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ON GENERALIZATION OF BI-PSEUDO-STARLIKE FUNCTIONS

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Abstract. We introduce certain subclasses of bi-univalent functions related to the strongly Janowski functions and discuss the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for the newly defined classes. Also, we deduce certain new results and known results as special cases of our investigation.

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1. Introduction

An analytic function f in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

is said to be in the class $\mathcal{A}$. We denote by $\mathcal{S}$, $\mathcal{S}^*$ and $\mathcal{P}$ the classes of functions $f \in \mathcal{A}$ that are univalent, starlike and Carathodory functions, respectively, in $\mathcal{U}$.

We say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwartz function w in $\mathcal{U}$ such that $f(z) = g(w(z))$. In addition, if $g \in \mathcal{S}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$. Using the concept of subordination, Janowski [8] introduced the class $\mathcal{P}[A, B]$ of analytic functions p such that $p(z) \prec (1 + Az) / (1 + Bz)$, for $-1 \leq B < A \leq 1$, $z \in \mathcal{U}$.

Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. Then $p \in \mathcal{P}_\alpha[A, B]$, if and only if,

$$p(z) \prec \left(\frac{1 + Az}{1 + Bz}\right)^\alpha, \quad \alpha \in (0, 1], \ -1 \leq B < A \leq 1, \ z \in \mathcal{U}.$$

where $p_1, p_2 \in \mathcal{P}[A, B]$. Furthermore, let $p \in \mathcal{P}_{m,\alpha}[A, B]$, if and only if,

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$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z),$$

where $p_1, p_2 \in \mathcal{P}_\alpha[A, B]$ and $m \geq 2$.

Particularly, for $\alpha = 1$ the class $\mathcal{P}_{m,\alpha}[A, B]$ coincides with the class $\mathcal{P}_m[A, B]$ introduced in [14], whereas, for $\alpha = 1$, $A = 1 - 2\beta$ and $B = -1$, the class $\mathcal{P}_{m,\alpha}[A, B]$ reduces to the class $\mathcal{P}_m(\beta)$ of analytic univalent functions p, normalized with $p(0) = 1$ and satisfying

$$\int_0^{2\pi} \left| \frac{\Re(p(z)) - \beta}{1 - \beta} \right| d\theta \leq m\pi,$$

where $m \geq 2$, $\beta \in [0, 1)$ and $z \in \mathcal{U}$, we refer to [15]. Moreover, for $\beta = 0$, we have the class $\mathcal{P}_m(0) = \mathcal{P}_m$, introduced by Pinchuk [16]. Furthermore, for $m = 2$ we have well known class $\mathcal{P}$ of Caratheodory functions. Also, we note that, when $m = 2$, $A = 1$ and $B = -1$, then $p \in P_{2,\alpha}[1, -1]$ implies $|\arg p(z)| \leq \frac{\alpha\pi}{2}$.

It is well known by Koebe one quarter theorem [7] that the image of $\mathcal{U}$ under every function $f \in \mathcal{S}$ contains a disc of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \quad (z \in \mathcal{U})$$

and

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f),\ r_0(f) \geq 1/4\right).$$

The following is the series expansion of the inverse of f, (we say, $g(w) = f^{-1}(w)$),

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \ldots \quad (2)$$

A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathcal{U}$ if there exists a function $g \in \mathcal{S}$ such that $g(z)$ is an univalent extension of f^{-1} to $\mathcal{U}$. We denote by $\sum$ the class of bi-univalent in $\mathcal{U}$. The functions $\frac{z}{1-z}$, $-\log(1 - z)$ and $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ are in the class $\sum$; see [18]. However, the familiar Koebe function is not bi-univalent. Various classes of bi-univalent functions were introduced and studied in recent times, the study of bi-univalent functions gained momentum mainly due to the work of Srivastava et al. [18]. Many researchers [1, 2, 3, 4, 5, 6, 9, 11, 12, 13] recently investigated several interesting subclasses of the class $\sum$ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients.

