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SOME RESULTS FOR THE FRACTIONAL INTEGRAL OPERATOR DEFINED ON THE SOBOLEV SPACES

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Abstract. We investigated the invariant subspaces of the fractional integral operator in the Sobolev space $W_p^k[0,1]$ and prove unicellularity of the operator J^α by using the Duhamel product.

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1. Introduction and Background

In this manuscript, we consider unicellularity problem for the fractional integral operator

$$J^\alpha f(x) = \frac{1}{\Gamma(x)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \ \ \operatorname{Re}\alpha > 0$$

which is the complex powers of the integration operator $J^1 = \int_0^x f(t)\,dt$, where $f \in W_p^k[0,1]$ and $W_p^k[0,1] = \{f : f$ has absolutely continuous derivatives on $[0,1]$ up to order $k-1$ and have the derivative $f^{(k)}(x) \in L_p[0,1], p > 1\}$. If $k = 0$ we set $W_p^0[0,1] = L_p[0,1]$. A linear bounded operator A which is defined on $W_p^k[0,1]$ is said to be unicellular if it lattice of invariant subspaces is totally ordered with respect to the inclusion operation, i.e. if $E_1, E_2 \in \operatorname{Lat} A$ then $E_1 \subset E_2$ or $E_2 \subset E_1$. Note that an integration operator J^1 is an unicellular operator on the Banach spaces (see Brodskii [3], Nikolskii [20]). In [21], it was shown that J^1 is unicellular on the space $C^{(n)}[0,1]$.

It is known that [7, 20] the fractional integral operator is unicellular on $L_p[0,1], p \in [1,\infty)$. In other words, the lattices of invariant and hyperinvariant

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subspaces of the operator J^α are of the form

$$\text{Lat } J^\alpha = \text{HypLat } J^\alpha = \left\{E_a := X_{[0,1]} L_p [0,1] : 0 \le a \le 1\right\}.$$

In [25] invariant subspaces were investigated for the integration operator J^k defined on the Sobolev space $W_2^k[0,1]$. Domonov and Malamud [5] have extended these results for the fractional integral operator defined in the Sobolev spaces $W_p^k[0,1]$. Also, various applications of fractional integral operator can be found in [1, 12, 18, 27].

Some results related with non-trivial invariant subspaces and unicellularity problem for the integration operator $V = \int_0^x f(t)\,dt$ in various spaces have been obtained with application of the Duhamel product in papers [4, 9, 10, 11, 13, 14, 15, 16, 22, 23, 24]. It arises the question of study of the invariant subspaces of the operator J^α on the Sobolev spaces $W_p^k[0,1]$ with application of the Duhamel product. Answering this question in this paper we investigate unicellularity problem for the fractional integral operator defined on the space $W_p^k[0,1]$ and describe the lattice $\text{Lat} J^\alpha$ of invariant subspaces.

Consider the fractional order operator

$$J^\alpha : f \to \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt, \tag{1}$$

where $\Gamma(.)$ is the Euler Gamma function and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$. Here we suppose $f(x) \in W_p^k[0,1]$. The norm law on the space $W_p^k[0,1]$ is defined as

$$\|f\|_{W_p^k} = \sum_{i=0}^{k-1} \left| f^{(i)}(0) \right| + \left\| f^{(k)} \right\|_{L_p}$$

Lemma 1.1. *(*[2]*) Let $n \in \mathbb{N}_0$ The space $W_p^k[0,1]$ consists of those and only those functions f which are represented in the form*

$$f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} \varphi(t)dt + \sum_{k=0}^{n-1} c_k x^k, \tag{2}$$

where $\varphi(t) \in L_p[0,1]$ and c_k $(k = 0, 1, 2, ..., n-1)$ are constants such that $\varphi(t) = f^{(n)}(t)$, $c_k = \frac{f^{(k)}(0)}{k!}$.

