mathcal{I}$-equivalence by Hazarika and Kumar [15] and the concept of asymptotical double statistical equivalence by Esi and Açıkgöz [11].

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Over the years, many authors have been studying on the notions of various convergence for set sequences. One of them, handled in this study, is the notion of Wijsman convergence ([2, 3, 45]). Using the concepts of statistical convergence, lacunary sequence and $\mathcal{I}$-convergence, the notion of Wijsman convergence was extended to new convergence concepts for double set sequences by many authors ([9, 24, 25, 27, 42]).

For double set sequences, the notions of asymptotical equivalence in Wijsman sense were introduced by Nuray et al. [26] and studied by many authors ([40, 41]). Ulusu and Dündar [40] introduced the concepts of Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalence and Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalence for double set sequences. Recently, new notions of asymptotical equivalence of order α for double set sequences was studied by Gülle [14], Ulusu and Gülle [44].

More study on notions of convergence or asymptotical equivalence for real sequences or set sequences can be found in [5, 8, 12, 13, 16, 30, 32, 35, 36, 37, 38, 39, 43].

2. Preliminaries

For the study to be more understandable, some basic definitions and notations are needed. First of all, let's give these concepts below ([2, 3, 7, 17, 26, 28, 29, 31, 40, 44, 45]).

A double sequence (a_{mn}) is called convergent to L (in Pringsheim sense) if every $\xi > 0$, there exist $N_\xi \in \mathbb{N}$ (the set of natural numbers) such that $|a_{mn} - L| < \xi$, when ever $m, n > N_\xi$.

A family of sets $\mathcal{I} \subseteq P_{\mathbb{N}}$ (the power set of $\mathbb{N}$) is said to be an ideal iff

(i) $\emptyset \in \mathcal{I}$, (ii) $E \cup F \in \mathcal{I}$ for each $E, F \in \mathcal{I}$, (iii) $F \in \mathcal{I}$ for each $E \in \mathcal{I}$ and $F \subseteq E$.

An ideal $\mathcal{I} \subseteq P_{\mathbb{N}}$ is said to be non-trivial if $\mathbb{N} \notin \mathcal{I}$ and a non-trivial ideal $\mathcal{I} \subseteq P_{\mathbb{N}}$ is said to be admissible if $\{m\} \in \mathcal{I}$ for each $m \in \mathbb{N}$.

A non-trivial ideal $\mathcal{I}_2 \subseteq P_{\mathbb{N}\times\mathbb{N}}$ is said to be strong admissible if $\{m\} \times \mathbb{N}$ and $\mathbb{N} \times \{m\}$ belongs to $\mathcal{I}_2$ for each $m \in \mathbb{N}$. Obviously, a strong admissible ideal is admissible.

Throughout the study, $\mathcal{I}_2 \subseteq P_{\mathbb{N}\times\mathbb{N}}$ will taken a strong admissible ideal.

Two non-negative double sequences (a_{mn}) and (b_{mn}) are called asymptotical equivalent if

$$\lim_{m,n\to\infty} \frac{a_{mn}}{b_{mn}} = 1$$

and denoted by $a_{mn} \sim b_{mn}$.

Let Y be non-empty set. A function $g : \mathbb{N} \to P_Y$ is defined $g(m) = V_m \in P_Y$ for each $m \in \mathbb{N}$. The sequence $\{V_m\} = \{V_1, V_2, \ldots\}$, which the range' elements of g, is called sequences of sets.

Let (Y, d) be metric space. For any $y \in Y$ and any non-empty $V \subseteq Y$, the distance from y to V is defined

$$\rho(y, V) = \inf_{v \in V} d(y, v).$$

A double sequence $\{V_{mn}\}$ is called Wijsman convergent to V if each $y \in Y$,

$$\lim_{m,n \to \infty} \rho(y, V_{mn}) = \rho(y, V).$$

Throughout the study, (Y, d) will taken a metric space and U_{mn}, V_{mn} will taken any non-empty closed subsets of Y.

The term $\rho_y\Big(\dfrac{U_{mn}}{V_{mn}}\Big)$ is defined as follows:

$$\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) = \begin{cases} \dfrac{\rho(y, U_{mn})}{\rho(y, V_{mn})} & , \quad y \notin U_{mn} \cup V_{mn} \\ \lambda & , \quad y \in U_{mn} \cup V_{mn}. \end{cases}$$

Double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are called Wijsman asymptotical equivalent if each $y \in Y$,

$$\lim_{m,n \to \infty} \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) = 1.$$

Double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are called Wijsman asymptotical $\mathcal{I}_2$-equivalent of multiple λ if every $\xi > 0$ and each $y \in Y$,

$$\Big\{(m, n) \in \mathbb{N} \times \mathbb{N} : \Big|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\Big| \geq \xi\Big\} \in \mathcal{I}_2.$$

Double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are called Wijsman asymptotical $\mathcal{I}_2$-statistical equivalent of multiple λ if every ξ, $\delta > 0$ and each $y \in Y$,

$$\Big\{(i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{ij}\Big|\Big\{(m, n) : m \leq i, n \leq j, \ \Big|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\Big| \geq \xi\Big\}\Big| \geq \delta\Big\} \in \mathcal{I}_2.$$