Motivated by the work on bi-univalent functions in [11], we define a new subclass $\sum\mathcal{B}_{[A,B]}^{\gamma,\lambda,\alpha}(m, \mu)$ and determine the bounds for initial Taylor-Maclaurin coefficients of $|a_2|$ and $|a_3|$ for $f \in \sum\mathcal{B}_{[A,B]}^{\gamma,\lambda,\alpha}(m, \mu)$.

Definition 1.1. A function $f \in \sum$ is said to be in the class $\sum\mathcal{B}_{[A,B]}^{\gamma,\lambda,\alpha}(m, \mu)$ if the following conditions are satisfied

$$1 + \frac{1}{\gamma}\left[\frac{z\left(f'(z)\right)^\lambda}{(1 - \mu)z + \mu f(z)} - 1\right] \in \mathcal{P}_{m,\alpha}[A, B], \quad (z \in \mathcal{U})$$

and

$$1+\frac{1}{\gamma}\left[\frac{z\left(g'\left(w\right)\right)^{\lambda}}{\left(1-\mu\right)w+\mu g\left(w\right)}-1\right]\in\mathcal{P}_{m,\alpha}\left[A,B\right],\quad\left(w\in\mathcal{U}\right),$$

where $-1\leq B<A\leq 1$, $m\geq 2$, $\lambda\geq 1$, $\alpha\in(0,1]$, $\mu\in[0,1]$ and $\gamma\in\mathbb{C}\backslash\{0\}$, and $g\left(w\right)$ is given by (2).

Special cases:

(i) We note that, for $\gamma=1$ we get a new class $\sum\mathcal{B}_{[A,B]}^{1,\lambda,\alpha}\left(m,\mu\right)=\sum\mathcal{B}_{[A,B]}^{\lambda,\alpha}\left(m,\mu\right)$ of functions $f\in\sum$ satisfying the following two conditions

$$\frac{z\left(f'\left(z\right)\right)^{\lambda}}{\left(1-\mu\right)z+\mu f\left(z\right)}\in\mathcal{P}_{m,\alpha}\left[A,B\right],\quad\left(z\in\mathcal{U}\right)$$

and

$$\frac{z\left(g'\left(w\right)\right)^{\lambda}}{\left(1-\mu\right)w+\mu g\left(w\right)}\in\mathcal{P}_{m,\alpha}\left[A,B\right],\quad\left(w\in\mathcal{U}\right),$$

where $-1\leq B<A\leq 1$, $m\geq 2$, $\lambda\geq 1$, $\mu\in[0,1]$ and $\alpha\in(0,1]$, and $g\left(w\right)$ is given by (2).

(ii) For $\alpha=\gamma=1$, we obtain a new class $\sum\mathcal{B}_{[A,B]}^{1,\lambda,1}\left(m,\mu\right)=\sum\mathcal{B}_{[A,B]}^{\lambda}\left(m,\mu\right)$ of functions $f\in\sum$ such that

$$\frac{z\left(f'\left(z\right)\right)^{\lambda}}{\left(1-\mu\right)z+\mu f\left(z\right)}\in\mathcal{P}_{m}\left[A,B\right],\quad\left(z\in\mathcal{U}\right)$$

and

$$\frac{z\left(g'\left(w\right)\right)^{\lambda}}{\left(1-\mu\right)w+\mu g\left(w\right)}\in\mathcal{P}_{m}\left[A,B\right],\quad\left(w\in\mathcal{U}\right),$$

where $-1\leq B<A\leq 1$, $m\geq 2$, $\mu\in[0,1]$ and $\lambda\geq 1$, and $g\left(w\right)$ is given by (2).

(iii) For $m=2$ and $\gamma=1$, we obtain a new class $\sum\mathcal{B}_{[A,B]}^{1,\lambda,\alpha}\left(2,\mu\right)=\sum\mathcal{B}_{[A,B]}^{\lambda,\alpha}\left(\mu\right)$ of functions $f\in\sum$ such that

$$\frac{z\left(f'\left(z\right)\right)^{\lambda}}{\left(1-\mu\right)z+\mu f\left(z\right)}\in\mathcal{P}_{\alpha}\left[A,B\right],\quad\left(z\in\mathcal{U}\right)$$

and

$$\frac{z\left(g'\left(w\right)\right)^{\lambda}}{\left(1-\mu\right)w+\mu g\left(w\right)}\in\mathcal{P}_{\alpha}\left[A,B\right],\quad\left(w\in\mathcal{U}\right),$$

where $-1\leq B<A\leq 1$, $\alpha\in(0,1]$, $\mu\in[0,1]$ and $\lambda\geq 1$, and $g\left(w\right)$ is given by (2).