Note that for $\operatorname{Re}\alpha > 0$ we have that the fractional integral operator J^α is a bounded operator on the space $L_p[0,1]$ In this case we have

$$\|J^\alpha\|_{L_p} \le C \|f\|_{L_p} \tag{3}$$

where $C = \frac{1}{|\Gamma(\alpha)| \operatorname{Re}\alpha}$. From the above lemma we have immediately

$$\|J^\alpha f\|_{W_p^k} \le C_1 \|f\|_{W_p^k}, \tag{4}$$

where $C_1 > 0$ is a constant.

2. Unicellularity of the fractional integration operator in the Sobolev spaces

We consider the operator $J_{k,0}^{\alpha} := J^{\alpha}$ acting on the space

$$W_{p,0}^{k}[0,1] = \left\{ f(x) \in W_{p}^{k}[0,1] : f^{(i)} = 0,\ i = 0,1,...,k-1 \right\}.$$

In this case $f(x) = J^{k} f^{(k)}(x)$ and $f^{(k)} \in L_{p}[0,1]$, where J^{k} is k^{th} power of the integration operator. Consequently, we have

$$J^{\alpha} f(x) = J^{\alpha+k} f^{(k)}(x) = J^{k}(J^{\alpha} f^{(k)}(x)).$$

Further, if we denote

$$U_{k} = \frac{d_{k}}{dx^{k}} : W_{p}^{k}[0,1] \to L_{p}[0,1],$$

then $J^{\alpha} f(x) = \left(U_{k}^{-1} J_{0}^{\alpha} U_{k}\right) f(x)$ which implies that $J^{\alpha} = U^{-1} J_{0}^{\alpha} U$. Moreover if we define the norm in the space $W_{p}^{k}[0,1]$ as

$$\|f\|_{W_{p}^{k}[0,1]} = \left[\sum_{k=0}^{k-1} \left| f^{(i)}(0) \right|^{p} + \int_{0}^{1} \left| f_{(x)}^{(k)} \right|^{p} dx \right]^{\frac{1}{p}}$$

then the operator U_{k} is an isometry by this norm. Indeed, if $f \in W_{p,0}^{k}[0,1]$ then

$$\begin{aligned} \|U_{k} f\|_{L_{P}} &= \int_{0}^{1} \left| f^{(k)}(t) \right|^{p} dt = \left[\sum_{i=0}^{k-1} \left| f^{(i)}(0) \right|^{p} + \int_{0}^{1} \left| f^{(k)}(t) \right|^{p} dt \right]^{\frac{1}{p}} \\ &= \|f\|_{W_{p}^{k}[0,1]}. \end{aligned}$$

Now let $J_{k,l}^{\alpha}$ is the operator J^{α} acting on the subspaces

$$E_{l}^{k} = \left\{ f \in W_{p}^{k}[0,1] : f^{(i)}(0) = 0,\ i = 0,1,...,k-l-1 \right\}$$

if $f(x) \in E_{l}^{k}$ then

$$f(x) = \frac{1}{(k-l-1)!} \int_{0}^{x} (x-t)^{k-l-1} f^{(k-l)}(t)\, dt,$$

where $f^{(k-l)}(x) \in W_{p}^{l}[0,1]$. By this we have for $f \in E_{l}^{k}$:

$$J_{k,l}^{\alpha} f(x) = J^{\alpha} J^{k-l} f^{(k-l)}(x) = J^{k-l} J^{\alpha} D^{k-l} f(x) = U_{k-l}^{-1} J_{l}^{\alpha} U_{k-l} f(x).$$

Therefore

$$J_{k,l}^{\alpha} = U_{k-l}^{-1} J_{l}^{\alpha} U_{k-l},$$

where $U_{k-l} = D^{k-l}$ is the differentiation operator and it maps E_l^k to $W_p^l[0,1]$. Moreover, if $f \in E_l^k$ we have

$$\|U_{k-l}f\|_{W_p^l[0,1]} = \left[\sum_{m=0}^{l-1}\left|f^{(k-l+m)}(0)\right|^p + \int_0^1 \left|f^{(k)}(t)\right|^p dt\right]^{\frac{1}{p}}$$
$$= \left[\sum_{j=k-l}^{k-1}\left|f^{(j)}(0)\right|^p + \int_0^1 \left|f^{(k)}(t)\right|^p dt\right]^{\frac{1}{p}} = \|f\|_{E_l^k}.$$