A double sequence $\theta_2 = \{(j_s, k_t)\}$ is said to be double lacunary sequence if there exists increasing sequences of the integers such that

$$j_0 = 0, \ h_s = j_s - j_{s-1} \to \infty \ \text{ and } \ k_0 = 0, \ \bar{h}_t = k_t - k_{t-1} \to \infty \ \text{ as } \ s, t \to \infty.$$

Throughout the study, regarding lacunary sequence $\theta_2 = \{(j_s, k_t)\}$, we will use the following notations:

$$\ell_{st} = j_s k_t, \quad h_{st} = h_s \bar{h}_t, \quad I_{st} = \{(m, n) : j_{s-1} < m \leq j_s \ \text{ and } \ k_{t-1} < n \leq k_t\}$$

$$q_s = \frac{j_s}{j_{s-1}} \ \text{ and } \ q_t = \frac{k_t}{k_{t-1}}.$$

Throughout the study, $\theta_2 = \{(j_s, k_t)\}$ will taken a double lacunary sequence.

Double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are called Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalent of multiple λ if every ξ, $\delta > 0$ and each $y \in Y$,

$$\Big\{(s, t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{st}}\Big|\Big\{(m, n) \in I_{st} : \Big|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\Big| \geq \xi\Big\}\Big| \geq \delta\Big\} \in \mathcal{I}_2.$$

The class of Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalent double sequences is denoted by $S_\theta\left(\mathcal{I}_{W_2}^{\lambda}\right)$.

Double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are called Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalent of multiple λ if every $\xi > 0$ and each $y \in Y$,

$$\left\{(s,t) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{st}} \sum_{(m,n)\in I_{st}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \geq \xi\right\} \in \mathcal{I}_2.$$

The class of Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalent double sequences is denoted by $N_\theta\left[\mathcal{I}_{W_2}^{\lambda}\right]$.

Double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are called Wijsman asymptotical $\mathcal{I}_2$-statistical equivalent to multiple λ of order η if every $\xi, \delta > 0$ and each $y \in Y$,

$$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(ij)^\eta} \left|\left\{(m,n) : m \leq i, n \leq j,\ \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \geq \xi\right\}\right| \geq \delta\right\} \in \mathcal{I}_2$$

and denoted by $U_{mn} \overset{\mathcal{I}_2^W(S_\lambda^\eta)}{\sim} V_{mn}$.

Double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are called Wijsman asymptotical strong $\mathcal{I}_2$-Cesàro equivalent to multiple λ of order η if every $\xi > 0$ and each $y \in Y$,

$$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(ij)^\eta} \sum_{m,n=1,1}^{i,j} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \geq \xi\right\} \in \mathcal{I}_2$$

and denoted by $U_{mn} \overset{\mathcal{I}_2^W[C_\lambda^\eta]}{\sim} V_{mn}$.

3. New Concepts

In this section, for double set sequences, we introduce notion of Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalence of order η and notions of Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalence of order η where $0 < \eta \leq 1$.

Definition 3.1. Let $0 < \eta \leq 1$ and $\theta_2 = \{(j_s, k_t)\}$ be double lacunary sequence. Double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalent to multiple λ of order η if every $\xi, \delta > 0$ and each $y \in Y$,

$$\left\{(s,t) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{st}^\eta} \left|\left\{(m,n) \in I_{st} : \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \geq \xi\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

Also, we write $U_{mn} \overset{\mathcal{I}_{\theta_2}^W(S_\lambda^\eta)}{\sim} V_{mn}$ and Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalent of order η if $\lambda = 1$.

The class of Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalent to multiple λ of order η double sequences will denoted by $\mathcal{I}_{\theta_2}^W(S_\lambda^\eta)$.

Example 3.2. Let $Y = \mathbb{R}^2$ and double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ be defined as following:

$$U_{mn} := \begin{cases} \{(a,b) \in \mathbb{R}^2 : (a+1)^2 + b^2 = \dfrac{1}{mn}\} & , \text{ if } \begin{array}{l} j_{s-1} < m < j_{s-1} + [\sqrt{h_s}], \\ k_{t-1} < n < k_{t-1} + [\sqrt{h_t}], \\ mn \text{ is a square integer.} \end{array} \\ \{(0,0)\} & , \text{ otherwise.} \end{cases}$$

and

$$V_{mn} := \begin{cases} \{(a,b) \in \mathbb{R}^2 : (a-1)^2 + b^2 = \dfrac{1}{mn}\} & , \text{ if } \begin{array}{l} j_{s-1} < m < j_{s-1} + [\sqrt{h_s}], \\ k_{t-1} < n < k_{t-1} + [\sqrt{h_t}], \\ mn \text{ is a square integer.} \end{array} \\ \{(0,0)\} & , \text{ otherwise.} \end{cases}$$

If we take $\mathcal{I}_2 = \mathcal{I}_2^f$, ($\mathcal{I}_2^f$ is the class of finite subsets of $\mathbb{N} \times \mathbb{N}$), the double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalent of order η.

Remark 3.1. For $\eta = 1$, concept of Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalence to multiple λ of order η coincides with concept of Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalence of multiple λ for double set sequences in [40].