(iv) For $\gamma=\alpha=1$, $A=1-2\beta$ and $B=-1$, we get the class $\sum\mathcal{B}^{\lambda}\left(m,\mu\right)$ introduced in [11].

(v) For $\gamma=\alpha=\mu=1$, $m=2$, $A=1-2\beta$ and $B=-1$, we get the class $\sum\mathcal{B}^{\lambda}\left(\beta\right)$ introduced in [10].

(vi) For $\gamma=\mu=1$, $m=2$, $A=1$ and $B=-1$, we get the class $\sum\mathcal{B}^{\lambda}\left(\alpha\right)$ introduced in [10].

2. Main Results

The following lemmas are required to prove our investigations.

Lemma 2.1. [17] *Let* $q(z)=1+\sum_{n=1}^{\infty}q_nz^n$ *be subordinate to* $Q(z)=\sum_{n=1}^{\infty}Q_nz^n$. *If* $Q(z)$ *is univalent in* $\mathcal{U}$ *and* $Q(\mathcal{U})$ *is convex, then*

$$|q_n|\leq|Q_1|\,,\quad \text{for } n\geq 1.$$

The following lemma can be easily proved by using Lemma 2.1 along with the definition of $\mathcal{P}_{\alpha}[A,B]$.

Lemma 2.2. *Let* $p\in\mathcal{P}_{\alpha}[A,B]$ *with* $p(z)=1+\sum\limits_{n=1}^{\infty}p_nz^n$. *Then, for* $\alpha\in(0,1]$, $-1\leq A<B\leq 1$ *and* $n\geq 1$,

$$|p_n|\leq\alpha\,(A-B)\,,\ \ \text{for } n\geq 1.$$

Lemma 2.3. *Let* $m\geq 2$, $\alpha\in(0,1]$, $-1\leq A<B\leq 1$ *and let* $p\in\mathcal{P}_{m,\alpha}[A,B]$ *with* $p(z)=1+\sum\limits_{n=1}^{\infty}p_nz^n$. *Then*

$$|p_n|\leq\frac{m\alpha}{2}\,(A-B)\,,\ \ \text{for } n\geq 1.$$

Proof. This proof is straight forward by using Lemma 2.2 along with the definition of $\mathcal{P}_{m,\alpha}[A,B]$. □

Theorem 2.4. *Let* $f\in\sum\mathcal{B}_{[A,B]}^{\gamma,\lambda,\alpha}(m,\mu)$ *be given by* (1). *Then*

$$|a_2|\leq\min\left\{\sqrt{\frac{m\alpha\,(A-B)\,|\gamma|}{2\,[2\lambda^2+\lambda\,(1-2\mu)-\mu\,(1-\mu)]}};\frac{m\alpha\,(A-B)\,|\gamma|}{2\,(2\lambda-\mu)}\right\}$$

and

$$|a_3|\leq\min\left\{\begin{array}{c}\frac{m\alpha(A-B)|\gamma|}{2(3\lambda-\mu)}+\frac{m\alpha(A-B)|\gamma|}{2[2\lambda^2+\lambda(1-2\mu)-\mu(1-\mu)]};\\ \frac{m\alpha(A-B)|\gamma|}{2(3\lambda-\mu)}\left[1+\frac{m\alpha(A-B)|\gamma|\left\{2\lambda^2-2\lambda(\mu+1)+\mu^2\right\}}{2(2\lambda-\mu)^2}\right];\\ \frac{m\alpha(A-B)|\gamma|}{2(3\lambda-\mu)}\left[1+\frac{m\alpha(A-B)|\gamma|\left\{2\lambda^2+(2\lambda-\mu)(2-\mu)\right\}}{2(2\lambda-\mu)^2}\right]\end{array}\right\},$$