Hence we have proved the following lemma :

Lemma 2.1. *The operator $J_{k,l}^{\alpha}$ acting on the subspace E_l^k is isometrically equivalent to the operator J_l^{α} defined on $W_p^l[0,1]$ $(l = 0, 1, ..., k-1)$.*

From the Lemma 1.1 we have that following theorem.

Theorem 2.2. *If $\operatorname{Re}\alpha > 0$ then the operator $J_{k,0}^{\alpha}$ acting on the space $W_p^k[0,1]$ is unicellular and*

$$\operatorname{Lat} J_{k,0}^{\alpha} = \left\{E_a^k : 0 \le a \le 1\right\}$$

where

$$E_a^k = \left\{f \in W_{p,0}^k[0,1] : f(x) = 0 \text{ for } x \in [0,a]\right\}.$$

Consider the Duhamel product (see [26])

$$(f \circledast g)(x) = \frac{d}{dx}\int_0^x f(x-t)\,g(t)dt = \int_0^x f'(x-t)g(t)dt + f(0)g(x) \tag{5}$$

where $f, g \in W_p^k[0,1]$. It is easy to obtain

$$(f \circledast g)^{(m)}(x) = \int_0^x f^{(m)}(x-t)\,g'(t)dt + \sum_{i=0}^{m-1} f^{(i)}(0)\,g^{(m-i)}(x) + f^{(m)}(x)\,g(0) \tag{6}$$

where $m = 0, 1, ..., k$. From Equation (6) we can write

$$(f \circledast g)^{(m)}(x) = \int_0^x f'(t)\,g^{(m)}(x-t)\,dt + \sum_{i=0}^{m-1} f^{(m-i)}(x)\,g^{(i)}(0) + f(0)g^{(m)}(x). \tag{7}$$

Now (6) and (7) imply

$$(f \circledast g)^{(m)}(x) = \frac{1}{2}\int_0^x f^{(m)}(x-t)\,g'(t)dt + \frac{1}{2}\int_0^x f'(t)\,g^{(m)}(x-t)\,dt$$
$$+ \frac{1}{2}\sum_{i=0}^{m-1} f^{(i)}(0)\,g^{(m-i)}(x) + \frac{1}{2}\sum_{i=0}^{m-1} f^{(m-i)}(x)\,g^{(i)}(0)$$

$$+\frac{1}{2}f^{(m)}(x)\,g(0)+\frac{1}{2}f(0)\,g^{(m)}(x)\,.$$

Consequently,

$$\begin{aligned}(f\circledast g)^{(m)}(0)&=\frac{1}{2}\sum_{i=0}^{m}f^{(i)}(0)\,g^{(m-i)}(0)+\frac{1}{2}\sum_{i=0}^{m}f^{(m-i)}(0)\,g^{(i)}(0)\\&=\sum_{i=0}^{m}f^{(i)}(0)\,g^{(m-i)}(0)\,,\end{aligned}$$

and we can compute the following norm:

$$\begin{aligned}&\|(f\circledast g)\|^{p}_{W_p^k[0,1]}(x)\\&=\sum_{k=0}^{m-1}\left|(f\circledast g)^{(m)}(0)\right|^{p}+\int_0^1\left|(f\circledast g)^{(k)}(x)\right|^{p}dx\\&=\sum_{m=0}^{k-1}\left|\sum_{i=0}^{m}f^{(i)}(0)\,g^{(m-i)}(0)\right|^{p}\\&+\int_0^1\left|\frac{1}{2}\int_0^x f^{(k)}(x-t)\,g'(t)dt+\frac{1}{2}\int_0^x f^{'}(x)\,g^{(k)}(x-t)\,dt+\frac{1}{2}\sum_{i=0}^{k-1}f^{(i)}(0)\,g^{(k-i)}(x)\right.\\&\left.+\frac{1}{2}\sum_{i=0}^{k-1}f^{(k-i)}(x)\,g^{(i)}(0)+\frac{1}{2}f^{(k)}_{(x)}g(0)+\frac{1}{2}f(0)\,g^{(k)}(x)\right|^{p}dx.\end{aligned}\tag{8}$$