Definition 3.3. Let $0 < \eta \leq 1$ and $\theta_2 = \{(j_s, k_t)\}$ be double lacunary sequence. Double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical $\mathcal{I}_2$-lacunary equivalent to multiple λ of order η if every $\xi > 0$ and each $y \in Y$,

$$\left\{(s,t) \in \mathbb{N} \times \mathbb{N} : \left| \frac{1}{h_{st}^{\eta}} \sum_{(m,n) \in I_{st}} \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \geq \xi \right\} \in \mathcal{I}_2.$$

Also, we write $U_{mn} \overset{\mathcal{I}_{\theta_2}^W(N_\lambda^\eta)}{\sim} V_{mn}$ and Wijsman asymptotical $\mathcal{I}_2$-lacunary equivalent of order η if $\lambda = 1$.

Definition 3.4. Let $0 < \eta \leq 1$ and $\theta_2 = \{(j_s, k_t)\}$ be double lacunary sequence. Double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalent to multiple λ of order η if every $\xi > 0$ and each $y \in Y$,

$$\left\{(s,t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{st}^{\eta}} \sum_{(m,n) \in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \geq \xi \right\} \in \mathcal{I}_2.$$

Also, we write $U_{mn} \overset{\mathcal{I}_{\theta_2}^W[N_\lambda^\eta]}{\sim} V_{mn}$ and Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalent of order η if $\lambda = 1$.

The class of Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalent to multiple λ of order η double sequences will denoted by $\mathcal{I}_{\theta_2}^W[N_\lambda^\eta]$.

Example 3.5. Let $Y = \mathbb{R}^2$ and double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ be defined as following:

$$U_{mn} := \begin{cases} \left\{(a,b) : \dfrac{a^2}{2mn} + \dfrac{(b+\sqrt{mn})^2}{mn} = 1\right\} & , \text{ if } \begin{array}{l} j_{s-1} < m < j_{s-1} + [\sqrt{h_s}], \\ k_{t-1} < n < k_{t-1} + [\sqrt{\overline{h}_t}]. \end{array} \\ \{(1,1)\} & , \text{ otherwise.} \end{cases}$$

and

$$V_{mn} := \begin{cases} \left\{(a,b) : \dfrac{a^2}{2mn} + \dfrac{(b-\sqrt{mn})^2}{mn} = 1\right\} & , \text{ if } \begin{array}{l} j_{s-1} < m < j_{s-1} + [\sqrt{h_s}], \\ k_{t-1} < n < k_{t-1} + [\sqrt{\overline{h}_t}]. \end{array} \\ \{(1,1)\} & , \text{ otherwise.} \end{cases}$$

If we take $\mathcal{I}_2 = \mathcal{I}_2^f$, the double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalent of order η.

Remark 3.2. For $\eta = 1$, concept of Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalence to multiple λ of order η coincides with concept of Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalence of multiple λ for double set sequences in [40].

Definition 3.6. Let $0 < \eta \leq 1$, $0 < p < \infty$ and $\theta_2 = \{(j_s, k_t)\}$ be double lacunary sequence. Double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical strong $p - \mathcal{I}_2$-lacunary equivalent to multiple λ of order η if every $\xi > 0$ and each $y \in Y$,

$$\left\{(s,t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{st}^{\eta}} \sum_{(m,n) \in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right|^p \geq \xi \right\} \in \mathcal{I}_2.$$

Also, we write $U_{mn} \overset{\mathcal{I}_{\theta_2}^W [N_\lambda^\eta]^p}{\sim} V_{mn}$ and Wijsman asymptotical strong $p - \mathcal{I}_2$-lacunary equivalent of order η if $\lambda = 1$.

4. Inclusion Theorems

In this section, firstly, we investigate some properties of the new asymptotical equivalence concepts that introduced in Section 3 and some relationships between them.

Theorem 4.1. *Let $\theta_2 = \{(j_s, k_t)\}$ be double lacunary sequence. Then,*

i. *If* $0 < \eta \leq \mu \leq 1$, *then* $\mathcal{I}_{\theta_2}^W(S_\lambda^\eta) \subseteq \mathcal{I}_{\theta_2}^W(S_\lambda^\mu)$.
ii. *Particularly, for* $\mu = 1$, $\mathcal{I}_{\theta_2}^W(S_\lambda^\eta) \subseteq S_\theta(\mathcal{I}_{W_2}^\lambda)$.