with $-1\leq B<A\leq 1$, $m\geq 2$, $\lambda\geq 1$, $\alpha\in(0,1]$, $\mu\in[0,1]$ *and* $\gamma\in\mathbb{C}\backslash\{0\}$. *Moreover,*

$$|a_3-\vartheta a_2|\leq\frac{m\alpha\,(A-B)\,|\gamma|}{2\,(3\lambda-\mu)},$$

where $\vartheta=\frac{2\lambda^2+(2\lambda-\mu)(2-\mu)}{(3\lambda-\mu)}$.

Proof. Let $f\in\sum\mathcal{B}_{[A,B]}^{\gamma,\lambda}(m,\phi)$ be given by (1). Then there exists two analytic functions $p,q\in\mathcal{P}_{m,\alpha}[A,B]$ with

$$p(z)=1+p_1z+p_2z^2+... \tag{3}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + ... \tag{4}$$

such that

$$1 + \frac{1}{\gamma}\left[\frac{z\left(f'\left(z\right)\right)^{\lambda}}{\left(1-\mu\right)z + \mu f\left(z\right)} - 1\right] = p(z) \tag{5}$$

and

$$1 + \frac{1}{\gamma}\left[\frac{z\left(g'\left(w\right)\right)^{\lambda}}{\left(1-\mu\right)w + \mu g\left(w\right)} - 1\right] = q(w), \tag{6}$$

where $g(w)$ is given by (2).

On the other hand

$$1 + \frac{1}{\gamma}\left[\frac{z\left(f'\left(z\right)\right)^{\lambda}}{\left(1-\mu\right)z + \mu f\left(z\right)} - 1\right] = 1 + \frac{\left(2\lambda-\mu\right)}{\gamma}a_2 z$$
$$+ \frac{1}{\gamma}\left[\left\{2\lambda^2 - 2\lambda\left(\mu+1\right) + \mu^2\right\}a_2^2 + \left(3\lambda-\mu\right)a_3\right]z^2 + ... \tag{7}$$

and

$$1 + \frac{1}{\gamma}\left[\frac{z\left(g'\left(w\right)\right)^{\lambda}}{\left(1-\mu\right)w + \mu g\left(w\right)} - 1\right] = 1 - \frac{\left(2\lambda-\mu\right)}{\gamma}a_2 w$$
$$+ \frac{1}{\gamma}\left[\left\{2\lambda^2 + \left(2\lambda-\mu\right)\left(2-\mu\right)\right\}a_2^2 - \left(3\lambda-\mu\right)a_3\right]w^2 + ... \tag{8}$$

From (3), (4), (7) and (8) comparing the coefficients of z, w, z^2 and w^2, we obtain

$$\frac{\left(2\lambda-\mu\right)}{\gamma}a_2 = p_1 \tag{9}$$

$$\frac{1}{\gamma}\left[\left\{2\lambda^2 - 2\lambda\left(\mu+1\right) + \mu^2\right\}a_2^2 + \left(3\lambda-\mu\right)a_3\right] = p_2 \tag{10}$$

$$-\frac{\left(2\lambda-\mu\right)}{\gamma}a_2 = q_1 \tag{11}$$

and

$$\frac{1}{\gamma}\left[\left\{2\lambda^2 + \left(2\lambda-\mu\right)\left(2-\mu\right)\right\}a_2^2 - \left(3\lambda-\mu\right)a_3\right] = q_2. \tag{12}$$

From (9) and (11), we can write

$$a_2 = \frac{\gamma p_1}{\left(2\lambda-\mu\right)} = -\frac{\gamma q_1}{\left(2\lambda-\mu\right)}. \tag{13}$$

From Lemma 2.3, it follows that

$$|a_2| \leq \frac{m\alpha\left(A-B\right)|\gamma|}{2\left(2\lambda-\mu\right)}. \tag{14}$$