Since $\left|\sum_{i=1}^{n}x_i\right|^{p}\leq n^{p-1}\sum_{i=0}^{n}|x_i|^{p}$ is satisfied for any collection of complex numbers $x_1,x_2,...,x_n$ from Eq. (8) we have the following estimations:

$$\begin{aligned}\|(f\circledast g)\|_{W_p^k}(x)\leq&\sum_{m=0}^{k-1}(m+1)^{p-1}\sum_{i=0}^{m}\left|f^{(i)}(0)\right|^{p}\left|g^{(m-i)}(0)\right|^{p}\\&+(2k+n)^{p-1}\int_0^1\left[\frac{i}{2p}\left|\int_0^x f^{(n)}(x-t)\,g'(t)\,dt\right|^{p}\right.\\&+\frac{1}{2p}\left|\int_0^x f'(t)\,g^{(k)}(x-t)\,dt\right|^{p}+\frac{1}{2p}\sum_{i=0}^{k-1}\left|f^{(i)}(0)\right|^{p}\left|g^{(k-i)}(x)\right|^{p}\\&+\frac{1}{2p}\sum_{i=0}^{k-1}\left|f^{(k-i)}(x)\right|^{p}\left|g^{(i)}(0)\right|^{p}\\&\left.+\frac{1}{2p}\left|f^{(k)}(x)\,g(0)\right|^{p}+\frac{1}{2}\left|f(0)\,g^{(k)}(x)\right|^{p}\right]dx.\end{aligned}$$

Now using the generalized Minkowski inequality we have

$$\|(f \circledast g)\|_{W_p^k}(x) \leq k^{p-1}\sum_{i=0}^{k-1}\left|f^{(i)}(0)\right|^p \sum_{m=0}^{k-1}\left|g^{(i)}(0)\right|^p +$$

$$+\frac{(2k+n)^{p-1}}{2p}\left(\int_0^1 |g'(s)|\,ds\left(\int_0^1 \left|f^{(k)}(x)\right|^p dx\right)^{1/p}\right)^p$$

$$+\frac{(2k+n)^{p-1}}{2p}\left(\int_0^1 \left|f'(s)\right|ds\left(\int_0^1 \left|g^{(k)}(x)\right|^p dx\right)^{1/p}\right)^p$$

$$+\frac{(2k+n)^{p-1}}{2p}\left(\sum_{i=0}^{k-1}\left|f^{(i)}(0)\right|^p\int_0^1\left|g^{(k-i)}(x)\right|^p dx\right)$$

$$+\frac{(2k+n)^{p-1}}{2p}\left(\sum_{i=0}^{k-1}\left|g^{(i)}(0)\right|^p\int_0^1\left|f^{(k-i)}(x)\right|^p dx\right)$$

$$+\frac{(2k+n)^{p-1}}{2p}|g(0)|^p\int_0^1\left|f^{(k)}(x)\right|^p dx+$$

$$+\frac{(2k+n)^{p-1}}{2p}|f(0)|^p\int_0^1\left|g^{(k)}(x)\right|^p dx. \tag{9}$$

Since $f(x) \in W_p^k[0,1]$ then

$$f(x)=\frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} f^{(k)}(t)\,dt+\sum_{m=0}^{k-1}\frac{f^{(m)}(0)}{m!}x^m, \tag{10}$$

where $f^{(k)}(t) \in L_p[0,1]$. We can also write

$$f^{(k-i)}(x)=\frac{1}{(i-1)!}\int_0^x (x-t)^{i-1} f^{(k)}(t)\,dt+\sum_{m=0}^{i-1}\frac{f^{(m+k-i)}(0)}{m!}x^m \tag{11}$$

where $i=1,2,...,k$.