Proof. i.) Suppose that $0 < \eta \leq \mu \leq 1$ and $U_{mn} \overset{\mathcal{I}_{\theta_2}^W(S_\lambda^\eta)}{\sim} V_{mn}$. For every $\xi > 0$ and each $y \in Y$, we have

$$\frac{1}{h_{st}^{\mu}}\left|\left\{(m,n)\in I_{st}:\left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|\geq\xi\right\}\right| \leq \frac{1}{h_{st}^{\eta}}\left|\left\{(m,n)\in I_{st}:\left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|\geq\xi\right\}\right|$$

and so for every $\delta>0$,

$$\left\{(s,t)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{st}^{\mu}}\left|\left\{(m,n)\in I_{st}:\left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|\geq\xi\right\}\right|\geq\delta\right\} \subseteq\left\{(s,t)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{st}^{\eta}}\left|\left\{(m,n)\in I_{st}:\left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|\geq\xi\right\}\right|\geq\delta\right\}.$$

Since the set on right side belongs to the ideal $\mathcal{I}_2$ by our assumption, the set on left side also belongs to $\mathcal{I}_2$. Consequently, $\mathcal{I}_{\theta_2}^{W}(S_\lambda^{\eta})\subseteq\mathcal{I}_{\theta_2}^{W}(S_\lambda^{\mu})$.

ii.) If we take $\mu=1$ in item (i), it is obvious. □

Theorem 4.2. *Let $\theta_2=\{(j_s,k_t)\}$ be double lacunary sequence. Then,*

i. *If $0<\eta\leq\mu\leq1$ and $0<p<\infty$, then $\mathcal{I}_{\theta_2}^{W}[N_\lambda^{\eta}]^p\subseteq\mathcal{I}_{\theta_2}^{W}[N_\lambda^{\mu}]^p$.*
ii. *Particularly, for $\mu=1$ and $p=1$, $\mathcal{I}_{\theta_2}^{W}[N_\lambda^{\eta}]\subseteq N_\theta[\mathcal{I}_{W_2}^{\lambda}]$.*

Proof. i.) Suppose that $0<\eta\leq\mu\leq1$ and $U_{mn}\overset{\mathcal{I}_{\theta_2}^{W}[N_\lambda^{\eta}]^p}{\sim}V_{mn}$. For each $y\in Y$, we have

$$\frac{1}{h_{st}^{\mu}}\sum_{(m,n)\in I_{st}}\left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|^p\leq\frac{1}{h_{st}^{\eta}}\sum_{(m,n)\in I_{st}}\left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|^p$$

and so for every $\xi>0$,

$$\left\{(s,t)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{st}^{\mu}}\sum_{(m,n)\in I_{st}}\left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|^p\geq\xi\right\} \subseteq\left\{(s,t)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{st}^{\eta}}\sum_{(m,n)\in I_{st}}\left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|^p\geq\xi\right\}.$$

Since the set on right side belongs to the ideal $\mathcal{I}_2$ by our assumption, the set on left side also belongs to $\mathcal{I}_2$. Consequently, $\mathcal{I}_{\theta_2}^{W}[N_\lambda^{\eta}]^p\subseteq\mathcal{I}_{\theta_2}^{W}[N_\lambda^{\mu}]^p$.

ii.) If we take $\mu=1$ and $p=1$ in item (i), it is obvious. □

Now, we shall express a theorem. This theorem gives a relationship between $\mathcal{I}_{\theta_2}^{W}[N_\lambda^{\eta}]^p$ and $\mathcal{I}_{\theta_2}^{W}[N_\lambda^{\eta}]^q$ where $0<\eta\leq1$ and $0<p<q<\infty$.

Theorem 4.3. *If $0 < \eta \leq 1$ and $0 < p < q < \infty$, then $\mathcal{I}_{\theta_2}^{W}[N_\lambda^\eta]^q \subset \mathcal{I}_{\theta_2}^{W}[N_\lambda^\eta]^p$.*

Proof. Assume that $0 < p < q < \infty$ and $U_{mn} \overset{\mathcal{I}_{\theta_2}^{W}[N_\lambda^\eta]^q}{\sim} V_{mn}$. By the Hölder inequality, for each $y \in Y$ we have

$$\frac{1}{h_{st}^\eta} \sum_{(m,n)\in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right|^p < \frac{1}{h_{st}^\eta} \sum_{(m,n)\in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right|^q$$

and so for every $\xi > 0$,

$$\left\{(s,t) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{st}^\eta} \sum_{(m,n)\in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right|^p \geq \xi \right\}$$
$$\subset \left\{(s,t) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{st}^\eta} \sum_{(m,n)\in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right|^q \geq \xi \right\}.$$

Hence, by our assumption, we get $U_{mn} \overset{\mathcal{I}_{\theta_2}^{W}[N_\lambda^\eta]^p}{\sim} V_{mn}$. Consequently, $\mathcal{I}_{\theta_2}^{W}[N_\lambda^\eta]^q \subset \mathcal{I}_{\theta_2}^{W}[N_\lambda^\eta]^p$. □

Theorem 4.4. *If double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical strong $p - \mathcal{I}_2$-lacunary equivalent to multiple λ of order η, then the double sequences are Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalent to multiple λ of order μ where $0 < \eta \leq \mu \leq 1$ and $0 < p < \infty$.*

Proof. Assume that $0 < \eta \leq \mu \leq 1$ and double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical strong $p - \mathcal{I}_2$-lacunary equivalent to multiple λ of order η. For every $\xi > 0$ and each $y \in Y$, we have

$$\sum_{(m,n)\in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right|^p \geq \sum_{\substack{(m,n)\in I_{st} \\ \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right)-\lambda\right| \geq \xi}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right|^p$$
$$\geq \xi^p \left| \left\{ (m,n) \in I_{st} : \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \geq \xi \right\} \right|$$

and so

$$\frac{1}{\xi^p\, h_{st}^\eta} \sum_{(m,n)\in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right|^p \geq \frac{1}{h_{st}^\eta} \left| \left\{ (m,n) \in I_{st} : \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \geq \xi \right\} \right|$$
$$\geq \frac{1}{h_{st}^\mu} \left| \left\{ (m,n) \in I_{st} : \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \geq \xi \right\} \right|.$$