Adding (10) and (12), we get

$$\left\{4\lambda^2 + 2\lambda\left(1-2\mu\right) - 2\mu\left(1-\mu\right)\right\}a_2^2 = \gamma\left(p_2+q_2\right),$$

by applying Lemma 2.3 and simple calculations yields

$$|a_2| \leq \sqrt{\frac{m\alpha\,(A-B)\,|\gamma|}{2\left[2\lambda^2+\lambda\,(1-2\mu)-\mu\,(1-\mu)\right]}}. \tag{15}$$

Subtracting (10) from (12) to get

$$a_3 = \frac{\gamma\,(p_2-q_2)}{2\,(3\lambda-\mu)} + a_2^2.$$

Now, employing Lemma 2.3 and (14), we obtain

$$|a_3| \leq \frac{m\alpha\,(A-B)\,|\gamma|}{2\,(3\lambda-\mu)} + \frac{m\alpha\,(A-B)\,|\gamma|}{2\left[2\lambda^2+\lambda\,(1-2\mu)-\mu\,(1-\mu)\right]}. \tag{16}$$

On making use of (9) and (10), we can easily find

$$|a_3| \leq \frac{m\alpha\,(A-B)\,|\gamma|}{2\,(3\lambda-\mu)}\left[1+\frac{2m\alpha\,(A-B)\,|\gamma|\left\{2\lambda^2-2\lambda\,(\mu+1)+\mu^2\right\}}{4\,(2\lambda-\mu)^2}\right]. \tag{17}$$

Again, by using (9) and (12), we finally obtain

$$|a_3| \leq \frac{m\alpha\,(A-B)\,|\gamma|}{2\,(3\lambda-\mu)}\left[1+\frac{2m\alpha\,(A-B)\,|\gamma|\left\{2\lambda^2+(2\lambda-\mu)\,(2-\mu)\right\}}{4\,(2\lambda-\mu)^2}\right]. \tag{18}$$

From (12), we can write

$$\frac{2\lambda^2+(2\lambda-\mu)\,(2-\mu)}{(3\lambda-\mu)}a_2^2 - a_3 = \frac{\gamma q_2}{(3\lambda-\mu)}.$$

By employing Lemma 2.3, this implies

$$|a_3-\vartheta a_2| = \left|\frac{\gamma q_2}{(3\lambda-\mu)}\right| \leq \frac{m\alpha\,(A-B)\,|\gamma|}{2\,(3\lambda-\mu)}, \tag{19}$$

where $\vartheta = \frac{2\lambda^2+(2\lambda-\mu)(2-\mu)}{(3\lambda-\mu)}$. Hence, the inequalities (14) to (19) follows our required proof. □

We note that for specializing the parameters, as mentioned in special cases (i)-(iii) of Definition 1.1, we deduce the following new results.

Corollary 2.5. *Let* $f \in \sum\mathcal{B}_{[A,B]}^{\lambda,\alpha}(m,\mu)$ *be given by* (1). *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{m\alpha\,(A-B)}{2\left[2\lambda^2+\lambda\,(1-2\mu)-\mu\,(1-\mu)\right]}}; \frac{m\alpha\,(A-B)}{2\,(2\lambda-\mu)}\right\}$$

and

$$|a_3| \leq \min\left\{\begin{array}{c} \frac{m\alpha(A-B)}{2(3\lambda-\mu)} + \frac{m\alpha(A-B)}{2[2\lambda^2+\lambda(1-2\mu)-\mu(1-\mu)]}; \\ \frac{m\alpha(A-B)}{2(3\lambda-\mu)}\left[1+\frac{m\alpha(A-B)\left\{2\lambda^2-2\lambda(\mu+1)+\mu^2\right\}}{2(2\lambda-\mu)^2}\right]; \\ \frac{m\alpha(A-B)}{2(3\lambda-\mu)}\left[1+\frac{m\alpha(A-B)\left\{2\lambda^2+(2\lambda-\mu)(2-\mu)\right\}}{2(2\lambda-\mu)^2}\right] \end{array}\right\},$$

with $-1 \leq B < A \leq 1$, $m \geq 2$, $\lambda \geq 1$, $\alpha \in (0,1]$ *and* $\mu \in [0,1]$. *Moreover,*

$$|a_3 - \vartheta a_2| \leq \frac{m\alpha\,(A-B)}{2\,(3\lambda-\mu)},$$

where $\vartheta = \frac{2\lambda^2+(2\lambda-\mu)(2-\mu)}{(3\lambda-\mu)}$.