By the Eq. (10) and (11) we find

$$\int_0^1\left|f^{(k-i)}(x)\right|^p dx=\int_0^1\left|\frac{1}{(i-1)!}\int_0^x (x-t)^{i-1} f^{(k)}(t)\,dt+\sum_{m=0}^{i-1}\frac{f^{(m+k-i)}(0)}{m!}x^m\right|^p dx$$

$$\leq (i+1)^{(p-1)} \int_0^1 dx \left[\left| \int_0^x (x-t)^{(i-1)} f^{(k)}(t)\, dt \right|^p + \sum_{m=0}^{i-1} \left| f^{(m+k-i)}(0) \right|^p \left(\frac{x^m}{m!} \right)^p \right]$$

$$\leq (i+1)^{(p-1)} \left[\int_0^1 dt \left(\int_x^1 \left| t^{i-1} f^{(k)}(x) \right|^p dx \right)^{\frac{1}{p}} \right]^p$$

$$+ (i+1)^{(p+1)} \sum_{m=0}^{i-1} \left| f^{(m+k-i)}(0) \right|^p \int_0^1 \frac{x^{m^p} dx}{(m!)^p}$$

$$\leq (i+1)^{(p-1)} \left[\int_0^1 t^{i-1} dt \left(\int_0^1 \left| f^{(k)}(t) \right|^p dt \right)^{\frac{1}{p}} \right]^p$$

$$+ (i+1)^{p-1} \sum_{m=0}^{i-1} \left(f^{(m+k-i)}(0) \right)^p \frac{1}{(m!)^p (mp+1)}$$

$$\leq \frac{(i+1)^{(p-1)}}{i^p} \int_0^1 \left| f^{(k)}(t) \right|^p dt + (i+1)^{p-1} \sum_{m=0}^{k-1} \left| f^{(m)}(0) \right|^p,$$

i.e.

$$\int_0^1 \left| f^{(k-i)}(x) \right|^p dx \leq \frac{(i+1)^{(p-1)}}{i^p} \int_0^1 \left| f^{(k)}(x) \right|^p dx + (i+1)^{(p-1)} \sum_{m=0}^{k-1} \left| f^{(m)}(0) \right|^p. \tag{12}$$

Now we continue our estimations using (9) and (12) :

$$\|(f \circledast g)\|^p_{W^k_p} \leq k^{p-1} \sum_{i=0}^{k-1} \left| f^{(i)}(0) \right|^p \sum_{m=0}^{k-1} \left| g^{(i)}(0) \right|^p$$

$$+ \frac{(2k+n)^{p-1}}{2p} \left(\int_0^1 |g'(s)|\, ds \right)^p \int_0^1 \left| f^{(k)}(x) \right|^p dx$$

$$+ \frac{(2k+n)^{p-1}}{2p} \left(\int_0^1 |f'(s)|\, ds \right)^p \int_0^1 \left| g^{(k)}(x) \right|^p dx$$

$$+\frac{(2k+n)^{p-1}}{2p}\sum_{i=1}^{k-1}\left|f^{(i)}(0)\right|^p\left[\frac{(i+1)^{p-1}}{i^p}\int_0^1\left|g^{(k)}(x)\right|^p dx\right.$$

$$\left.+(i+1)^{p-1}\sum_{m=0}^{k-1}\left|g^{(m)}(0)\right|^p\right]+\frac{(2k+n)^{p-1}}{2p}|f(0)|^p\int_0^1\left|g^{(k)}(x)\right|^p dx$$

$$+\frac{(2k+n)^{p-1}}{2p}\sum_{i=1}^{k-1}\left|g^{(i)}(0)\right|^p\left[\frac{(i+1)^{p-1}}{i^p}\int_0^1\left|f^{(k)}(x)\right|^p dx\right.$$