Then for any $\delta > 0$,

$$\left\{(s,t) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{st}^{\mu}}\left|\left\{(m,n) \in I_{st} : \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \geq \xi\right\}\right| \geq \delta\right\}$$

$$\subseteq \left\{(s,t) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{st}^{\eta}} \sum_{(m,n)\in I_{st}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right|^p \geq \xi^p\,\delta\right\}.$$

Since the set on right side belongs to the ideal $\mathcal{I}_2$ by our assumption, the set on left side also belongs to $\mathcal{I}_2$. Consequently, we get that the double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalent to multiple λ of order μ. □

If $\mu = \eta$ is taken in the Theorem 4.4, then the following corollary is obtained.

Corollary 4.5. *If double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical strong $p-\mathcal{I}_2$-lacunary equivalent to multiple λ of order η, then the double sequences are Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalent to multiple λ of order η where $0 < \eta \leq 1$ and $0 < p < \infty$.*

Now, secondly, we investigate the relationships between the new asymptotical equivalence concepts that introduced in Section 3 and previously studied some asymptotical equivalence concepts for double set sequences.

Theorem 4.6. *Let $\theta_2 = \{(j_s, k_t)\}$ be double lacunary sequence. If $\liminf_s q_s^{\eta} > 1$ and $\liminf_t q_t^{\eta} > 1$ where $0 < \eta \leq 1$, then*

$$U_{mn} \overset{\mathcal{I}_2^W(S_\lambda^\eta)}{\sim} V_{mn} \Rightarrow U_{mn} \overset{\mathcal{I}_{\theta_2}^W(S_\lambda^\eta)}{\sim} V_{mn}.$$

Proof. Let double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical $\mathcal{I}_2$-statistical equivalent to multiple λ of order η. Also assume that $\liminf_s q_s^{\eta} > 1$ and $\liminf_t q_t^{\eta} > 1$. Then, there exist $\alpha, \beta > 0$ such that $q_s^{\eta} \geq 1+\alpha$ and $q_t^{\eta} \geq 1+\beta$ for all s, t, which implies that

$$\frac{h_{st}^{\eta}}{\ell_{st}^{\eta}} \geq \frac{\alpha\beta}{(1+\alpha)(1+\beta)}.$$

For every $\xi > 0$ and each $y \in Y$, we have

$$\frac{1}{\ell_{st}^{\eta}}\left|\left\{(m,n) :\ m \leq j_s, n \leq k_t,\ \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \geq \xi\right\}\right|$$

$$\geq \frac{1}{\ell_{st}^{\eta}}\left|\left\{(m,n) \in I_{st} : \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \geq \xi\right\}\right|$$

$$= \frac{h_{st}^{\eta}}{\ell_{st}^{\eta}}\,\frac{1}{h_{st}^{\eta}}\left|\left\{(m,n) \in I_{st} : \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \geq \xi\right\}\right|$$

$$\geq \frac{\alpha\beta}{(1+\alpha)(1+\beta)}\,\frac{1}{h_{st}^{\eta}}\left|\left\{(m,n) \in I_{st} : \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \geq \xi\right\}\right|$$

and so for any $\delta > 0$,

$$\left\{(s,t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{st}^{\eta}} \left| \left\{ (m,n) \in I_{st} : \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \geq \xi \right\} \right| \geq \delta \right\}$$

$$\subseteq \left\{(s,t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\ell_{st}^{\eta}} \left| \left\{ (m,n) : \, m \leq j_s, n \leq k_t, \; \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \geq \xi \right\} \right| \geq \frac{\alpha\, \beta\, \delta}{(1+\alpha)(1+\beta)} \right\}.$$

Since the set on right side belongs to the ideal $\mathcal{I}_2$ by our assumption, the set on left side also belongs to $\mathcal{I}_2$. Consequently, $U_{mn} \overset{\mathcal{I}_{\theta_2}^{W}(S_{\lambda}^{\eta})}{\sim} V_{mn}$. □

Theorem 4.7. *Let $\theta_2 = \{(j_s, k_t)\}$ be double lacunary sequence. If $\limsup_s q_s < \infty$ and $\limsup_t q_t < \infty$, then*

$$U_{mn} \overset{\mathcal{I}_{\theta_2}^{W}(S_{\lambda}^{\eta})}{\sim} V_{mn} \Rightarrow U_{mn} \overset{\mathcal{I}_{2}^{W}(S_{\lambda}^{\eta})}{\sim} V_{mn}$$

where $0 < \eta \leq 1$.