Corollary 2.6. *Let* $f \in \sum \mathcal{B}^{\lambda}_{[A,B]}(m,\mu)$ *be given by* (1). *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{m\,(A-B)}{2\,[2\lambda^2+\lambda\,(1-2\mu)-\mu\,(1-\mu)]}};\frac{m\,(A-B)}{2\,(2\lambda-\mu)}\right\}$$

and

$$|a_3| \leq \min\left\{\begin{array}{c} \frac{m(A-B)}{2(3\lambda-\mu)} + \frac{m(A-B)}{2[2\lambda^2+\lambda(1-2\mu)-\mu(1-\mu)]}; \\ \frac{m(A-B)}{2(3\lambda-\mu)}\left[1+\frac{m(A-B)\left\{2\lambda^2-2\lambda(\mu+1)+\mu^2\right\}}{2(2\lambda-\mu)^2}\right]; \\ \frac{m(A-B)}{2(3\lambda-\mu)}\left[1+\frac{m(A-B)\left\{2\lambda^2+(2\lambda-\mu)(2-\mu)\right\}}{2(2\lambda-\mu)^2}\right] \end{array}\right\},$$

with $-1 \leq B < A \leq 1$, $m \geq 2$, $\lambda \geq 1$ *and* $\mu \in [0,1]$. *Moreover,*

$$|a_3 - \vartheta a_2| \leq \frac{m\,(A-B)}{2\,(3\lambda-\mu)},$$

where $\vartheta = \frac{2\lambda^2+(2\lambda-\mu)(2-\mu)}{(3\lambda-\mu)}$.

Corollary 2.7. *Let* $f \in \sum \mathcal{B}^{\lambda,\alpha}_{[A,B]}(\mu)$ *be given by* (1). *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{\alpha\,(A-B)}{[2\lambda^2+\lambda\,(1-2\mu)-\mu\,(1-\mu)]}};\frac{\alpha\,(A-B)}{(2\lambda-\mu)}\right\}$$

and

$$|a_3| \leq \min\left\{\begin{array}{c} \frac{\alpha(A-B)}{(3\lambda-\mu)} + \frac{\alpha(A-B)}{[2\lambda^2+\lambda(1-2\mu)-\mu(1-\mu)]}; \\ \frac{\alpha(A-B)}{(3\lambda-\mu)}\left[1+\frac{\alpha(A-B)\left\{2\lambda^2-2\lambda(\mu+1)+\mu^2\right\}}{(2\lambda-\mu)^2}\right]; \\ \frac{\alpha(A-B)}{(3\lambda-\mu)}\left[1+\frac{\alpha(A-B)\left\{2\lambda^2+(2\lambda-\mu)(2-\mu)\right\}}{(2\lambda-\mu)^2}\right] \end{array}\right\},$$

with $-1 \leq B < A \leq 1$, $\lambda \geq 1$, $\alpha \in (0,1]$ *and* $\mu \in [0,1]$. *Moreover,*

$$|a_3 - \vartheta a_2| \leq \frac{\alpha\,(A-B)}{(3\lambda-\mu)},$$

where $\vartheta = \frac{2\lambda^2+(2\lambda-\mu)(2-\mu)}{(3\lambda-\mu)}$.

Taking $A = 1-2\beta$ and $B = -1$ in Corollary 2.6, we obtain the following result proved in [11].