$$\left.+(i+1)^{p-1}\sum_{m=0}^{k-1}\left|f^{(m)}(0)\right|^p\right]+\frac{(2k+n)^{p-1}}{2p}|g(0)|^p\int_0^1\left|f^{(k)}(x)\right|^p dx$$

$$+\frac{(2k+n)^{p-1}}{2p}|g(0)|^p\int_0^1\left|f^{(k)}(x)\right|^p dx$$

$$+\frac{(2k+n)^{p-1}}{2p}|f(0)|^p\int_0^1\left|g^{(k)}(x)\right|^p dx$$

$$\leq L(k,p)\left[\sum_{i=0}^{k-1}\left|f^{(i)}(0)\right|^p\sum_{i=0}^{k-1}\left|g^{(i)}(0)\right|^p+\int_0^1\left|f^{(k)}(x)\right|^p dx\left(\int_0^1|g'(s)|\,ds\right)^p\right.$$

$$+\int_0^1\left|g^{(k)}(x)\right|^p dx\left(\int_0^1|f'(s)|\,ds\right)^p$$

$$\left.+\int_0^1\left|g^{(k)}(x)\right|^p dx\sum_{i=0}^{k-1}\left|f^{(i)}(0)\right|^p+\int_0^1\left|f^{(k)}(x)\right|^p dx\sum_{i=0}^{k-1}\left|g^{(i)}(0)\right|^p\right]. \qquad (13)$$

Here $L(k,p)$ is a constant. Since

$$f'(x)=\frac{1}{(k-2)!}\int_0^x(x-t)^{k-2}f^{(k)}(t)\,dt+\sum_{m=1}^{k-1}\frac{f^{(m)}(0)}{(m-1)!}x^{m-1},$$

we have

$$\left(\int_0^1\left|f'(x)\right|^p dx\right)^p\leq\int_0^1\left|f'(x)\right|^p dx$$

$$\leq\frac{k^{p-1}}{(k-1)^p}\int_0^1\left|f^{(k)}(t)\right|^p dt+k^{p-1}\sum_{m=0}^{k-1}\left|f^{(m)}(0)\right|^p. \qquad (14)$$

Now from the last two inequalities we find that

$$\|(f \circledast g)\|_{W_p^k[0,1]}(x) \le L(k,p)\|f\|_{W_p^k[0,1]}\|g\|_{W_p^k[0,1]}. \tag{15}$$

The inequality

$$\|(f \circledast g)(x)\|_{W_p^k[0,1]} = L\|f\|_{W_p^k}\|g\|_{W_p^k}$$

shows that the operator

$$D_f g := f \circledast g,\ \ g \in W_p^k[0,1]$$

is continuous in the space $W_p^k[0,1]$. We also obtain that

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha+1)}\frac{d}{dx}\int_0^x (x-t)^\alpha f(t)\,dt = \frac{x^\alpha}{\Gamma(\alpha+1)}??f(x)$$

for $f(x) \in W_p^k[0,1]$.

Now we will prove the following Lemma.

Lemma 2.3. *Let $f \in W_p^k[0,1]$. Then f is $\circledast-$invertible in $W_p^k[0,1]$ if and only if $f(0) \neq 0$.*

Proof. If f is $\circledast-$invertible then $(f \circledast g)(0) = f(0)g(0) = 1$ which implies $f(0) \neq 0$. Let $f(0) \neq 0$. Prove that f is $\circledast-$invertible in the space $W_p^k[0,1]$. The operator D_f can be rewritten as

$$D_f = f(0)I + D_{f.,f(0)}$$

where I is an identity operator on $W_p^k[0,1]$. Let $h(x) = f(x) - f(0)$. Then

$$D_f = f(0)I + D_h$$

we have $h(0) = 0$ and consequently

$$(D_h g)(x) = \frac{d}{dx}\int_0^x h(x-t)g(t)\,dt = \int_0^x h'(x-t)g(t)\,dt. \tag{16}$$