Proof. Let double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalent to multiple λ of order η. Also assume that $\limsup_s q_s < \infty$ and $\limsup_t q_t < \infty$. Then, there exist $M, N > 0$ such that $q_s < M$ and $q_t < N$ for all s, t. Let

$$\kappa_{st} := \left| \left\{ (m,n) \in I_{st} : \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \geq \xi \right\} \right|.$$

Since $U_{mn} \overset{\mathcal{I}_{\theta_2}^{W}(S_{\lambda}^{\eta})}{\sim} V_{mn}$; for every ξ, $\delta > 0$, each $y \in Y$ we have

$$\left\{(s,t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{st}^{\eta}} \left| \left\{ (m,n) \in I_{st} : \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \geq \xi \right\} \right| \geq \delta \right\}$$

$$= \left\{ (s,t) \in \mathbb{N} \times \mathbb{N} : \frac{\kappa_{st}}{h_{st}^{\eta}} \geq \delta \right\} \in \mathcal{I}_2.$$

Hence, we can choose $s_0, t_0 \in \mathbb{N}$ such that

$$\frac{\kappa_{st}}{h_{st}^{\eta}} < \delta$$

for all $s \geq s_0$, $t \geq t_0$. Now take the value γ as

$$\gamma := \max\{\kappa_{ru} : 1 \leq s \leq s_0, \, 1 \leq t \leq t_0\}$$

and let i and j be integers satisfying $j_{s-1} < i \leq j_s$ and $k_{t-1} < j \leq k_t$. Then, we have

$$\begin{aligned}
&\frac{1}{(ij)^\eta}\left|\left\{(m,n):\, m\le i, n\le j,\ \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|\ge\xi\right\}\right| \\
&\quad\le \frac{1}{\ell^\eta_{(s-1)(t-1)}}\left|\left\{(m,n):\, m\le j_s, n\le k_t,\ \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|\ge\xi\right\}\right| \\
&\quad= \frac{1}{\ell^\eta_{(s-1)(t-1)}}\left\{\kappa_{11}+\kappa_{12}+\kappa_{21}+\kappa_{22}+\cdots+\kappa_{s_0t_0}+\cdots+\kappa_{st}\right\} \\
&\quad\le \frac{s_0\,t_0}{\ell^\eta_{(s-1)(t-1)}}\left(\max_{\substack{1\le m\le s_0\\1\le n\le t_0}}\{\kappa_{mn}\}\right)+\frac{1}{\ell^\eta_{(s-1)(t-1)}}\left\{h^\eta_{s_0(t_0+1)}\frac{\kappa_{s_0(t_0+1)}}{h^\eta_{s_0(t_0+1)}}\right. \\
&\qquad\left.+h^\eta_{(s_0+1)t_0}\frac{\kappa_{(s_0+1)t_0}}{h^\eta_{(s_0+1)t_0}}+h^\eta_{(s_0+1)(t_0+1)}\frac{\kappa_{(s_0+1)(t_0+1)}}{h^\eta_{(s_0+1)(t_0+1)}}+\cdots+h^\eta_{st}\frac{\kappa_{st}}{h^\eta_{st}}\right\} \\
&\quad\le \frac{s_0\,t_0\,\gamma}{\ell^\eta_{(s-1)(t-1)}}+\frac{1}{\ell^\eta_{(s-1)(t-1)}}\left(\sup_{\substack{s>s_0\\t>t_0}}\left\{\frac{\kappa_{st}}{h^\eta_{st}}\right\}\right)\left(\sum_{\substack{m\ge s_0\\n\ge t_0}}^{s,t}h_{mn}\right) \\
&\quad\le \frac{s_0\,t_0\,\gamma}{\ell^\eta_{(s-1)(t-1)}}+\frac{1}{\ell_{(s-1)(t-1)}}\left(\sup_{\substack{s>s_0\\t>t_0}}\left\{\frac{\kappa_{st}}{h^\eta_{st}}\right\}\right)\left(\sum_{\substack{m\ge s_0\\n\ge t_0}}^{s,t}h_{mn}\right) \\
&\quad\le \frac{s_0\,t_0\,\gamma}{\ell^\eta_{(s-1)(t-1)}}+\delta\,\frac{(j_s-j_{s_0})(k_t-k_{t_0})}{\ell_{(s-1)(t-1)}} \\
&\quad\le \frac{s_0\,t_0\,\gamma}{\ell^\eta_{(s-1)(t-1)}}+\delta\,q_s\,q_t \\
&\quad\le \frac{s_0\,t_0\,\gamma}{\ell^\eta_{(s-1)(t-1)}}+\delta\,M\,N.
\end{aligned}$$

Since $j_{s-1}, k_{t-1}\to\infty$ as $i,j\to\infty$, it follows that for each $y\in Y$

$$\frac{1}{(ij)^\eta}\left|\left\{(m,n):\, m\le i, n\le j,\ \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|\ge\xi\right\}\right|\to 0$$

and so for any $\delta_1>0$

$$\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{1}{(ij)^\eta}\left|\left\{(m,n):\, m\le i, n\le j,\ \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big)-\lambda\right|\ge\xi\right\}\right|\ge\delta_1\right\}\in\mathcal{I}_2.$$