Corollary 2.8. *Let $f \in \sum \mathcal{B}^{\lambda}(m,\mu)$ be given by* (1). *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{m(1-\beta)}{[2\lambda^2+\lambda(1-2\mu)-\mu(1-\mu)]}}; \frac{m(1-\beta)}{(2\lambda-\mu)}\right\}$$

and

$$|a_3| \leq \min\left\{\begin{array}{c} \frac{m(1-\beta)}{(3\lambda-\mu)} + \frac{m(1-\beta)}{[2\lambda^2+\lambda(1-2\mu)-\mu(1-\mu)]}; \\ \frac{m(1-\beta)}{(3\lambda-\mu)}\left[1+\frac{m(1-\beta)\left\{2\lambda^2-2\lambda(\mu+1)+\mu^2\right\}}{(2\lambda-\mu)^2}\right]; \\ \frac{m(1-\beta)}{(3\lambda-\mu)}\left[1+\frac{m(1-\beta)\left\{2\lambda^2+(2\lambda-\mu)(2-\mu)\right\}}{(2\lambda-\mu)^2}\right] \end{array}\right\},$$

with $\beta \in [0,1)$, $m \geq 2$, $\lambda \geq 1$ and $\mu \in [0,1]$. Moreover,

$$|a_3 - \vartheta a_2| \leq \frac{m(1-\beta)}{(3\lambda-\mu)},$$

where $\vartheta = \frac{2\lambda^2+(2\lambda-\mu)(2-\mu)}{(3\lambda-\mu)}$.

If we set $\mu = 1$ and $m = 2$ in the previous corollary, we deduce the following.

Corollary 2.9. *Let $f \in \sum \mathcal{B}^{\lambda}(\beta)$ be given by* (1). *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\beta)}{\lambda(2\lambda-1)}}; \frac{2(1-\beta)}{2\lambda-1}\right\}$$

and

$$|a_3| \leq \min\left\{\begin{array}{c} \frac{2(1-\beta)}{(3\lambda-1)} + \frac{2(1-\beta)}{\lambda(2\lambda-1)}; \\ \frac{2(1-\beta)}{(3\lambda-1)}\left[1+\frac{2(1-\beta)\left\{2\lambda^2-4\lambda+1\right\}}{(2\lambda-1)^2}\right]; \\ \frac{2(1-\beta)}{(3\lambda-1)}\left[1+\frac{2(1-\beta)\left(2\lambda^2+2\lambda-1\right)}{(2\lambda-1)^2}\right] \end{array}\right\},$$

with $\beta \in [0,1)$ and $\lambda \geq 1$. Moreover,

$$|a_3 - \vartheta a_2| \leq \frac{2(1-\beta)}{3\lambda-1},$$

where $\vartheta = \frac{2\lambda^2+(2\lambda-1)}{(3\lambda-1)}$.

Taking $m = 2$, $\mu = 1$, $A = 1$ and $B = -1$ in Corollary 2.5, we get the following.

Corollary 2.10. *Let $f \in \sum \mathcal{B}^{\lambda}(\alpha)$ be given by* (1). *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2\alpha}{\lambda(2\lambda-1)}}; \frac{2\alpha}{2\lambda-1}\right\}$$

and

$$|a_3| \leq \min \left\{ \begin{array}{c} \frac{2\alpha}{(3\lambda-1)} + \frac{2\alpha}{\lambda(2\lambda-1)}; \\ \frac{2\alpha}{(3\lambda-1)} \left[1 + \frac{2\alpha\left\{2\lambda^2-4\lambda+1\right\}}{(2\lambda-1)^2} \right]; \\ \frac{2\alpha}{(3\lambda-1)} \left[1 + \frac{2\alpha\left(2\lambda^2+2\lambda-1\right)}{(2\lambda-1)^2} \right] \end{array} \right\},$$

with $\lambda \geq 1$ *and* $\alpha \in (0, 1]$. *Moreover,*

$$|a_3 - \vartheta a_2| \leq \frac{2\alpha}{3\lambda - 1},$$

where $\vartheta = \frac{2\lambda^2+2\lambda-1}{3\lambda-1}$.

Remark 2.1. The estimates obtained in the Corollary 2.9 and Corollary 2.10 are the improvements of the estimates proved by the authors, as Theorem 1 and Theorem 2, in [10].

3. Conclusion

The main aim of this paper is to estimate the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for the subclass of analytic functions associated with generalized strongly Janowski functions. Several new and known results are derived from our main investigations.

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