It is easy to show that operator D_h is a bounded operator on $W_p^k[0,1]$. By using the inequality (15) we also obtain that D_h is a compact operator on $W_p^k[0,1]$. On other hand if $g(x) \in \ker\{f(0)I + D_h\}$ then $(f \circledast g)(x) = 0$. Therefore by the Titchmarch's convolution theorem [19] we have $\ker\{D_f\} = \{0\}$. Thus, by the well-known Fredholm alternative [6] D_f is an invertible on $W_p^k[0,1]$. □

Lemma 2.4. *Let $g \in E_l^{(k)}$, $l = 0,1,...,k-1$. If $g(x) \neq 0$ in any right neighborhood of zero, then*

$$\operatorname{span}\{(J^\alpha)^m g : m \ge 0\} = E_l^{(k)}.$$

Proof. We know that if $g(x) \in E_l^{(k)}$ then

$$g(x) = \frac{1}{(k-l-1)!} \int_0^x (x-t)^{x-l-g(k-l)}(t)\,dt,$$

where $g^{(k-l)}(x) \in W_p^l[0,1]$. Therefore we have

$$\begin{aligned}(J^\alpha)^m g(x) &= \frac{1}{\Gamma(k-l+\alpha_m+1)} x^{k-l+\alpha_m} \circledast g^{(k-l)}(x) \\ &= D_{g(k-l)} \frac{x^{k-l+\alpha_m}}{\Gamma(k-l+\alpha_m+1)}.\end{aligned}$$

Consequently we obtain

$$\begin{aligned}\operatorname{span}\{(J^\alpha)^m g : m \geq 0\} &= \overline{D_{g(k-l)} \operatorname{span}\left\{\frac{x^{k-l+\alpha_m}}{\Gamma(k-l+\alpha_m+1)}\right\}} \\ &= \overline{D_{g(k-l)} \operatorname{span}\left\{\frac{x^{k-l+m}}{m!} : m \geq 0\right\}} = E_l^{(k)}.\end{aligned}$$

□

The following two lemmas can be proved by the similar arguments (see [24]).

Lemma 2.5. *If* $\operatorname{Re}\alpha > 1-p$ *then* $f \in \operatorname{Cyc}(J^\alpha/E_\lambda)$ *in* $W_p^l[0,1]$ *if and only if* $f \in E_\lambda \backslash E_\mu$ *for every* $\mu > \lambda$.

Lemma 2.6. *If* $\operatorname{Re}\alpha > k - \frac{1}{p}$ $(k \geq 2)$ *or* $\alpha \in \mathbb{Z}_+ \backslash \{0\}$ *then* $f \in \operatorname{Cyc}(J^\alpha/E_\lambda)$ *in* $W_p^k[0,1]$ *if and only if* $\alpha = 1$ *and* $f \in E_\lambda \backslash E_\mu$ *for every* $\lambda < \mu$.

From the Lemmas 1.1, 2.1, 2.3 and 2.4, we have the following theorems.

Theorem 2.7. *The operator* $J_1^\alpha = J^\alpha$ *is unicellular in* $W_p^1[0,1]$ *if* $\operatorname{Re}\alpha > 1 - \frac{1}{p}$ *and* $\operatorname{Lat}J_1^\alpha = \left\{E_a^1 : 0 \leq a \leq 1\right\} \cup W_p^1[0,1]$.

Theorem 2.8. *If* $k \geq 2$ *and* $\operatorname{Re}\alpha > k - \frac{1}{p}$ *or* $\alpha = m \in \mathbb{Z}$, $m \neq 0$ *then* J_k^α *is unicellular in* $W_p^k[0,1]$ *if and only if* $\alpha = 1$.

3. Conclusion

In this paper we investigate the invariant subspaces of the fractional integral operator in the Sobolev space $W_p^k[0,1]$ and unicellularity of the operator J^α by using the Duhamel product and describe the lattice $\operatorname{Lat}J^\alpha$ of invariant subspaces.

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