Consequently, $U_{mn}\overset{\mathcal{I}_2^W(S_\lambda^\eta)}{\sim}V_{mn}$. □

Theorem 4.8. *Let $\theta_2 = \{(j_s, k_t)\}$ be double lacunary sequence. If*

$$1 < \liminf_s q_s^\eta \leq \limsup_s q_s < \infty \quad \text{and} \quad 1 < \liminf_t q_t^\eta \leq \limsup_t q_t < \infty$$

where $0 < \eta \leq 1$, then

$$U_{mn} \overset{\mathcal{I}_2^W(S_\lambda^\eta)}{\sim} V_{mn} \Leftrightarrow U_{mn} \overset{\mathcal{I}_{\theta_2}^W(S_\lambda^\eta)}{\sim} V_{mn}$$

Proof. This can be obtained from Theorem 4.6 and Theorem 4.7, immediately. □

Theorem 4.9. *Let $\theta_2 = \{(j_s, k_t)\}$ be double lacunary sequence. If $\liminf_s q_s^\eta > 1$ and $\liminf_t q_t^\eta > 1$ where $0 < \eta \leq 1$, then*

$$U_{mn} \overset{\mathcal{I}_2^W[C_\lambda^\eta]}{\sim} V_{mn} \Rightarrow U_{mn} \overset{\mathcal{I}_{\theta_2}^W[N_\lambda^\eta]}{\sim} V_{mn}.$$

Proof. Let double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical strong $\mathcal{I}_2$-Cesàro equivalent to multiple λ of order η. Also suppose that $\liminf_s q_s^\eta > 1$ and $\liminf_t q_t^\eta > 1$. Then, there exist $\alpha, \beta > 0$ such that $q_s^\eta \geq 1 + \alpha$ and $q_t^\eta \geq 1 + \beta$ for all s, t, which implies that

$$\frac{\ell_{st}^\eta}{h_{st}^\eta} \leq \frac{(1+\alpha)(1+\beta)}{\alpha\beta} \quad \text{and} \quad \frac{\ell_{(s-1)(t-1)}^\eta}{h_{st}^\eta} \leq \frac{1}{\alpha\beta}.$$

For each $y \in Y$, we have

$$\begin{aligned}
\frac{1}{h_{st}^\eta} \sum_{(m,n)\in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| &= \frac{1}{h_{st}^\eta} \sum_{r,u=1,1}^{j_s,k_t} \left| \rho_y\Big(\frac{U_{ru}}{V_{ru}}\Big) - \lambda \right| \\
&\quad - \frac{1}{h_{st}^\eta} \sum_{r,u=1,1}^{j_{s-1},k_{t-1}} \left| \rho_y\Big(\frac{U_{ru}}{V_{ru}}\Big) - \lambda \right| \\
&= \frac{\ell_{st}^\eta}{h_{st}^\eta} \left(\frac{1}{\ell_{st}^\eta} \sum_{r,u=1,1}^{j_s,k_t} \left| \rho_y\Big(\frac{U_{ru}}{V_{ru}}\Big) - \lambda \right| \right) \\
&\quad - \frac{\ell_{(s-1)(t-1)}^\eta}{h_{st}^\eta} \left(\frac{1}{\ell_{(s-1)(t-1)}^\eta} \sum_{r,u=1,1}^{j_{s-1},k_{t-1}} \left| \rho_y\Big(\frac{U_{ru}}{V_{ru}}\Big) - \lambda \right| \right).
\end{aligned}$$

Since $U_{mn} \overset{\mathcal{I}_2^W[C_\lambda^\eta]}{\sim} V_{mn}$, for each $y \in Y$ the following limits are hold

$$\frac{1}{\ell_{st}^\eta} \sum_{r,u=1,1}^{j_s,k_t} \left| \rho_y\Big(\frac{U_{ru}}{V_{ru}}\Big) - \lambda \right| \to 0 \quad \text{and} \quad \frac{1}{\ell_{(s-1)(t-1)}^\eta} \sum_{r,u=1,1}^{j_{s-1},k_{t-1}} \left| \rho_y\Big(\frac{U_{ru}}{V_{ru}}\Big) - \lambda \right| \to 0.$$

Thus, when the above equality is considered, it follows that for each $y \in Y$

$$\frac{1}{h_{st}^{\eta}} \sum_{(m,n)\in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \to 0$$

and so for any $\xi > 0$

$$\left\{ (s,t) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{st}^{\eta}} \sum_{(m,n)\in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \geq \xi \right\} \in \mathcal{I}_2.$$

Consequently, $U_{mn} \overset{\mathcal{I}_{\theta_2}^{W}[N_\lambda^\eta]}{\sim} V_{mn}$. □

Theorem 4.10. *Let $\theta_2 = \{(j_s, k_t)\}$ be double lacunary sequence. If $\limsup_s q_s < \infty$ and $\limsup_t q_t < \infty$, then*

$$U_{mn} \overset{\mathcal{I}_{\theta_2}^{W}[N_\lambda^\eta]}{\sim} V_{mn} \Rightarrow U_{mn} \overset{\mathcal{I}_{2}^{W}[C_\lambda^\eta]}{\sim} V_{mn}$$

where $0 < \eta \leq 1$.

Proof. Let double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalent to multiple λ of order η. Also suppose that $\limsup_s q_s < \infty$ and $\limsup_t q_t < \infty$. Then, there exist $M, N > 0$ such that $q_s < M$ and $q_t < N$ for all s, t. Since $U_{mn} \overset{\mathcal{I}_{\theta_2}^{W}[N_\lambda^\eta]}{\sim} V_{mn}$; for a given $\xi > 0$ and each $y \in Y$ we can find $s_0, t_0 > 0$ and $\vartheta > 0$ such that

$$\sup_{\substack{m \geq s_0 \\ n \geq t_0}} \tau_{mn} < \xi \quad \text{and} \quad \tau_{mn} < \vartheta \quad \text{for all} \quad m, n = 1, 2, \ldots$$

where

$$\tau_{st} = \frac{1}{h_{st}^{\eta}} \sum_{(m,n)\in I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right|.$$

If i and j are any integers satisfying $j_{s-1} < i \leq j_s$ and $k_{t-1} < j \leq k_t$ where $s > s_0$ and $t > t_0$, then for each $y \in Y$ we have

$$\begin{aligned}
\frac{1}{(ij)^{\eta}} \sum_{m,n=1,1}^{i,j} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| &\leq \frac{1}{\ell_{(s-1)(t-1)}^{\eta}} \sum_{m,n=1,1}^{j_s,k_t} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \\
&= \frac{1}{\ell_{(s-1)(t-1)}^{\eta}} \Bigg(\sum_{I_{11}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \\
&\quad + \sum_{I_{12}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| + \sum_{I_{21}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \\
&\quad + \sum_{I_{22}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| + \cdots + \sum_{I_{st}} \left| \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda \right| \Bigg)
\end{aligned}$$

$$= \frac{h_{11}^{\eta}}{\ell_{(s-1)(t-1)}^{\eta}}\,\tau_{11} + \frac{h_{12}^{\eta}}{\ell_{(s-1)(t-1)}^{\eta}}\,\tau_{12} + \frac{h_{21}^{\eta}}{\ell_{(s-1)(t-1)}^{\eta}}\,\tau_{21}$$

$$+ \frac{h_{22}^{\eta}}{\ell_{(s-1)(t-1)}^{\eta}}\,\tau_{22} + \cdots + \frac{h_{st}^{\eta}}{\ell_{(s-1)(t-1)}^{\eta}}\,\tau_{st}$$

$$\leq \sum_{m,n=1,1}^{s_0,t_0} \frac{h_{mn}}{\ell_{(s-1)(t-1)}}\,\tau_{mn} + \sum_{m,n=s_0+1,t_0+1}^{s,t} \frac{h_{mn}}{\ell_{(s-1)(t-1)}}\,\tau_{mn}$$

$$\leq \Bigg(\sup_{\substack{1\leq m\leq s_0\\ 1\leq n\leq t_0}} \tau_{mn}\Bigg)\frac{\ell_{s_0t_0}}{\ell_{(s-1)(t-1)}} + \Bigg(\sup_{\substack{m\geq s_0\\ n\geq t_0}} \tau_{mn}\Bigg)\frac{(j_s-j_{s_0})(k_t-k_{t_0})}{\ell_{(s-1)(t-1)}}$$

$$\leq \vartheta\,\frac{\ell_{s_0t_0}}{\ell_{(s-1)(t-1)}} + \xi\,M\,N.$$

Since $j_{s-1}, k_{t-1} \to \infty$ as $i, j \to \infty$, it follows that for each $y \in Y$

$$\frac{1}{(ij)^{\eta}} \sum_{m,n=1,1}^{i,j} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \to 0$$

and so for any $\xi > 0$

$$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{(ij)^{\eta}} \sum_{m,n=1,1}^{i,j} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \geq \xi\right\} \in \mathcal{I}_2.$$

Consequently, $U_{mn} \overset{\mathcal{I}_2^W[C_\lambda^\eta]}{\sim} V_{mn}$. □

Theorem 4.11. *Let $\theta_2 = \{(j_s, k_t)\}$ be double lacunary sequence. If*

$$1 < \liminf_s q_s^{\eta} \leq \limsup_s q_s < \infty \quad \textit{and} \quad 1 < \liminf_t q_t^{\eta} \leq \limsup_t q_t < \infty$$

where $0 < \eta \leq 1$, then

$$U_{mn} \overset{\mathcal{I}_2^W[C_\lambda^\eta]}{\sim} V_{mn} \Leftrightarrow U_{mn} \overset{\mathcal{I}_{\theta_2}^W[N_\lambda^\eta]}{\sim} V_{mn}.$$

Proof. This can be obtained from Theorem 4.9 and Theorem 4.10, immediately. □

5. Conclusions and Future Work

We presented new convergence concepts for double set sequences which are called Wijsman asymptotical $\mathcal{I}_2$-lacunary statistical equivalence of order η and Wijsman asymptotical strong $\mathcal{I}_2$-lacunary equivalence of order η where $0 < \eta \leq 1$. Also, we studied the relationships between them. Using the concepts of invariant mean and modulus function, these concepts can also be extended to more general convergence concepts for double set sequences in the future.

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